PHYSICAL REVIEW B VOLUME 43, NUMBER 7 1 MARCH 1991

Renormalization-group theory of the incommensurate pinning transition and threshold dynamics

Leigh Sneddon
The Analytic Sciences Corporation, 55 Walkers Brook Drive, Reading, Massachusetts 01867

Sen Liu
Martin Fisher School of Physics, Brandeis University, Waltham, Massachusetts 02254

Allen J. Kassman
Philip Morris Research Center, P.O. Box 26583, Richmond, Virginia 23261

(Received 1 August 1990; revised manuscript received 29 October 1990)

We introduce a renormalization-group approach to incommensurate systems. The transformation can treat both static and dynamic properties. A symmetric fixed point of this transformation is found and is seen to determine the critical properties of the depinning transition. The universality class of this transition is found to include higher harmonics of the pinning potential, both those that preserve the inversion symmetry and those that break it. The theory is shown to yield converging estimates of the correlation-length exponent. The results for the dynamics show that systems near the threshold flow toward a strong-pinning threshold fixed point on the boundary of the physical Hamiltonian space.

INTRODUCTION

While experimental results have provided a major motivation for the theoretical study of charge-density-wave (CDW) conductors,[1] it has long been recognized[2] that the dynamic threshold of these nonlinear conductors constitutes an important theoretical challenge in its own right. Classical models of CDW conductors have three attributes that make their solution an interesting problem. First, they have no unit cell, so that there are an infinite number of inequivalent degrees of freedom that can affect the bulk properties. Second, they are dynamical systems far from equilibrium, and third, the dynamics is nonlinear.

Progress has been made in understanding these systems when the nonlinearity is weak, for example, when the CDW is in a dc field much larger than threshold. The random pinning potential of the Fukuyama-Lee-Rice (FLR) model[3] can then be treated within perturbation theory[4] and qualitative insight, for example concerning the role of dissipative screening currents, can be obtained. When the nonlinearity is stronger, further progress has been made with models in which the pinning potential is periodic, but incommensurate with the CDW.[4,5,7] For these incommensurate or Freukel-Kontorova (FK) models,[6] it is possible to transform the steady-state dc dynamics to a static system.[7] Truncating this static problem in Fourier space then showed that the dynamic properties of the incommensurate models describe a variety of experimentally observed properties of CDW conductors.[4,5,7] Some information was also obtained about the threshold region; the closer one is to it, the slower is the convergence of successive truncations. This is the expected behavior for a critical point:[1] an infinite number of degrees of freedom are contributing significantly to the macroscopic properties.

It is then natural, given the success of the ideas of Fisher, Kadanoff, and Wilson[8] in understanding equilibrium critical phenomena, to attempt to apply the concepts of the renormalization-group (RG) theory here. There has been a conjecture[9] as to what the RG flows for a CDW system might be like, but so far there has been no success in constructing a working RG transformation in this many-body dynamical system.

To describe the deterministic dynamics, we cannot make use of partition function, but the basic RG concepts of the successive removal of degrees of freedom, and the resulting flow of the system in some parameter space, may still be useful. Which degrees of freedom should be removed and how; and what is the correct parameter space?

These questions apply equally to random and incommensurate pinning. Solutions of incommensurate systems give a good account of a wide range of CDW experiments.[4,5,7] There is, thus, a variety of these experiments for which the precise form of the pinning potential is not crucial. This paper shows how the concepts of the RG can be applied to the pinning transition and dynamic threshold in incommensurate systems. The goals include an understanding of the critical properties, for example, a way to determine critical exponents without explicit numerical simulation of the critical quantities, and a determination of both the extent and the origin of the universality. It is hoped that, in addition to its intrinsic interest, the solution of this problem will also teach us how to think about the dynamic threshold in randomly pinned systems.

Dimensionless equations of motion for a generalized FK model can be written:

$$du_j/dt = P(\tau j + u_j + \alpha) + 2\pi \sum_p D_p u_{j-p} + E \ , \qquad (1)$$

where u_j is the displacement of the jth particle in the chain; $j=1,2,\ldots,N$; $\tau=M/N$; M and N have no common factor and approach infinity so that τ approaches a fixed irrational number; the D_p are the elastic coefficients; E is a uniform electric field; $P(x)$ is a periodic pinning force with period 1; and α is an adjustable phase.

Unlike ferromagnetic critical behavior, CDW dynamics has an important underlying geometry: The critical nature of the threshold arises from the mismatch between the spatial structures of the pinning potential and the periodic CDW. While decimation and site-cell transformations have been useful for ferromagnets, it would seem wise here to choose a basis for the internal degrees of freedom which respects the underlying geometry. For incommensurate systems with $P(x+1)=P(x)$, a natural choice is

$$|W_m| = \left| 1/N \sum_j e^{-i2\pi m \tau j}(u_j - \langle u \rangle) \right| . \tag{2}$$

In addition these are the variables which, by proper choice of their phase, map the dc dynamics to a static problem.[7] To see this, note that a dc solution to Eq. (1) is given by

$$u_j = \langle u \rangle + g(\tau j + \langle u \rangle) , \tag{3}$$

where $g(x)=g(x+1)$ is a periodic time-independent function satisfying

$$v\left[1+\frac{dg}{dx}\right] = P[x+g(x)+\alpha] + 2\pi \sum_p D_p g(x-\tau p) + E , \tag{4}$$

where $v=\langle du_j/dt \rangle$. We can then write

$$g(x) = \sum_m W_m e^{i2mx} \tag{5}$$

and the W_m are also constant in time.

This transformation of the dc dynamics to a static problem is clearly of use in a RG theory where we seek a renormalized Hamiltonian with no explicit time dependence.

The use of the Fourier basis W_m is also suggested by the work of Shenker and Kadanoff (SK).[10] A theory of FK systems should include the special case of $E=0$. Here, there is a critical pinning transition which has been carefully studied.[11] Shenker and Kadanoff calculated the numerical values of the Fourier components and found that these variables revealed structure at all scales at the pinning transition: Within a power-law envelope, there is an asymptotic self-similarity in successive logarithmic intervals of the harmonic number, m.

This in turn means an asymptotic scale invariance in real space. The mth component of the chain's distortion has the form, for particle j, of $e^{im2\pi\tau j}$. The inverse wavelength of this distortion is thus the smallest absolute residue, R_m, between $m\tau$ and an integer. Now the integers (and, hence, the components W_m) can be divided into sequences such that R_m decreases, by a factor which approaches τ, from one member of a sequence to the next.[12] That is, the wavelength of each component is asymptotically a factor of τ greater than that of the preceding member of the sequence. These sequences are precisely those which show up in SK's results. Each successive element of a sequence is in the next logarithmic "period" of SK's self-similarity. This means that, as the length scale is increased by a factor of τ, the amplitude of the distortion diminishes by the same factor resulting in an asymptotic scale invariance in real space.

These observations then suggest the basis for a renormalization transformation which removes the shortest-length scales from the problem. The distortions with the shortest wavelengths are just those components which are the first members of all the sequences. Consider then a bare or starting incommensurate system. Determine the Fourier components of the distortions and arrange them into these special sequences. Discard the first member of each sequence. Relabel the modes so that the second member of each sequence becomes the first, the third becomes the second, etc. Rescale each component by a factor m/m', where m was its original harmonic number and m' is its new harmonic number. Now regard the new sets of modes as the distortions of a renormalized system, which must be determined.

We have examined several implementations of these ideas to check that there is nothing artifactual in the results. Here, we outline one of these implementations.

Fourier transforming Eq. (2) we obtain the time-independent equations of state for the W_m:

$$2\pi\left[\sum_{p>0} 2D_p[1-\cos(2\pi m\tau p)] + imv\right] W_m = \mathcal{F}_m(W_m), \tag{6}$$

where $v=E+\mathcal{F}_0(W_m)$ and

$$\mathcal{F}_m(W_m) = \int_{-1/2}^{1/2} dx\, e^{-i2\pi mx} P(x+g(x)+\alpha) . \tag{7}$$

The starting system is defined by specifying the pinning force $P(x)$, the elastic constants D_p, and the electric field E. The transformation can then be executed as follows:[13]

(1) Truncate and solve: Retain $m_{\max}$ complex components W_m, and find them by solving the first $m_{\max}$ of Eq. (6): $m=1,2,\ldots,m_{\max}$.

(2) Discard all modes W_m for which m is the first member of a sequence.

(3) Relabel the remaining modes m with the mode number m' which preceded m in its sequence. Each sequence now starts with its first member again.

(4) Rescale each remaining mode by multiplying by m/m'. Thus

$$W'_{m'} = (m/m')W_m . \tag{8}$$

(5) To determine the renormalized system, we note that splitting the equations of state into real and imaginary parts gives a set of linear, homogeneous real equations for the parameters which define the system. That is, the inverted equations have the form:[14]

$$\sum_p A_{n,p} S'_p = 0 \tag{9}$$

where $\mathbf{S}'$ is a vector containing the parameters which define the renormalized system and A is a real rectangular matrix, with $n=1,2,\ldots,(2m_{\max})$. We can then keep enough components in the truncation so that there are more equations than system parameters, and determine the latter by minimizing the sum of the squares of the residues to Eqs. (9). The renormalized system is then given by the eigenvector with the smallest eigenvalue of the real, symmetric, positive-definite matrix B, $B_{p,q} = \sum_{n1,n2} A_{n1,p} A_{n2,q}$. A measure of the uniqueness of this choice is given by the ratio of the two smallest eigenvalues.

(6) The adjustable phase α can be chosen,[15] for example, to minimize the smallest eigenvalue of the matrix B. For the case of a bare system with left-right symmetry, $\alpha=0$ and the transformation preserves the symmetry.[16]

MacKay[11] has developed a renormalization group to extract the critical exponents at the pinning transition. It is very similar to the one used by Kadanoff and his colleagues[17,18] in their calculation of the critical behavior of a Kudanoff-Arnold-Moser KAM surface and the quasiperiodicity in a dissipative system. However, it is very difficult to extend MacKay's method to include the threshold transition as well.[19]

RESULTS

The following results were obtained for the case of purely nearest-neighbor elastic forces ($D_p = 0$, $p > 1$).

The pinning transition

If we consider first a left-right-symmetric pinning potential, the pinning force $P(x)$ is then antisymmetric and can be expanded in a sine series: $P(x) = \sum_p P_{sp} \sin(2\pi p x)$. The W_m and the $\mathcal{F}_m$ are then imaginary, and $v=0$, so that Eq. (6) becomes real.

(a) Figure 1 shows the RG flows which were obtained in the $P_{s1} - P_{s2}$ subspace. We see that there is a fixed-point pinning force $P^*(x)$. At a fixed point one expects total scale invariance. Indeed we find that, as we approach the fixed point, the scale invariance that Shenker and Kadanoff found asymptotically at the pinning transition extends here to all Fourier components. That is at the fixed-point scale invariance occurs at all length scales, and not just asymptotically at large length scales.

(b) Next we see that the fixed point has a stable manifold or critical curve, and that the pinning transition lies on this critical curve. The intersection of this manifold with the P_{s1} line converges to the known critical value. This fixed point thus determines the critical properties of the pinning transition.

Including the higher harmonic P_{s3} then showed that the fixed point was also stable to this harmonic so that the critical curve shown in Fig. 1 is the intersection with the $P_{s1} - P_{s2}$ plane of a critical surface in the space of pinning forces $P(x)$. This in turn indicates that the universality class of this transition includes pinning potentials with all symmetric harmonic components. Table I shows the critical value of P_{s1} as a function of the number of harmonic components in the pinning force. The rows of the table correspond to the number of degrees of freedom, W_m, used in the calculation, and the columns correspond to the number of sine components, n_{harm}, used to parametrize the pinning force $P(x)$. It could be seen that one harmonic component would give a very poor approx-

TABLE I. RG estimates of the critical value of P_{s1}.

m_{max} \ n_{harm}	1	2	3
8	−1.502	−0.975	−0.972
13	−1.480	−0.973	−0.971

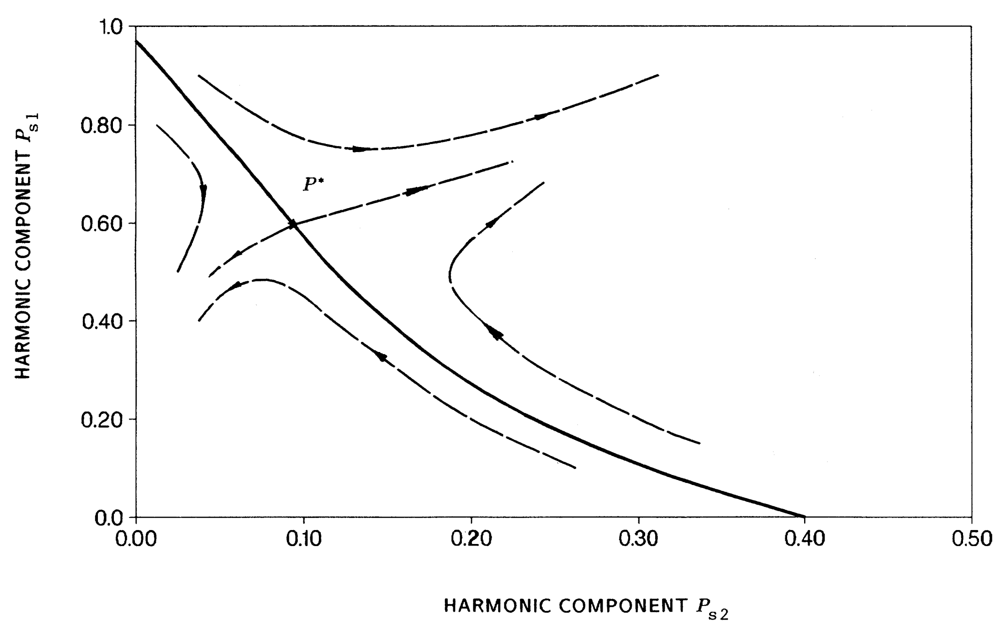

FIG. 1. Renormalization-group flows in the symmetric subspace. P_{s1} and P_{s2} are, respectively, the fundamental and first harmonic of the pinning force $P(x)$. (D_1, the nearest-neighbor spring constant, is normalized to 2.) The heavy line is the critical curve. The arrows show the direction of the RG flows. The point P_c(FK) is the Frenkel-Kontorova pinning transition (see Ref. 11), and P^* is the fixed point which determines its critical properties.

imation. While the result for those for $n_{\text{harm}} \geq 2$ have no substantial change as n_{harm} increases.

The critical surface divides the space of pinning forces into two parts. Those points on the weak-pinning side flow to zero pinning so that their long-range properties are those of a free harmonic chain. Those points on the other side flow to very strong pinning, so that their long-range behavior is of a system that is not free to slide but consists, asymptotically, of single particles pinned in the potential wells.

(c) Next we studied the effect of breaking the symmetry of the pinning potential. That is we broke the antisymmetry of the pinning force, introducing cosine components of $P(x)$. We found that these cosine components decayed under iteration of the transformation. The fixed-point pinning force $P^*(x)$ thus is stable with respect to asymmetric perturbations. Equivalently, when the transformation is linearized about the fixed point in the space of smooth pinning potentials, it is found to have only one relevant eigenoperator.[14] Thus, the universality class of the pinning transition includes both symmetric and asymmetric pinning potentials.

(d) Consider a displacement-displacement correlation function:

$$C(j) = \sum_p (u_{p+j} - \langle u \rangle)(u_p - \langle u \rangle) .$$

Since, for $E=0$, the fixed point has only one relevant eigenvalue, following the RG analysis of Wilson, Fisher, and Kadanoff, we find that the length scale, ξ, which characterizes the long-ranged correlations diverges according to a power law $\xi \sim |P - P_c|^{-\nu}$, where the critical exponent ν is related to the relevant eigenvalue λ in the usual way:

$$\nu = \frac{\ln \tau}{\ln \lambda} . \tag{10}$$

Some estimates of this exponent are given in the Table II. These estimates are consistent with convergence to the known[11] value of 0.987. It is interesting that the RG transformation provides estimates of the critical exponent accurate to within 2% when the number of degrees of freedom used, m_{max}, is still rather small.

Crossover exponent

The variables W_m are time independent even when the system is sliding, so that this transformation can also be applied to the dc dynamics.

An electric field E breaks the left-right symmetry so that both real and imaginary parts of the components W_m are generally nonzero, as is the adjustable phase α.

TABLE II. RG estimates of the correlation-length critical exponent.

m_{max} \ n_{harm}	2	3
8	1.02	1.02
13	1.01	0.99

TABLE III. RG estimates of the crossover exponent.

m_{max} \ n_{harm}	2	3
8	3.04	3.34
13	3.02	3.16

Because the symmetry is broken, we also now include a cosine component to the pinning force $P(x)$ for every sine component retained. The system is thus now flowing in a Hamiltonian space of twice the dimensionality necessary to study the pinning transition. The RG transformation could be linearized about the pinning-transition fixed point in the asymmetric subspace of parameters. It is found that there is one relevant eigenoperator which will move system toward the strong-pinning threshold fixed point. Table III gives some estimates of the crossover exponent.

Threshold transition

With a nonzero electric field E, we find that flows exhibit a well-defined flow pattern in the space of parameters. In Fig. 2 we show flows, projected onto the E, D_1 plane. The topologies of flows for number of harmonic components of two and of three are almost the same. However, they are different from the flows with only one harmonic component. As shown in Fig. 2(a), there is no flow boundary which would separate the sliding phase from the pinned phase for one-harmonic-component pinning force. A system will eventually flow to the weak-pinned fixed point under iteration of the RC transformation. It appeared to be a poor approximation as well as pinning transition for $n_{\text{harm}}=1$. So we report the results for two harmonic components of the pinning force.

A RG flow for FLR model has been conjectured. Threshold systems were conjectured to flow to a finite fixed point on the threshold curve, i.e., one with a finite pinning strength. Thresholds at strong pinning would then have their pinning strength diminished under iteration as they flowed to this finite fixed point. It is clear that the flow topologies of conjecture and the present results are different. In particular, the present results show that the threshold dynamic critical behavior is determined by a strong-pinning-threshold fixed point at the edge of the physical parameter space.

Figure 3 shows the pinning forces at the threshold fixed point. The amplitude of the second harmonic component, P_2, of pinning force and the phase difference between the fundamental and the second harmonic, α_2, have the values which minimizes the maximum pinning force.

The flow boundary is at a slightly different place on the threshold curve, which separates the pinned static system from the unpinned dynamic system. It is found that as the number of modes m_{max} increases. the flow boundary approaches the threshold curve from the pinned side of the curve. As shown in Fig. 2(b), flows move away from the strong-pinning threshold fixed point along the E axis with an infinite pinning strength. It is also found that the

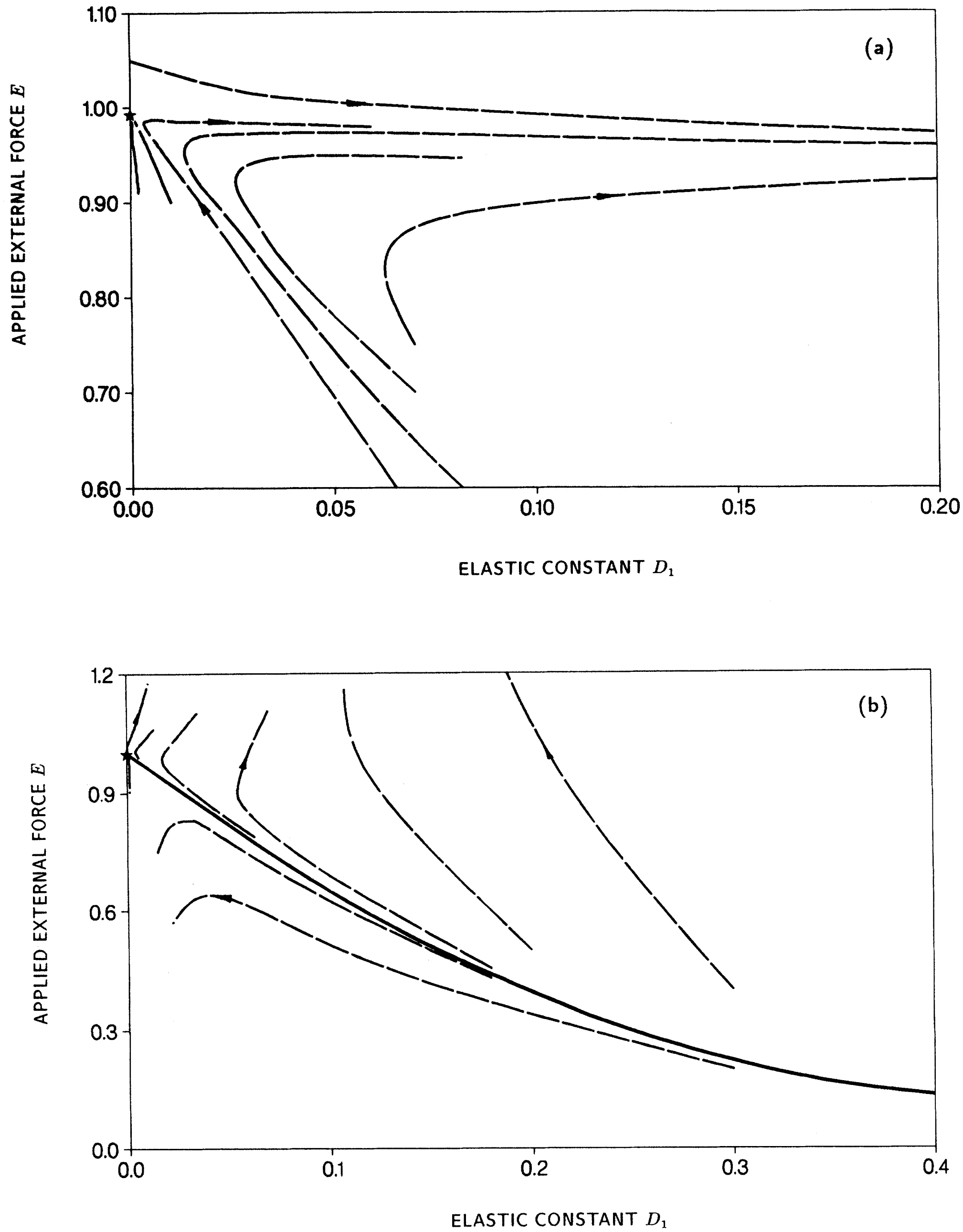

FIG. 2. Renormalization-group flows near the dynamic threshold, projected onto the E-D_1 plane. In this figure the flows each start with a pure sine-wave pinning force, but flow in a Hamiltonian space with sine and cosine components for $P(x)$. The normalization is $\max[-P(x)]=1$, so that as D_1 vanishes, the threshold field approaches 1.

points on the hypersurface of flow boundary in parameter space, in spite of distances between them, would flow into a limited vicinity of the threshold fixed point after iterations of the RG transformation. This critical surface divides the space of parameters into two parts as well as it did in $E=0$ subspace. Those points on the strong-pinning side flow to the fixed point with very strong pinning and a zero electric field. Those points on the other side flow to zero pinning.

The long-range properties of points on the critical surface should be determined by the properties of the strong-pinning threshold fixed point. At the fixed point, the incommensurate chain becomes a bunch of independent particles. Most of them are located near bottoms of the pinning force. If one could use the results of a single particle to describe the threshold dynamics of the chain at this fixed point, then the averaged velocity would be given by $v^{-1} \sim \int dx/[E+P^*(x)]$ when the driving elec-

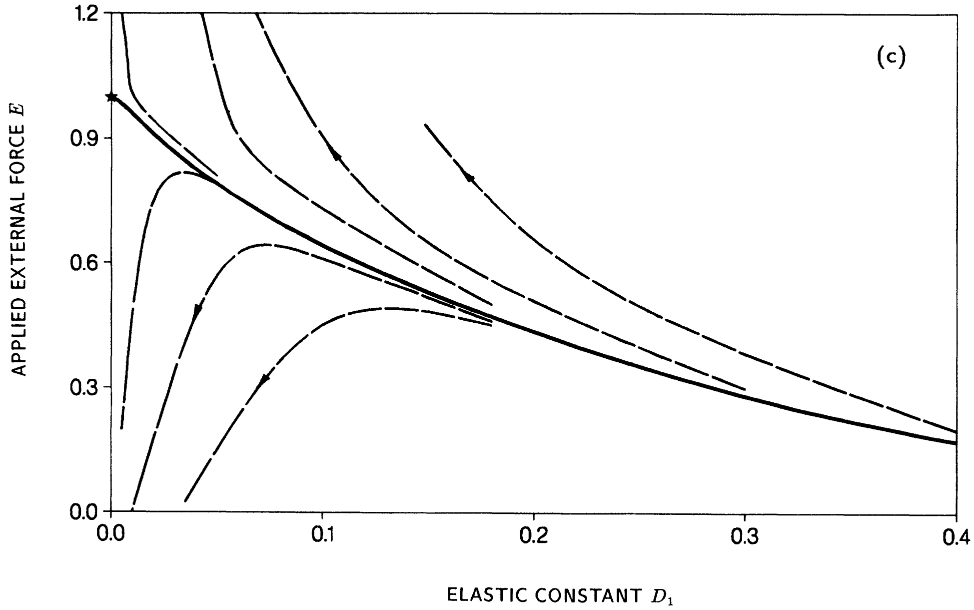

FIG. 2. (*Continued*).

tric field E is greater than the threshold field E_t for a single particle in a periodic potential which is equal to the maximum pinning force. If E is very close to E_t, then the averaged velocity could be written as $v \sim (E - E_t)^{\zeta}$. The exponent ζ is determined by the shape of the pinning force $P^*(x)$ at the fixed point, especially the shape of bottom of the pinning force. For example, if $P(x) = \sin(2\pi x + \alpha)$, for $n_{\text{harm}} = 1$, the exponent is equal to $\frac{1}{2}$. For a square wavelike pinning force, the exponent is equal to 1. These two numbers provide us the up-limit and the low-limit for the value of the exponent ζ. Coppersmith and Fisher[2,20] calculated the exponent ζ by direct simulation. They found ζ should be equal to 0.68. We could estimate the value of ζ from the pinning force at the fixed point. As the system flowed to the strong threshold fixed point along the trajectory on the critical surface, we found that the renormalized pinning force had a flatter bottom than the bare pinning force with which we started the iteration. Furthermore, this tendency increases as one increases the number of harmonic

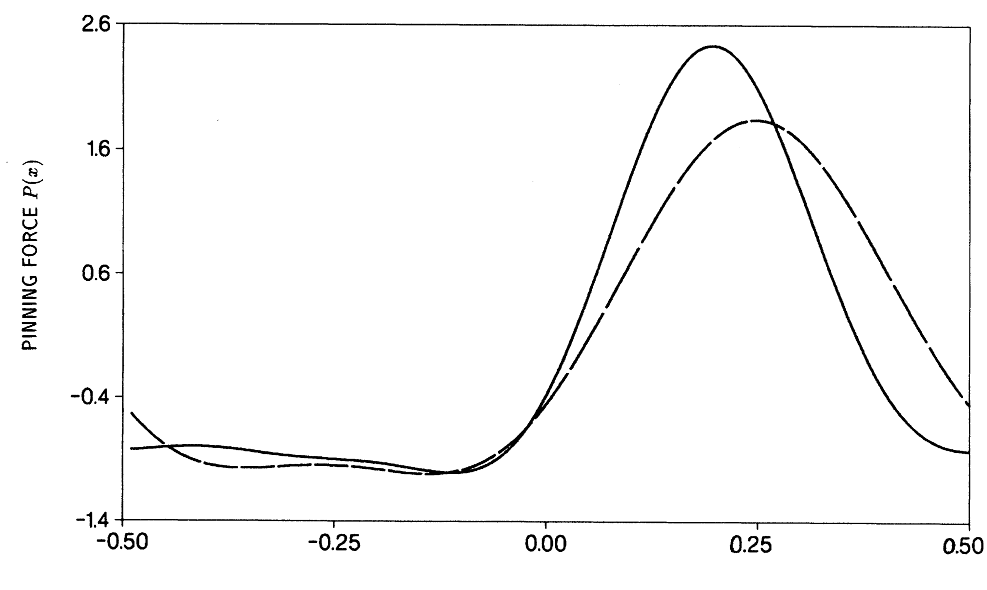

FIG. 3. Pinning forces at the threshold fixed point. The solid line is the curve of the pinning force for the system with $n_{\text{harm}} = 3$, while the dashed line is the curve for the system with $n_{\text{harm}} = 2$. The curve of the pinning force for $n_{\text{harm}} = 1$ is a pure sine curve and not shown here.

TABLE IV. RG estimates of the relevant eigenvalue λ_t at the threshold fixed point.

n_{harm}	Pinned side	Unpinned side
2	1.20	2.56
3	1.26	2.64

components in pinning force. The exponent ζ extracted from the pinning force $P^*(x)$ (see Fig. 3) is about 0.73 for $n_{\text{harm}}=2$, and 0.66 for $n_{\text{harm}}=3$. The numerical uncertainty is about 0.04. These results are consistent with the value given by Coppersmith and Fisher.

The transformation could be linearized at the fixed point, relevant eigenoperators are found on both sides of critical surface. Table IV gives some estimates of the eigenvalues of the relevant eigenoperators. The eigenvalue on the weak-pinning scale is twice as large as the one on the other side. This suggests that scaling functions may be different above and below the threshold.

CONCLUSION AND DISCUSSION

The new RG theory of the incommensurate chain can treat both static and dynamic properties. The depinning transition and the strong-pinning threshold fixed points have been found. The universality class of the depinning transition is found to include higher harmonics of the pinning potential.

The universality class associated with the introduction of other short-range elastic forces is an interesting one. Our results show that the universality class of the depinning transition does not include systems with more elastic forces other than the nearest-neighbor interactions. This suggests that changing the short-range interaction between particles will change the long-range critical behaviors.

With more CPU power, one could systematically study the effects of increasing the wavelength and complexity of pinning force. One may be able to approach the random-pinning threshold as the limit of incommensurate systems.

[1]See, for example, *Charge Density Waves in Solids,* Vol. 217 of *Lecture Notes in Physics* (Springer-Verlag, Berlin, 1985).

[2]See, for example, D. S. Fisher, Phys. Rev. Lett. **50**, 1486 (1983); S. N. Coppersmith, Phys. Rev. B **30**, 410 (1984).

[3]H. Fukuyama and P. A. Lee, Phys. Rev. B **17**, 535 (1977); P. A. Lee and T. M. Rice, *ibid.* **29**, 719 (1984); **29**, 725 (1984).

[4]Leigh Sneddon, M. C. Cross, and D. S. Fisher, Phys. Rev. Lett. **49**, 292 (1982); Leigh Sneddon, Phys. Rev. B **29**, 719 (1984); **29**, 725 (1984).

[5]For a discussion of incommensurate systems as models of CDW dynamics see also Leigh Sneddon (unpublished); Phys. Rev. B **30**, 2974 (1984); Phys. Rev. Lett. **56**, 1195 (1986); Leigh Sneddon and Kenneth A. Cox, *ibid.* **58**, 1903 (1987); Sen Liu and Leigh Sneddon, Phys. Rev. B **35**, 7745 (1987).

[6]These models are generalizations of those introduced by Y. I. Frenkel and T. Kontorova, Zh. Eksp. Teor. Fiz. **18**, 1340 (1938).

[7]Leigh Sneddon, Phys. Rev. Lett. **52**, 65 (1984).

[8]For reviews see, for example, K. G. Wilson and J. Kogut, Phys. Rep. C **12**, 75 (1974); S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, MA, 1976).

[9]D. S. Fisher, Phys. Rev. B **31**, 1396 (1985).

[10]S. Shenker and L. P. Kadanoff, J. Stat. Phys. **27**, 631 (1982).

[11]See, for example, R. S. Mackay, Ph.D. thesis, Princeton University, 1982 (unpublished); M. Peyrard and S. Aubry, J. Phys. C **16**, 1593 (1983); S. N. Coppersmith and D. S. Fisher, Phys. Rev. B **28**, 2566 (1983); D. F. Escande, Phys. Rep. **121**, 165 (1985).

[12]We consider τ equal to the "golden mean" $[\sqrt{5}+1]/2$, for which the first of these sequences is the Fibonacci sequence: 1,2,3,5,8,13,21,35, ...

[13]The pinning force $P(x)$ can be parametrized by its Fourier sine and cosine components.

[14]See, Sen Liu, Ph.D. thesis, Brandeis University, 1990 (unpublished).

[15]The dependence of W_m on the adjustable phase is simply

$$W_m=\exp(i2\pi m\alpha)W_m(\alpha=0)\ .$$

[16]The need for this adjustable phase α is determined by the transformation to commute with arbitrary phase shifts.

[17]M. Feigenbaum, L. P. Kadanoff, and S. J. Shenker, Physica **5D**, 370 (1982).

[18]Leo P. Kadanoff (unpublished).

[19]One possible reason may be that the applied external driving field breaks the right-left symmetry of the system, while the symmetry-broken mechanism was not preserved by the kind of renormalization-group transformation of MacKay.

[20]S. N. Coppersmith and D. S. Fisher, Phys. Rev. A **38**, 6338 (1988).