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General solution of the Landau-Lifshitz-Gilbert equations linearized around a Bloch wall

J. S. Helman
Centro Brasileiro de Pesquisas Fisicas (CBPF), Rua Dr. Xavier Sigaud 150, Urca, Rio de Janeiro, RJ 22290, Brazil

H. B. Braun, J. S. Broz, and W. Baltensperger
Theoretische Physik, Eidgenössische Technische Hochschule (ETH), Hönggerberg, Zürich 8093, Switzerland
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The Landau-Lifshitz-Gilbert equations are linearized around a Bloch wall. A general planar solution of these equations is found. Using this, the time evolution of small planar deviations of arbitrary shape can be evaluated. The example of a wall moving in a magnetic field which is suddenly turned off is developed in some detail. The results are compared with existing computer experiments. The effects of a space- and time-dependent magnetic field and of anisotropy changes are also considered.

INTRODUCTION

The motion of Bloch walls is phenomenologically described by the nonlinear Landau-Lifshitz-Gilbert (LLG) equations.[1,2] For a 180° wall with uniaxial anisotropy in a static external field $\mathbf{H}$ the exact solution is known.[3] Here we have linearized the LLG equations around the static solution ($H=0$), and these equations have been solved exactly for any initial planar deformation of the wall. This describes the time evolution of small planar deviations of arbitrary shape, with and without damping. Of course, only in a few cases can prescribed deformations be realized in practice. A wall moving in a magnetic field which is suddenly turned off is treated as an example.

LLG EQUATIONS

The LLG equations can be written in the form

$$\frac{\partial \mathbf{M}}{\partial t} = -\gamma(\mathbf{M}\times\mathbf{H}_{\text{eff}}) + \frac{\alpha}{M_0}\left[\mathbf{M}\times\frac{\partial \mathbf{M}}{\partial t}\right],$$

with (1)

$$\mathbf{H}_{\text{eff}}(x,t) = -\frac{\delta W}{\delta \mathbf{M}(\mathbf{x},t)} .$$

Here W is the energy of the magnetic system expressed as a functional of the magnetization $\mathbf{M}(\mathbf{x},t)$, its gradients, and the external magnetic field $\mathbf{H}$; α is a dimensionless phenomenological damping coefficient, γ (>0) is the gyromagnetic factor, and M_0 is the saturation magnetization. $\delta/\delta\mathbf{M}$ denotes functional derivation.

For a ferromagnet with uniaxial anisotropy along the z direction and planar magnetization depending on y, W is

$$W=\int_{-\infty}^{\infty} dy\left[\frac{A}{2M_0^2}\left|\frac{\partial \mathbf{M}}{\partial y}\right|^2 - \frac{K}{2M_0^2}M_z^2 + 2\pi M_y^2 - HM_z\right], \tag{2}$$

where A is the exchange constant, K the anisotropy constant, and H the external field taken along the z direction. In the following we shall use dimensionless units defined by

$$\begin{aligned} &\Delta=(A/K)^{1/2}=1 \quad \text{(static wall width, length)},\\ &M_0=1 \quad \text{(magnetization)},\\ &\gamma M_0/(1+\alpha^2)=1 \quad \text{(frequency)}. \end{aligned} \tag{3}$$

In the new units Eqs. (1) reduce to

$$\dot{\mathbf{M}} = -(1+\alpha^2)\mathbf{M}\times\mathbf{H}_{\text{eff}} + \alpha(\mathbf{M}\times\dot{\mathbf{M}}), \tag{4}$$

with

$$\begin{aligned} &H_{\text{eff}x} = KM_x'',\\ &H_{\text{eff}y} = KM_y'' - 4\pi M_y,\\ &H_{\text{eff}z} = KM_z'' + KM_z + H. \end{aligned} \tag{5}$$

Here $\mathbf{M}'$ means derivative with respect to y and $\dot{\mathbf{M}}$ denotes time derivative.

For completeness, we state here the following equation for the energy change which follows from (4) and (2):

$$\dot{w} = K(\mathbf{M}'\cdot\dot{\mathbf{M}})' + \mathbf{H}_{\text{eff}}\cdot\dot{\mathbf{M}}, \tag{6}$$

where w is the integrand of (2). Also from Eq. (4) it follows that $\mathbf{H}_{\text{eff}}\cdot\dot{\mathbf{M}} = \alpha\dot{\mathbf{M}}^2/(1+\alpha^2)$.

The unit vector $\mathbf{M}$ can be described by two variables. The parametrization

$$\begin{aligned} &M_x = \cos(\varphi)\text{sech}(u),\\ &M_y = \sin(\varphi)\text{sech}(u),\\ &M_z = -\tanh(u), \end{aligned} \tag{7}$$

is convenient since the static Bloch wall corresponds to $\varphi=0$ and $u=y$. The LLG equations (4) read

$$\begin{aligned} &\dot{\varphi} = -F + \alpha G,\\ &\dot{u} = G + \alpha F, \end{aligned} \tag{8}$$

where

$$F=Ku''+\{K[(\varphi')^2-(u')^2]+K+4\pi\sin^2(\varphi)\}\tanh(u)-H\ ,$$
$$G=K\varphi''-2K\varphi'u'\tanh(u)-4\pi\sin(\varphi)\cos(\varphi)\ . \quad (9)$$

LINEARIZATION

Without damping ($\alpha=0$), the eigenstates of the linearized equations were obtained by Thiele, even for the case of a moving wall.[4] The LLG equations with damping will now be linearized around the static wall to first order in the azimuthal angle $\varphi(y,t)$ and $\eta(y,t)$ defined by

$$u=y+\eta\ . \quad (10)$$

The linearized version of (9) is

$$F=-K\mathcal{D}\eta-H\ ,$$
$$G=-K\mathcal{D}\varphi-4\pi\varphi\ , \quad (11)$$

where $\mathcal{D}$ stands for the linear differential operator

$$\mathcal{D}=-\frac{d^2}{dy^2}+2\tanh(y)\frac{d}{dy}\ . \quad (12)$$

In this section the external field **H** will be considered to be zero. From (7), (10), and (11) the linear equations read

$$\dot{\phi}=\underline{\mathcal{T}}\phi\ , \quad (13)$$

where

$$\phi=\begin{bmatrix}\varphi\\ \eta\end{bmatrix},\quad \underline{\mathcal{T}}=\begin{bmatrix}-\alpha(4\pi+K\mathcal{D}) & K\mathcal{D}\\ -(4\pi+K\mathcal{D}) & -\alpha K\mathcal{D}\end{bmatrix}. \quad (14)$$

A formal solution can be obtained by casting the linearized equations with their initial conditions in the form of integral equations and perceiving that they involve only convolutions and consequently are solvable by Laplace transforms. The solution is

$$\phi=\underline{\mathcal{M}}\phi_0\ , \quad (15)$$

where

$$\underline{\mathcal{M}}=e^{-\alpha(2\pi+K\mathcal{D})t}\begin{bmatrix}C-2\pi\alpha S & SK\mathcal{D}\\ -S(4\pi+K\mathcal{D}) & C+2\pi\alpha S\end{bmatrix},$$
$$\phi_0=\begin{bmatrix}\varphi_0\\ \eta_0\end{bmatrix}. \quad (16)$$

Here $\varphi_0=\varphi(y,0)$ and $\eta_0=\eta(y,0)$ are the initial deviations, and

$$C=\cosh(\mathcal{N}t),\quad S=\frac{\sinh(\mathcal{N}t)}{\mathcal{N}}\ , \quad (17)$$

with

$$\mathcal{N}^2=-K\mathcal{D}(4\pi+K\mathcal{D})+(2\pi\alpha)^2\ . \quad (18)$$

It is easy to check that $\dot{\underline{\mathcal{M}}}=\underline{\mathcal{T}}\,\underline{\mathcal{M}}$, $\underline{\mathcal{M}}=e^{\underline{\mathcal{T}}t}$, and $\det(\underline{\mathcal{M}})=e^{-2\alpha(2\pi+K\mathcal{D})t}$. The normal modes of the wall are given by the eigenstates of $\underline{\mathcal{T}}$.

SPECTRUM OF $\mathcal{D}$

The general solution (16) is useful if φ_0 and η_0 can be developed in eigenstates of $\mathcal{D}$. Hence we look for solutions of

$$-\frac{d^2\psi}{dy^2}+2\tanh(y)\frac{d\psi}{dy}=E\psi\ . \quad (19)$$

Substituting

$$\psi=\theta(y)\cosh(y)\ , \quad (20)$$

Eq. (19) looks like a Schrödinger equation with a potential $-2\,\mathrm{sech}^2(y)$,

$$-\theta''-2\,\mathrm{sech}^2(y)\theta=(E-1)\theta\ . \quad (21)$$

It has one localized state

$$E_0=0,\quad \theta_0(y)=2^{-1/2}\mathrm{sech}(y)\ , \quad (22)$$

and running states

$$E_k=1+k^2\ ,$$
$$\theta_k(y)=[-ik+\tanh(y)]e^{iky}/(1+k^2)^{1/2}\ , \quad (23)$$

where k is a real number. Equations (23) are the classical spin waves on a Bloch wall[4] and (22) is the zero-energy Goldstone mode and therefore represents a translation of the wall.

The functions described in (22) and (23) form an orthogonal and complete set, since they are eigenfunctions of a self-adjoint operator:

$$\int_{-\infty}^{+\infty}dy\ \theta_k^*(y)\theta_{k'}(y)=2\pi\delta(k-k')\ ,$$
$$\int_{-\infty}^{+\infty}dy\ \theta_k(y)\theta_0(y)=0\ ; \quad (24)$$

$$\sum_{n=0,k}^{k=\infty}\theta_n^*(y)\theta_n(y')\equiv\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\theta_k^*(y)\theta_k(y')+\theta_0(y)\theta_0(y')=\delta(y-y')\ . \quad (25)$$

To simplify the notation in the following, we use the index n to denote the running states and the localized one as well.

EIGENSTATES OF $\underline{\mathcal{T}}$

We look for solutions of

$$\underline{\mathcal{T}}\phi_n=\gamma_n\phi_n\ . \quad (26)$$

Trying the ansatz

$$\phi_n=\begin{bmatrix}\theta_n\\ p_n\theta_n\end{bmatrix}\cosh(y)\ , \quad (27)$$

where p_n is a number, consistency requires

$$\gamma_n^{\pm}=-\alpha(2\pi+KE_n)\pm\mathcal{N}_n\ ,$$
$$p_n^{\pm}=(2\pi\alpha\pm\mathcal{N}_n)/(KE_n)\ , \quad (28)$$

where

$$\mathcal{N}_n^2=(2\pi\alpha)^2-(4\pi+KE_n)KE_n\ . \quad (29)$$

Note that the $\gamma_n^\pm < 0$ always holds. This shows the stability of the static Bloch wall in the presence of damping with respect to one dimensional deformations.

For the localized state $E_0=0$, $N_0=2\pi\alpha$. Then the solution (28) with the plus sign

$$\gamma_0^+=0, \quad p_0^+=\infty, \quad \varphi_0=0, \quad \eta_0=\text{const} \tag{30}$$

represents a translation of the wall by the amount $-\eta_0$. The other solution (28) with the minus sign

$$\gamma_0^-=-4\pi\alpha, \quad p_0^-=1/\alpha, \quad \varphi_0=\text{const}, \quad \eta_0=\varphi_0/\alpha \tag{31}$$

contains a rotation and a translation. If the change of wall thickness with velocity is neglected, this mode by itself describes the relaxation of a wall when the field is suddenly turned off.

For the running states, a purely damped solution exists if

$$(2\pi/K)[(1+\alpha^2)^{1/2}-1]>1+k^2 . \tag{32}$$

Otherwise $\gamma_k^\pm$ is complex

$$\begin{aligned} \mathrm{Re}\gamma_k^\pm &= -\alpha[2\pi+K(1+k^2)] , \\ \mathrm{Im}\gamma_k^\pm &= \pm f(k) , \end{aligned} \tag{33}$$

where

$$f(k)=\{[4\pi+K(k^2+1)]K(k^2+1)-(2\pi\alpha)^2\}^{1/2} \tag{34}$$

is the spin-wave dispersion in the presence of anisotropy and demagnetization energy[5] and damping. These eigenstates of $\mathcal{T}$ can be used to construct special solutions of the equations of motion since

$$\phi_n(t)=\phi_n e^{\gamma_n^\pm t} \tag{35}$$

satisfies (13).

GENERAL SOLUTION OF THE HOMOGENEOUS LINEAR EQUATIONS

Now the initial deviations $\varphi(y,0)$ and $\eta(y,0)$ are developed in eigenfunctions of $\mathcal{D}$

$$\begin{aligned} \varphi(y,0) &= \sum_{n=0,k} a_n\theta_n(y)\cosh(y) , \\ \eta(y,0) &= \sum_{n=0,k} b_n\theta_n(y)\cosh(y) , \end{aligned} \tag{36}$$

where

$$\begin{aligned} a_n &= \int_{-\infty}^{\infty} \theta_n^*(y)\varphi(y,0)\mathrm{sech}(y)dy , \\ b_n &= \int_{-\infty}^{\infty} \theta_n^*(y)\eta(y,0)\mathrm{sech}(y)dy . \end{aligned} \tag{37}$$

Hence the solution (15) reads

$$\begin{aligned} \varphi(y,t) &= \sum_{n=0,k} e^{-\alpha(2\pi+KE_n)t}[(C_n-2\pi\alpha S_n)a_n \\ &\qquad +S_nKE_nb_n]\psi_n(y) , \\ \eta(y,t) &= \sum_{n=0,k} e^{-\alpha(2\pi+KE_n)t}[(C_n+2\pi\alpha S_n)b_n \\ &\qquad -S_n(4\pi+KE_n)a_n]\psi_n(y) . \end{aligned} \tag{38}$$

Here C_n and S_n represent the eigenvalues of the respective operators (17) in the state n.

From (7) and (38), the time-dependent part of the magnetization is

$$\begin{aligned} \delta m_x(y,t) &= -\tanh(y)\sum_{n=0,k} e^{-\alpha(2\pi+KE_n)t} \\ &\qquad \times[(C_n+2\pi\alpha S_n)b_n \\ &\qquad -S_n(4\pi+KE_n)a_n]\theta_n(y) , \\ \delta m_y(y,t) &= \sum_{n=0,k} e^{-\alpha(2\pi+KE_n)t} \\ &\qquad \times[(C_n-2\pi\alpha S_n)a_n+S_nKE_nb_n]\theta_n(y) , \\ \delta m_z(y,t) &= \mathrm{csch}(y)\delta m_x(y,t) . \end{aligned} \tag{39}$$

EXAMPLE: RELAXATION OF THE MOVING WALL WHEN THE EXTERNAL FIELD IS SUDDENLY TURNED OFF

Let us consider a wall moving in a static external field $\mathbf{H}$ in the positive z direction. The stationary Walker solution (see Ref. 6) to order φ^2 is

$$\varphi=\varphi_0=H/(4\pi\alpha), \quad u=(y-vt)(1+\chi_0) , \tag{40}$$

where $\chi_0=2\pi\varphi_0^2/K$ and $v=(1+\alpha^2)H/\alpha$. If at $t=0$ the magnetic field is suddenly turned off, the wall is left out of equilibrium with initial conditions

$$\varphi(y,0)=\varphi_0, \quad \eta(y,0)=\chi_0 y . \tag{41}$$

Then, using (36), taking into account that sech(y) is an eigenfunction of Eq. (21), and that the functions $\theta_n(y)$, given by Eqs. (22) and (27), form an orthonormal set, we obtain

$$a_n=2^{1/2}\varphi_0\delta_{n,0} \tag{42}$$

and

$$b_k=\frac{\pi\chi_0}{(1+k^2)^{1/2}}\mathrm{sech}(\pi k/2) . \tag{43}$$

Let us consider first δm_y. We now assume a sufficiently small α, i.e., $2\pi\alpha<\sqrt{K(4\pi+K)}$. Then using (39), (22), and (23) we obtain

$$\begin{aligned} \delta m_y(y,t) &= K\chi_0 e^{-\alpha(2\pi+K)t} \\ &\qquad \times\int_0^\infty dk\, e^{-\alpha Kk^2t}\mathrm{sech}(\pi k/2)\frac{\sin[f(k)t]}{f(k)} \\ &\qquad \times[k\sin(ky)+\tanh(y)\cos(ky)] \\ &\qquad +\varphi_0 e^{-4\pi\alpha t}\mathrm{sech}(y) . \end{aligned} \tag{44}$$

The integral can be evaluated in various limiting cases. Owing to the factor sech$(\pi k/2)$ the integrand in (44) only contributes appreciably for k up to $k_1\approx 1$.

For small t, $f(k_1)t \ll 1$, and $\alpha \ll 1$, Eq. (44) becomes

$$\begin{aligned} \delta m_y(y,t) &= \varphi_0\mathrm{sech}(y)(1-4\pi\alpha t) \\ &\qquad +2Kt\chi_0\mathrm{sech}(y)\tanh(y) . \end{aligned} \tag{45}$$

The first and second terms represent the damping and the forward movement, respectively.

For large t, $f(k_1)t \gg 1$. For this asymptotic limit we use the method of steepest descent. The integral

$$I_1 = \int_0^\infty dk\, \sin[f(k)t]\cos(ky) \tag{46}$$

becomes

$$I_1 \approx (\pi/[2tf''(k_0)])^{1/2}\sin[F(k_0)+\pi/4] , \tag{47}$$

with $F(k)=f(k)t-ky$ and where k_0 is the root of $f'(k)-y/t=0$. Here, the prime means derivative with respect to k. Analogously,

$$I_2 = \int_0^\infty dk\, \sin[f(k)t]\sin(ky) \approx (\pi/[2tf''(k_0)])^{1/2}\cos[F(k_0)+\pi/4] . \tag{48}$$

Since the remaining factors of Eq. (44) are slowly varying functions of k, the result is

$$\delta m_y(y,t) \approx \frac{K\chi_0}{f(k_0)}\left[\frac{\pi}{2tf''(k_0)}\right]^{1/2} e^{-\alpha(2\pi+K)t} e^{-\alpha K k_0^2 t} \times \operatorname{sech}(\pi k_0/2)\{k_0\cos[F(k_0)+\pi/4]+\tanh(y)\sin[F(k_0)+\pi/4]\}+\varphi_0 e^{-4\pi\alpha t}\operatorname{sech}(y) . \tag{49}$$

In the case $y/t \ll 1$, one obtains

$$\delta m_y(y,t) = K\chi_0\left[\frac{\pi}{2tC_2\omega_0^2}\right]^{1/2} e^{-\alpha\{2\pi+K[1+y^2/(C_2t)^2]\}t} \times\left[\frac{y}{C_2 t}\cos\left[\omega_0 t-\frac{y^2}{2C_2 t}+\frac{\pi}{4}\right]+\tanh(y)\sin\left[\omega_0 t-\frac{y^2}{2C_2 t}+\frac{\pi}{4}\right]\right]+\varphi_0 e^{-4\pi\alpha t}\operatorname{sech}(y) , \tag{50}$$

where $\omega_0 = f(0)$ and $C_2 = 2K(2\pi+K)/\omega_0$. The last term is the attenuated initial deformation. The first term describes a shift along y which oscillates in time. For large t, the time dependence is dominated by the lowest frequency ω_0. Note that the attenuation of the shift tends to be weaker than that of the initial deformation.

In the case $y/t \gg 1$ one obtains

$$\delta m_y(y,t) = K^2\chi_0\left[\frac{4\pi K t^3}{y^4}\right]^{1/2} e^{-\alpha[2\pi+K+y^2/(4Kt^2)]t} \times\operatorname{sech}\left[\frac{\pi y}{4Kt}\right]\left[\frac{y}{2Kt}\cos\left[\frac{y^2}{4Kt}-\frac{\pi}{4}\right]-\tanh(y)\sin\left[\frac{y^2}{4Kt}-\frac{\pi}{4}\right]\right]+\varphi_0 e^{-4\pi\alpha t}\operatorname{sech}(y) . \tag{51}$$

For a given time t, $\delta m_y(y,t)$ oscillates as function of y. The rather complicated behavior is due to the fact that the steepest-descent method selects a k vector which depends on y and t.

The calculation of δm_x is analogous. We only mention the interesting point that for $\alpha \neq 0$ the limit of δm_x for $t \to \infty$ is finite. From (39), (42), and (43) we obtain

$$\lim_{t\to\infty} \delta m_x = \frac{\varphi_0}{\alpha}\tanh(y)\operatorname{sech}(y) . \tag{52}$$

The wall comes to rest after a distance

$$d = \frac{H}{4\pi\alpha^2} \tag{53}$$

in the direction of motion.

It is also interesting to obtain an approximate expression for $\eta(y,t)$ in the limit of large t, which is related to an effective wall width, and can be compared with results from computer experiments. Starting from Eq. (37) with the initial conditions (41) and with the following approximations: $K \ll 4\pi$, $\omega_0 t \gg 1$, and $2\pi\alpha \ll \omega_0$, one obtains

$$\eta(y,t) \approx -\frac{\varphi_0}{\alpha}(1-e^{-4\pi\alpha t}) + \frac{\chi_0}{2}\left[\frac{\pi}{Kt^2}\right]^{1/4} e^{-2\pi\alpha t}\cos(\omega_0 t+\pi/4)\sinh(y) . \tag{54}$$

In the region where these expressions are meaningful ($y \leq 1$), $\sinh(y) \cong y$ and η describes a relative width change

$$\frac{\delta\Delta}{\Delta} = -\frac{\chi_0}{2}\left[\frac{\pi}{Kt^2}\right]^{1/4} e^{-2\pi\alpha t}\cos(\omega_0 t+\pi/4) . \tag{55}$$

COMPARISON WITH COMPUTER EXPERIMENTS

Schryer and Walker[6] studied with computer simulation the transients of Bloch walls when the external field is either suddenly switched on or off. They modeled various substances using the following constants: yttrium iron garnet (YIG): $K=0.59$, $4\pi M_0=1700$ Oe, $\alpha=0.001$ to 0.03; orthoferrite (material "O"): $K=1.6\times 10^4$,

$4\pi M_0=110$ Oe, $\alpha=0.01$; and iron: $K=0.29$, $4\pi M_0=2.15\times10^4$ Oe, $\alpha=0.8$. Here K is given in our units. The time t_{SW} used in Ref. 4 is given in units of $\gamma^{-1}\times(1\ \text{Oe})^{-1}$ $(=5.68\times10^{-8}\ \text{sec})$; thus $t=M_0t_{SW}/(1+\alpha^2)$ where M_0 is given in Oe.

Since Schryer and Walker noted that φ depended only weakly on y, they presented their time-dependent results using a mean value for φ. We shall therefore compare this time dependence with that of the space-independent part of our results.

First we compare relaxation times. The time dependence of φ corresponding to YIG with $\alpha=0.01$ reported in their Fig. 4 corresponds to an attenuation time $T_R=0.059$ in their units. This, according to Eq. (49), is to be identified with $(1+\alpha^2)/(4\pi M_0\alpha)=0.0588$. Analogously, for material O with $\alpha=0.01$ reported in Fig. 11 of Ref. 6, $T_R=0.914$ while we obtain 0.909.

According to (10) and (40), the wall velocity is minus the time derivative of the space-independent first term of Eq. (55),

$$v=4\pi\varphi_0e^{-4\pi\alpha t}\ , \tag{56}$$

while the second term in (54) alters the width of the wall. The relaxation (55) is consistent with the curve shown in Fig. 10 of Ref. 6. for material O.

Figure 9 of Ref. 6 shows a damped oscillation of the wall width for YIG with $\alpha=0.1$. An approximate period can be defined as the time between two maxima, which is $\Delta T_{SW}=0.0158$. This, according to Eq. (55), is to be compared with $2\pi(1+\alpha^2)/(M_0\omega_0)=0.017$. In the same Fig. 9, the decay of φ shows an oscillatory variation which has approximately the same period as the wall width. This period also appears in Eq. (50) in the range of small y. The most serious discrepancy is that the experimental time decay of the width $\delta\Delta$ in Fig. 9 is closer to $4\pi\alpha$ rather than to $2\pi\alpha$ as the linear theory predicts.

EXCITATION BY A TIME- AND SPACE-DEPENDENT MAGNETIC FIELD ALONG THE ANISOTROPY AXIS

Equations (8) with the linearized functions (11) read

$$\begin{aligned}\dot\varphi&=-\alpha(4\pi+K\mathcal{D})\varphi+K\mathcal{D}\eta+H\ ,\\ \dot\eta&=-(4\pi+K\mathcal{D})\varphi-\alpha K\mathcal{D}\eta-\alpha H\ .\end{aligned} \tag{57}$$

$H(y,t)$ is supposed to be small enough so that the linear approximation is justified.

The quantities $a_n(\omega)$, $b_n(\omega)$, and $H_n(\omega)$ are defined by

$$\begin{aligned}H(y,t)&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{n=0,k}H_n(\omega)\psi_n(y)e^{i\omega t}\ ,\\ \varphi(y,t)&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{n=0,k}a_n(\omega)\psi_n(y)e^{i\omega t}\ ,\\ \eta(y,t)&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{n=0,k}b_n(\omega)\psi_n(y)e^{i\omega t}\ .\end{aligned} \tag{58}$$

In Fourier space equations (57) read

$$\begin{aligned}i\omega a_n&=-\alpha(4\pi+KE_n)a_n+KE_nb_n+H_n\ ,\\ i\omega b_n&=-(4\pi+KE_n)a_n-\alpha KE_nb_n-\alpha H_n\ .\end{aligned} \tag{59}$$

The solutions are

$$a_n(\omega)=\frac{-i\omega H_n(\omega)}{(\omega+i\gamma_n^+)(\omega+i\gamma_n^-)}\ , \tag{60}$$

$$b_n(\omega)=\frac{(\alpha^2+1)(4\pi+KE_n)+i\alpha\omega}{(\omega+i\gamma_n^+)(\omega+i\gamma_n^-)}H_n(\omega)\ . \tag{61}$$

Here $\gamma_n^{\pm}$ is given by (28).

EXAMPLES

Homogeneous field: $H_n(\omega)=H_0(\omega)\delta_{n,0}$. According to (30) and (31), $\gamma_0^+=0$, $\gamma_0^-=-4\pi\alpha$, and Eqs. (60) and (61) reduce to

$$a_0(\omega)=\frac{-iH_0(\omega)}{(\omega-4\pi i\alpha)} \tag{62}$$

and

$$b_0(\omega)=\frac{4\pi(\alpha^2+1)+i\alpha\omega}{(\omega-i\epsilon)(\omega-4\pi i\alpha)}H_0(\omega)\ . \tag{63}$$

Here $\epsilon\to0$; causality requires $a_n(\omega)$ and $b_n(\omega)$ to be analytic in the lower half plane of ω.

A homogeneous field is turned on at $t=0$ with characteristic time λ^{-1}:

$$H(t)=H(1-e^{-\lambda t})\ , \tag{64}$$

$$H_0(\omega)=\sqrt{2}H\left[\frac{-i}{\omega-i\epsilon}-\frac{-i}{\omega-i\lambda}\right]\ . \tag{65}$$

Complex integration of (58) yields

$$\varphi(t)=H\left[\frac{1}{4\pi\alpha}(1-e^{-4\pi\alpha t})+\frac{e^{-4\pi\alpha t}-e^{-\lambda t}}{4\pi\alpha-\lambda}\right] \tag{66}$$

and

$$\eta(t)=H\left[-\frac{\alpha^2+1}{\alpha}t+\frac{1}{4\pi\alpha^2}(1-e^{-4\pi\alpha t})+\frac{\alpha^2+1}{\alpha\lambda}+\frac{[4\pi\alpha(\alpha^2+1)-\lambda\alpha^2]e^{-\lambda t}-\lambda e^{-4\pi\alpha t}}{\alpha\lambda(\lambda-4\pi\alpha)}\right]\ . \tag{67}$$

In the short-time limit

$$\varphi(t)\simeq\tfrac{1}{2}\lambda Ht^2,\quad \eta\simeq-\tfrac{1}{2}\lambda\alpha Ht^2\ , \tag{68}$$

while for long times the Walker solution

$$\varphi(\infty)=\frac{H}{4\pi\alpha},\quad v=-\dot\eta=\frac{1+\alpha^2}{\alpha}H \tag{69}$$

is recovered.

Space-dependent cases can be treated similarly. For instance, if $H(y)=He^{ipy}$ from (58) it is

$$H_n=H\int_{-\infty}^{\infty}dy\ \theta_n^*(y)e^{ipy}\text{sech}(y) \tag{70}$$

which can be exactly integrated, yielding

$$H_0=H\frac{\pi p}{\sqrt{2}}\operatorname{csch}\left[\frac{\pi p}{2}\right],$$
$$H_k=H\frac{i\pi p}{(1+k^2)^{1/2}}\operatorname{sech}\left[\frac{\pi(p-k)}{2}\right]. \tag{71}$$

ANISOTROPY CHANGE

Consider a small change $K\to K+\delta K$ in the anisotropy (which may result from a temperature change). This modifies K in Eq. (2); we use the unchanged K for the definition of units, so that the right-hand side of Eq. (9) contains an additional term $\delta K\tanh(y)$, which acts like an effective magnetic field

$$H'=-\delta K\tanh(y). \tag{72}$$

Now, $H'_0=0$ and

$$H'_k=-\delta K\int_{-\infty}^{\infty}dy\frac{ik+\tanh(y)}{(1+k^2)^{1/2}}e^{-iky}\tanh(y)\operatorname{sech}(y) \tag{73}$$

can be exactly integrated by parts, yielding

$$H'_k=-\frac{\pi}{2}\delta K(1+k^2)^{1/2}\operatorname{sech}(\pi k/2). \tag{74}$$

We consider explicitly the case of a sudden change $\theta(t)$ in the anisotropy, then

$$H'_k(\omega)=\frac{i\pi}{2}\delta K(1+k^2)^{1/2}\operatorname{sech}(\pi k/2)\frac{1}{\omega-i\epsilon}. \tag{75}$$

Using (60), (61), and (75) in (58), and performing complex integrations in ω, we get

$$\varphi(y,t)=-(\pi/2)\delta K\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-\alpha(2\pi+KE_k)t}\frac{\sin[f(k)t]}{f(k)}\operatorname{sech}(\pi k/2)[-ik+\tanh(y)]e^{iky}\cosh(y) \tag{76}$$

and

$$\eta(y,t)=-(\pi/2)\frac{\delta K}{K}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left[e^{-\alpha(2\pi+KE_k)t}\left[2\pi\alpha\frac{\sin[f(k)t]}{f(k)}+\cos[f(k)t]\right]-1\right]$$
$$\times\frac{\operatorname{sech}(\pi k/2)}{E_k}[-ik+\tanh(y)]e^{iky}\cosh(y). \tag{77}$$

To consider the limiting cases, the following exact integrals are useful

$$I_1\equiv\int_{-\infty}^{\infty}\frac{dk}{2\pi}\operatorname{sech}(\pi k/2)\frac{-ik+\tanh(y)}{1+k^2}\times e^{iky}\cosh(y)=y/\pi, \tag{78}$$

$$I_2=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\operatorname{sech}(\pi k/2)[-ik+\tanh(y)]e^{iky}\cosh(y)=(2/\pi)\tanh(y). \tag{79}$$

For $\omega_0 t\ll 1$ we get, using (79),

$$\varphi(y,t)=-\delta K\,t\tanh(y) \tag{80}$$

and

$$\eta(y,t)=\alpha\,\delta K\,t\tanh(y), \tag{81}$$

which could have been derived directly from (57) with H' instead of H. For $\omega_0 t\gg 1$, $\varphi(y,t)$ tends to zero while, according to (78):

$$\eta(y,\infty)=(\delta K/K)(y/2), \tag{82}$$

which gives the first-order correction to the wall width due to the new anisotropy $K+\delta K$:

$$y+\eta=y(1+\delta K/2K)=y/\Delta, \tag{83}$$

which agrees with the exact result up to order $\delta K/K$:

$$\Delta=[A/(K+\delta K)]^{1/2}\approx(A/K)^{1/2}-\tfrac{1}{2}(A/K)^{1/2}(\delta K/K)+\cdots \tag{84}$$

(in our units $A/K=1$). Asymptotic values of φ and η can be obtained by the method of steepest descent.

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APPENDIX

Since the magnetization is of constant length it is determined by two parameters. In the parametrization we followed Walker in directing the z axis along the external magnetic field. This parametrization, however, is singular when the magnetization points along the z direction in the sense that φ is not defined, a small deviation from the z direction is described by a finite φ, $0\le\varphi<2\pi$. The orientation along the z direction is realized far from the Bloch wall. The fact that the linearized equations correctly describe the spin-wave excitations may be fortuitous. In order to overcome this difficulty a coordinate system for which the singularity points into a different direction may be used.

Let us choose the z direction perpendicular to the plane of the Bloch wall, and the x axis along the easy direction of magnetization, so that the old system (x,y,z) goes into the new system (y,z,x). Then the new parametrization is

$$M_x' = -\sin\theta \tanh(u) \,,$$
$$M_y' = \sin\theta \,\mathrm{sech}(u) \,, \quad \text{(A1)}$$
$$M_z' = \cos\theta \,.$$

Linearizing around the static wall $\theta=\pi/2, u=z$:

$$\theta = \frac{\pi}{2} + \vartheta, \quad u = z + \eta \,, \quad \text{(A2)}$$

the linearized equations turn out to be

$$\dot{\psi} = (K\mathcal{R} + 4\pi)\vartheta - \alpha K\mathcal{R}\psi - \alpha H \,\mathrm{sech}(z) \,,$$
$$\dot{\vartheta} = -K\mathcal{R}\psi - \alpha(K\mathcal{R} + 4\pi)\vartheta - H \,\mathrm{sech}(z) \,, \quad \text{(A3)}$$

where $\psi = \eta \,\mathrm{sech}(z)$ and

$$\mathcal{R} = -\frac{d^2}{dz^2} + 1 - 2\,\mathrm{sech}^2(z) \,. \quad \text{(A4)}$$

On the other hand, from identifying the linearized versions of (A1) and (7), we see that $\vartheta = -\varphi \,\mathrm{sech}(z)$. Hence (A3) becomes equivalent to (13). This proves that (13) [and (57) respectively] correctly describe the spin-wave spectrum far from the Bloch wall.

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