Theory of unstable growth. II. Conserved order parameter

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The theory of domain growth developed previously to treat a nonconserved-order-parameter (NCOP) system is extended to treat the conserved-order-parameter (COP) case (spinodal decomposition). The theory here, as for the NCOP case, leads to universal scaling behavior for the order-parameter structure factor, which depends only on the spatial dimensionality of the system. The short-distance ordering in the system is found to be identical to that found for the NCOP case indicating that the structure near the interface is independent of the driving dynamics. Porod's law, signifying sharp interfaces, is of exactly the same form as for the NCOP case. There are significant differences between the two cases, however. In the COP case the growth mechanism works through a coupling between the ordering component and a diffusing component. This coupling increases the growth law, $L(t)$, from the rather slow surface-diffusion form, $t^{1/4}$, to the classical Lifshitz-Slyzov-Wagner form, $t^{1/3}$. In the NCOP case, the ordering component is strongly decoupled from any fluctuating component. While the structure factors for the two cases are the same for small values of the scaled lengths ($x=R/L\ll 1$) they differ significantly over the rest of the range of x. In the COP case the theoretical expression for the scaled structure factor $F(x)$ agrees well with the best available simulational results. A striking feature of the theory in the NCOP case was the existence of a nonlinear eigenvalue problem associated with the determination of the scaling function $F(x)$. In the COP case one has two such eigenvalues. The additional eigenvalue can be associated with the coefficient x^2 in the expansion of $F(x)$ in powers of x. The nonzero value of this coefficient renders invalid the symmetry $[1-F(x)]=-[1-F(-x)]$ found in the NCOP case.

I. INTRODUCTION

In paper I of this series,[1] a theory of domain growth[2] for systems with a nonconserved scalar order parameter (NCOP) was presented. This theory is extended here to the case of a system with a conserved order parameter.[3] The associated problem is commonly known as spinodal decomposition.[4] It has long been appreciated[5] that the conserved-order-parameter (COP) case is more involved theoretically than the NCOP case, but the precise nature[6] of the additional complexity of the COP case has been elusive. Essentially, all first-principles treatments[7-10] of spinodal decomposition have led to a growth law [dominant scaling length $L(t)$] with $L\sim t^{1/4}$, where t is the time after a quench into an unstable state. However, theories[11-16] which assume the existence of sharp interfaces (of whatever morphological structure) and the existence of local equilibrium [referred to here as Lifshitz-Slyzov-Wagner (LSW) theories] invariably lead to a growth law of $t^{1/3}$. The theory presented here leads unambiguously to the $t^{1/3}$ LSW growth law *and* an understanding of the elusive nature of this result from a first-principles point of view.

The theory developed in Refs. 1 and 3 is predicated on the separation of the order-parameter field $\psi(\mathbf{R},t)$ into the sum of an ordering field $\sigma(\mathbf{R},t)$ and a fluctuation field $\zeta(\mathbf{R},t)$. In the NCOP case, the theory is organized such that σ and ζ are completely decoupled after a relatively short time after a deep temperature quench. If one carries out the analogous decomposition in the COP case, as shown in Sec. II C, the ordering field σ is decoupled from the fluctuation field ζ after a short time, and the only ordering mechanism available is surface diffusion which is well known[17] to give a $t^{1/4}$ growth law. Clearly, from the classic LSW analysis, a faster ordering mechanism for the longest times in the problem is via bulk diffusion. It is physically clear that the bulk diffusion process requires a coupling between the ordering field $\sigma(\mathbf{R},t)$ and the fluctuation field $\zeta(\mathbf{R},t)$. As shown below, this coupling is via flow terms in the equations of motion for the σ and ζ variables not present in the NCOP case. It will be satisfying to find that, in the presence of this coupling, the fluctuation field ζ becomes a diffusion field in agreement with elementary treatments of Ostwald[18] ripening. Once the fields are coupled it becomes more physical to refer to σ as the interfacial field and to ζ as the diffusion field.[19] The organizing principle in treating the communication between the interfacial field σ and the diffusion field ζ is that the coupling be chosen to maximize the rate of ordering as measured by the growth law $L(t)$ in the system.

The theory developed in I supports the proposition that the scaling properties of growth kinetics have universal features. For the general class of field theoretical models treated in I, the scaling functions obtained depended *only* on the spatial dimensionality[20] of the system. The class of models treated in I included purely dissipative time-dependent Ginzburg-Landau (TDGL) models with a single nonconserved scalar order parameter. The driving potential was assumed to be a general symmetric degenerate double-well potential with two quadratic

TABLE I. Dimensionality dependent quantities entering the theory. β_2^* and μ^* are the eigenvalues associated with the scaling function $F(x)$. A_d is defined by (2.73), B_d by (3.48), C_d by (1.6), and z_0 by (2.70).

d	β_2^*	μ^*	A_d	B_d	C_d	z_0
2	−0.077 8	0.337 596	0.894	1.839	$4\sqrt{\pi}$	1.58
3	−0.035 6	0.175 171	0.910	3.284	8π	1.55

minima. While results were established in detail for quenches to zero temperature, it was conjectured[21] that nonzero temperatures would not influence the universal features.

A somewhat surprising result found in I is that the growth law $L(t)$, when measured in units of the equilibrium interfacial width, takes a universal form [see (I.1.51)]. This onclusion is connected to the result that the parameter $\mu = L\dot{L}$ approaches a fixed point value $\mu^*(d)$ as $L \to \infty$ which depends only on the spatial dimensionality d. It was, however, also pointed out that μ^* has another physical interpretation. For large scaled distances $x = R/L$, the order-parameter scaling function $F(x)$ approaches zero as $e^{-\mu^* x^2/2}$ and μ^* can be defined as the limit

$$\mu^* = \lim_{x\to\infty} -\frac{2}{x^2}\ln F(x) . \tag{1.1}$$

It seems that this last result is robust.[22] The result

$$\mu^* = \lim_{t\to\infty} L\dot{L} \tag{1.2}$$

seems less compelling. We shall see below in the COP case how the results (1.1) and (1.2) can become uncoupled.

While there are a number of differences between the NCOP and COP cases, there are also a number of important similarities. In both cases one obtains universal scaling functions $F(x)$ which depend only on spatial dimensionality. While the specific forms for $F(x)$ for the NCOP and COP cases for $x \gg 1$ are quite different, for intermediate distances $R \ll L$, the results are in precise agreement. In both cases, for $\xi \ll R \ll L$, where ξ is the interfacial width,

$$F(x) = 1 - \alpha x + \cdots , \tag{1.3}$$

where

$$\alpha = \sqrt{2/[\pi(d-1)]} \tag{1.4}$$

and Porod's law[23] holds in the same form[24] for both cases. Thus, the short-distance structure (which is determined by local equilibrium) is the same in the two cases. The behavior at very short distances $R \leq \xi$ agrees in detail with the results reported in I. These very short-distance findings, of course, depend on microscopic details of the potential, lattice type, etc.

One surprising result of the analysis for the COP case is the breakdown of the so-called "Tomita" sum rule.[25] In the NCOP case it was found, for small x, that the scaling function can be written in the form

$$F(x) = 1 - \alpha \sum_{n=1}^{\infty} \beta_n x^n , \tag{1.5}$$

where even powers of x do not appear beyond the leading term: $\beta_{2n} = 0$ for $n > 0$. Explicit expressions for β_{2n+1} can be found which are strictly functions of the spatial dimensionality. In the COP case, things are somewhat different. Assuming the form (1.5), one can obtain analytic expressions for β_n as shown in Sec. III C. One again obtains $\beta_1 = 1$, but now β_2 is undetermined by the power-series solution of the fundamental equation governing the scaling function due to the gradients associated with the conservation law. Indeed, all higher-order coefficients depend on β_2 and all even-order coefficients are proportional to β_2. β_2 is not, however, undetermined. Just as in the NCOP case, the differential equation and boundary conditions satisfied by $F(x)$ are over determined and lead to a nonlinear eigenvalue[26] problem. In the COP case, however, there are *two* eigenvalues to be determined. The first parameter is equivalent[27] to the parameter μ present in the NCOP case. The second eigenvalue can be related to β_2. This follows since, as will be shown below, $F(x)$ satisfies a fourth-order differential equation and the conditions $F(0) = 1$, $F'(0) = -\alpha$, $F''(0) = -2\alpha\beta_2$, $F'''(0) = -6\alpha\beta_3$, and $F(x)$ decay exponentially to zero as $x \to \infty$. These conditions uniquely determine μ and β_2 as functions of d as given in Table I for $d = 2$ and 3.

The solution of the nonlinear eigenvalue problem not only gives μ and β_2 but also the complete scaling function $F(x)$ shown in Fig. 1 for $d = 2$ and 3. The scaled structure factor is compared with the best available numerical simulation[28–30] results in Fig. 2. The comparison is quite

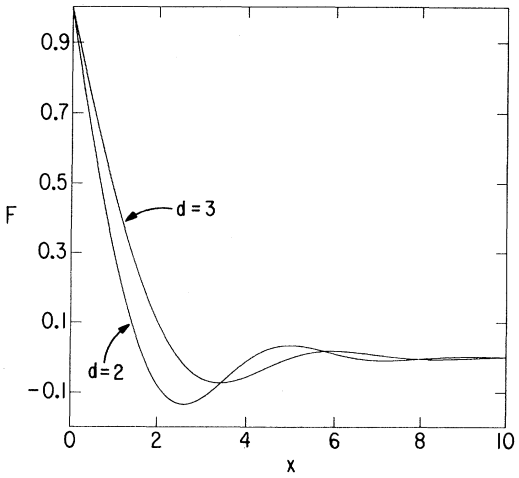

FIG. 1. The scaling function $F(x)$ for $d = 2$ and 3.

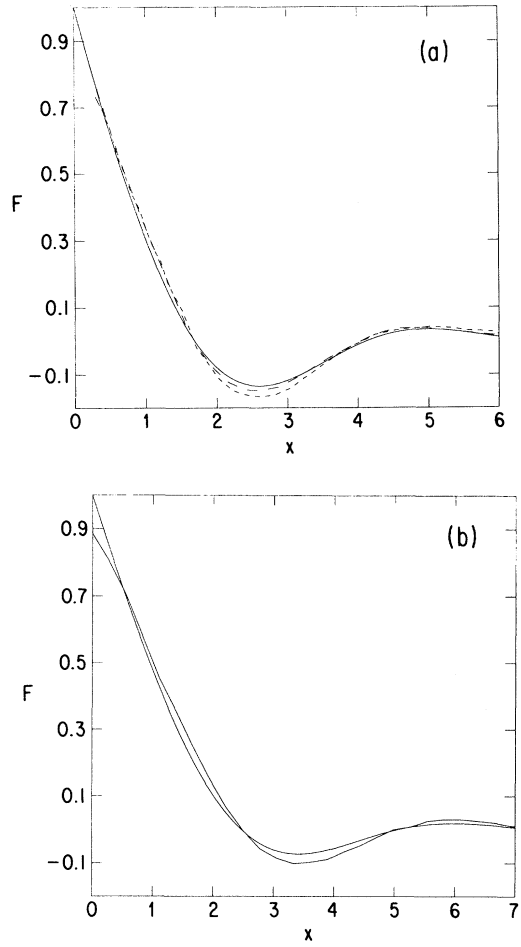

FIG. 2. (a) Comparison of $F(x)$ for $d=2$ with the numerical results from Ref. 28 (short-dashed line) and Ref. 29 (long-dashed line). The scaled distances have been chosen such that the first zeros for the calculations coincide. (b) Comparison of $F(x)$ for $d=3$ with the numerical results from Ref. 30. The x axis is chosen as in (a). The theory extends to 1 as $x\to 0$.

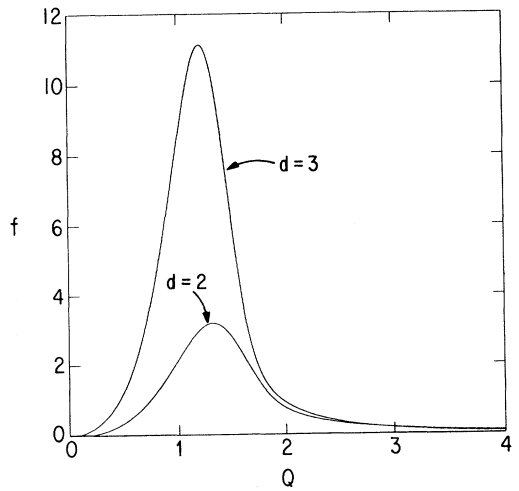

FIG. 3. Fourier transforms of $F(x)$ for $d=2$ and 3.

good.[31] The scaled Fourier transform $f(qL)$ can also be determined in the scaling regime (shown in Fig. 3) and Porod's law is explicitly confirmed in the form

$$f(Q)=\alpha C_d Q^{-(d+1)} , \tag{1.6}$$

where C_d as a function of d is given in Table I and $Q=qL$.

II. FORMAL DEVELOPMENT

A. Definition of problem

The general system treated here is the conserved TDGL model in the presence of Gaussian noise. In terms of a dimensionless length scale, the basic equation of motion satisfied by the order-parameter field $\psi(\mathbf{R},t)$ is

$$\frac{\partial\psi(\mathbf{R},t)}{\partial t}=\hat{\Gamma}(\mathbf{R})\frac{\delta F}{\delta\psi(\mathbf{R},t)}+\eta(\mathbf{R},t) , \tag{2.1}$$

where the Gaussian noise satisfies

$$\langle\eta(\mathbf{R},t)\eta(\mathbf{R}',t')\rangle=T\hat{\Gamma}(\mathbf{R})\delta(\mathbf{R}-\mathbf{R}')\delta(t-t') , \tag{2.2}$$

and

$$\hat{\Gamma}(\mathbf{R})=-D\nabla_R^2 , \tag{2.3}$$

where D is a bare diffusion coefficient. In (2.2), T is a dimensionless measure of the final temperature, and in (2.1) the driving effective Hamiltonian is assumed to be of the general form

$$F=\int d^dR\,[\tfrac{1}{2}(\nabla_R\psi)^2+V(\psi)] . \tag{2.4}$$

$V(\psi)$ is assumed to be a symmetric degenerate double-well potential with quadratic minima. A particular realization is the λ potential

$$V(\psi)=\tfrac{1}{4}(1-\psi^2)^2+\frac{\lambda}{6}\psi^6 \tag{2.5}$$

introduced in I. For $\lambda=0$ this reduces to the standard ψ^4 potential. In this paper the analysis is restricted to the case of a critical quench where the potential is symmetric. In the next paper in this series, the off-critical quench case will be discussed.

The equation of motion (2.1) must be supplemented by a set of initial conditions at time $t=t_0=0$ satisfied by $\psi(\mathbf{R},t_0)=\psi_0(\mathbf{R})$. For present purposes, a Gaussian initial probability distribution governing ψ_0 is assumed with second moment

$$\langle\psi_0(\mathbf{R})\psi_0(\mathbf{R}')\rangle=\epsilon_I\delta(\mathbf{R}-\mathbf{R}') . \tag{2.6}$$

This model is the same as studied in I except the bare kinetic coefficient Γ is replaced by the operator $\hat{\Gamma}$ which guarantees the conservation of the order parameter.

B. The functional-integral formulation of the problem

The functional-integral formulation for the problem defined in the last section follows the discussion in Mazenko, Valls, and Zannetti (MVZ)[10] and I. The action A has the same form as (I.2.7):

$$A[\hat{\psi},\psi,\psi_0]=A[\psi_0]+\int d1\{\hat{\psi}(1)2T\hat{\Gamma}(1)\hat{\psi}(1)+i\hat{\psi}(1)[B(1;\psi)-\delta(t_1-t_0)\psi_0(1)]\}\ , \tag{2.7}$$

where

$$B(1;\psi)=\Lambda(1)\psi(1)+\hat{\Gamma}(1)V'[\psi(1)]\ , \tag{2.8}$$

$$\Lambda(1)=\frac{\partial}{\partial t_1}+\hat{\Gamma}(1)(-\nabla_1^2)\ , \tag{2.9}$$

and $\int d1=\int d^dR_1\int_{t_0}^{+\infty}dt_1$. Averages correspond to performing functional integrations over the fields ψ_0, $\hat{\psi}$, and ψ as weighted by the factor e^{-A}. Normalizations and regularizations[32] have been chosen such that the Jacobian associated with the transformation from a functional integral over the noise to one over the field ψ is constant. The part of the action governing the initial values of ψ is given by

$$A_I[\psi_0]=\tfrac{1}{2}\int d^dR\frac{1}{\epsilon_I}\psi_0^2(\mathbf{R})\ . \tag{2.10}$$

As discussed in Sec. II C of I, the first step in the theory is to enlarge the function space from that spanned by $\psi(\mathbf{R},t)$ to a joint space spanned by $\psi(\mathbf{R},t)$ and a new scalar field $m(\mathbf{R},t)$ as indicated by (I.2.12). These spaces are coupled through the translation

$$\psi(1)=\sigma[m(1)]+\phi(1)\ , \tag{2.11}$$

where σ is the ordering or peak variable and ϕ is essentially[33] the fluctuating variable. The peak variable σ is chosen to satisfy the classical equation for a single interface

$$\frac{1}{2}\frac{d^2\sigma}{dm^2}=V'[\sigma] \tag{2.12}$$

with the boundary conditions

$$\lim_{|m|\to\infty}\frac{d\sigma}{dm}=0\ . \tag{2.13}$$

A key aspect of this development is that the coordinate m is a field. The fluctuation field $\phi(1)$, in order to decouple σ and the fluctuating component in the zeroth-order action, is written as

$$\phi(1)=\zeta(1)-\int d2\,G_F(12)iB(2)\ , \tag{2.14}$$

where $\zeta(1)$ is a new independent field, $B(1)=B(1;\sigma)$,

$$G_F^{-1}(12)=i\left[\frac{\partial}{\partial t_1}+\hat{\Gamma}(1)[-\nabla_1^2+q_0^2(1)]\right]\delta(12)\ , \tag{2.15}$$

$\delta(12)=\delta(\mathbf{R}_1-\mathbf{R}_2)\delta(t_1-t_2)$, and

$$q_0^2(1)=\left\langle\left[\frac{\partial^2}{\partial\phi^2}V(\sigma+\phi)\right]\Big|_{\phi=0}\right\rangle_0\ . \tag{2.16}$$

The total action can then be written in the form

$$A_T(\hat{\psi},\phi,\psi_0,m)=A_0(\hat{\psi},\phi,\psi_0,m)+A_I(\hat{\psi},\phi,\psi_0,m)\ , \tag{2.17}$$

where the zeroth-order action is given by

$$A_0(\hat{\psi},\phi,\psi_0,m)=A_I[\psi_0]+A[m]+\int d1\,\hat{\psi}(1)2T\hat{\Gamma}(1)\hat{\psi}(1)$$
$$+\int d1\,\hat{\psi}(1)\left[iB(1;\sigma)+\int d2\,G_F^{-1}(12)\phi(2)-i\delta(t_1-t_0)\psi_0(1)\right] \tag{2.18}$$

and the term which can be treated as a perturbation is

$$A_I(\hat{\psi},\phi,\psi_0,m)=\int d1\,\hat{\psi}(1)i\hat{\Gamma}(1)\tilde{V}_I(1;\sigma,\phi) \tag{2.19}$$

with

$$\tilde{V}_I(1;\sigma,\phi)=V'[\sigma(1)+\phi(1)]-V'[\sigma(1)]-q_0^2(1)\phi(1)\ . \tag{2.20}$$

The average in (2.16), $\langle\cdots\rangle_0$, is assumed to be over the zeroth-order probability distribution governed by the action A_0 given by (2.18). The action $A[m]$ in (2.18) is assumed to be quadratic in the field m, so $P_0[m]$ is a Gaussian probability distribution. Following the analysis in I, the variance

$$C_0(12)=\langle m(1)m(2)\rangle_0\ ,$$

is then determined by the fundamental equation

$$\langle B(1)\sigma(2)\rangle_0=\delta(t_1-t_0)\langle\sigma(1)\sigma(2)\rangle_0\ . \tag{2.21}$$

Using (2.8) with ψ replaced by σ and (2.12), one can rewrite (2.21) as

$$\Lambda(1)C(12)+\tfrac{1}{2}\hat{\Gamma}(1)C_{20}(12)=\delta(t_1-t_0)C(12)\ , \tag{2.22}$$

where $C_{20}(12)$ is a particular component of the matrix correlation function

$$C_{nl}(12)=\langle\sigma_n(1)\sigma_l(2)\rangle_0\ , \tag{2.23}$$

where

$$\sigma_n(1)=\frac{\partial^n}{\partial m(1)^n}\sigma(1) \tag{2.24}$$

and

$$C(12)\equiv C_{00}(12)\ . \tag{2.25}$$

After these transformations, the original fundamental field $\psi(1)$ is then expressed in terms of the new fields by

$$\psi(1)=\sigma(1)-\int d2\,iG_F(12)B(2)+\zeta(1)\ . \tag{2.26}$$

The evaluation of the structure factor

$$C_\psi(12)=\langle\psi(1)\psi(2)\rangle \tag{2.27}$$

can be carried out at zeroth order, since the fields m and ζ are uncoupled, to obtain

$$\begin{aligned}C_\psi^0(12)=&\,C(12)-i\int d\bar{2}\,G_F(2\bar{2})\delta(t_{\bar{2}}-t_0)C(\bar{2}1)\\&-i\int d\bar{1}\,G_F(1\bar{1})\delta(t_{\bar{1}}-t_0)C(\bar{1}2)\\&-\int d\bar{1}\,d\bar{2}G_F(1\bar{1})G_F(2\bar{2})C_B(\bar{1}\bar{2})+C_\zeta^0(12)\,,\end{aligned} \tag{2.28}$$

where

$$C_\zeta^0(12)=\langle\zeta(1)\zeta(2)\rangle_0 \tag{2.29}$$

and

$$C_B(12)=\langle B(1)B(2)\rangle_0\,. \tag{2.30}$$

The motivation for constructing the theory in this manner is described in some detail in I. At zeroth order the fundamental equation of motion governing the ordering field is (2.22). The fundamental equation of motion governing the Gaussian fluctuation field ζ is given by

$$\begin{aligned}\int d\bar{1}-iG_F^{-1}(1\bar{1})C_\zeta^0(\bar{1}2)=&-i2T\hat{\Gamma}(1)G_F(21)\\&+\delta(t_1-t_0)C_\zeta^0(12)\,.\end{aligned} \tag{2.31}$$

At equal times, $t_1=t_2=t>0$ and choosing time units where $2D=1$, (2.22) and (2.31) reduce to the equations of motion

$$\frac{\partial}{\partial t}C(\mathbf{R},t)=(-\nabla_R^2)[\tfrac{1}{2}C_{20}(\mathbf{R},t)+\nabla_R^2 C(\mathbf{R},t)] \tag{2.32}$$

and

$$\frac{\partial}{\partial t}C_\zeta^0(\mathbf{R},t)=(-\nabla_R^2)[q_0^2(t)+\nabla_R^2]C_\zeta^0(\mathbf{R},t)-2T\nabla_R^2\delta(\mathbf{R})\,, \tag{2.33}$$

where

$$C(\mathbf{R},0)=C_\zeta^0(\mathbf{R},0)=\epsilon_I\delta_{\mathbf{R},0} \tag{2.34}$$

and $\mathbf{R}=\mathbf{R}_1-\mathbf{R}_2$.

C. Direct implementation of the theory

In direct analogy with the development in I, one can show, for long times [where $S_0(1)=\langle m^2(1)\rangle\gg 1$], that (2.32) reduces to

$$\frac{\partial}{\partial t}\tilde{C}(\mathbf{R},t)=-\nabla_R^2\left[\frac{1}{L^2}\tan\left[\frac{\pi}{2}\tilde{C}(\mathbf{R},t)\right]+\nabla_R^2\tilde{C}(\mathbf{R},t)\right]\,, \tag{2.35}$$

where

$$\tilde{C}(\mathbf{R},t)=C(\mathbf{R},t)/\psi_0^2\,, \tag{2.36}$$

$$\sqrt{2}L=\xi/[1-\tilde{C}(\mathbf{0},t)]\,, \tag{2.37}$$

and ψ_0 is the ordered value of the order parameter and ξ the interfacial width defined (I.2.42). If $\tilde{C}$ is assumed to obey scaling at long times,

$$\tilde{C}(\mathbf{R},t)=F(\mathbf{R}/L)\,, \tag{2.38}$$

then (2.35) reduces to

$$-\frac{\dot{L}}{L}\mathbf{x}\cdot\nabla_x F(\mathbf{x})=-\frac{1}{L^4}\left[\nabla_x^2\tan\left[\frac{\pi}{2}F(x)\right]+\nabla_x^4F(x)\right]\,. \tag{2.39}$$

If (2.39) is to lead to a scaling solution with appropriate long-distance behavior, then $\mu_0\equiv L^4(\dot{L}/L)$ will be a constant as $L\to\infty$. This immediately gives a growth law $L\sim t^{1/4}$. The associated ordering process is quite slow and suggests a search for a faster mechanism. As mentioned in the Introduction, this search is guided by the physics of the situation. As the theory now stands, there is a very weak coupling between the ordering field σ and the fluctuating field ζ. There is, therefore, a *very* weak coupling between different interfaces. In the case of a COP, long-time growth is expected to be via bulk diffusion which couples interfaces. This requires that small portions of one phase (monomers in the physical realization[18] in terms of particles) swim across ordered regions of the opposite phase. In the development of this section, no such diffusion exists. If the theory developed here is to include this diffusion process, then it will be via the fluctuation field ζ. This diffusion must, in turn, be driven by a coupling between the σ and ζ fields which is sustained on a long distance and time scale. The appropriate coupling will be discussed below. At the formal level the origins of the diffusion process appear rather obscure, but in the end a physically appealing picture emerges.

D. Coupling σ and ζ through the noise

The first step in understanding how separated interfaces can communicate requires addressing the following formal question: Can part of the noise driving the fluctuating field ζ be transferred into the equation driving the ordering field σ? The answer is yes and involves the simultaneous translation of the fluctuating field

$$\zeta(1)=\int d\bar{1}\,d\bar{2}G_F(1\bar{1})\Pi(\bar{1}\bar{2})\hat{\psi}(\bar{2})+u(1)\,, \tag{2.40}$$

and a change in the defining equation for C from (2.21) to

$$\langle B(1)\sigma(2)\rangle=\delta(t_1-t_0)C(12)-i\int d\bar{1}\,G_F(2\bar{1})\Pi(1\bar{1})\,, \tag{2.41}$$

where, at this point, the function $\Pi(12)$ is arbitrary. $u(1)$, defined by (2.40), replaces $\zeta(1)$ as an independent field. With these replacements, the total action becomes

$$\begin{aligned}A_T(\hat{\psi},u,\psi_0,m)=&\,A(m)+A_0(\psi_0)+\int d1\,d2\hat{\psi}(1)[\pi(12)+\Pi(12)]\hat{\psi}(2)\\&+i\int d1\,\hat{\psi}(1)[-iG_F^{-1}(1\bar{1})u(\bar{1})+\hat{\Gamma}(1)\bar{V}_I(1;\sigma;\phi)-\delta(t_1-t_0)\psi_0(1)]\,,\end{aligned} \tag{2.42}$$

where $\pi(12)=2T\hat{\Gamma}(1)\delta(12)$, σ, u, and $\hat{\psi}$ are the independent fields, and ϕ is now given by

$$\phi(1)=-i\int d\bar{1}\,G_F(1\bar{1})\left[B(\bar{1})+\int d\bar{2}\,i\Pi(\bar{1}\bar{2})\hat{\psi}(\bar{2})\right]+u(1). \tag{2.43}$$

Clearly from the quadratic term in $\hat{\psi}$ in (2.42) one sees that $\pi(12)+\Pi(12)$ plays the role of the noise for the fields $\hat{\psi}$, u, and ψ_0. In particular, at zeroth order where the $\tilde{V}_I$ term is set to zero in (2.42),

$$\langle u(1)\hat{\psi}(2)\rangle_0=G_F(12) \tag{2.44}$$

and, using the results from Sec. II F in MVZ, one easily finds that

$$\begin{aligned}C_u(12)&\equiv\langle u(1)u(2)\rangle_0\\&=-\int d\bar{1}\,d\bar{2}G_F(1\bar{1})G_F(2\bar{2})\\&\quad\times[\,2\pi(\bar{1}\,\bar{2})+\delta(\bar{1}\,\bar{2})\delta(t_{\bar{1}}-t_0)\epsilon_I\\&\quad+\Pi(\bar{1}\bar{2})+\Pi(\bar{2}\bar{1})]\,.\end{aligned} \tag{2.45}$$

Computing $\langle\psi(1)\psi(2)\rangle$ at zeroth order, one obtains, using (2.28), (2.40), (2.41), (2.44), and (2.45), that

$$\begin{aligned}C^0_\psi(12)=&\,C(12)-i\int d\bar{1}\,G_F(1\bar{1})\delta(t_{\bar{1}}-t_0)C(\bar{1}2)\\&-i\int d\bar{2}\,G_F(2\bar{2})\delta(t_{\bar{2}}-t_0)C(\bar{2}1)\\&-\int d\bar{1}\,d\bar{2}G_F(1\bar{1})G_F(2\bar{2})C_B(\bar{1}\bar{2})+C_u(12)\,.\end{aligned} \tag{2.46}$$

Equation (2.46) is structurally of the same form as (2.28). The difference is that part[34] of the noise term driving the fluctuation field has been transferred to the ordering field now governed by (2.41).

The net result of the transformations (2.40) and (2.41) becomes more transparent when one looks at the equations of motion governing C and C_u. Equation (2.41) can be written as

$$\begin{aligned}\Lambda(1)C(12)+\tfrac{1}{2}\hat{\Gamma}(1)C_{20}(12)=&\,\delta(t_1-t_0)C(12)\\&-\int d\bar{2}\,iG_F(2\bar{2})\Pi(1\bar{2})\,,\end{aligned} \tag{2.47}$$

while (2.45) can be rewritten as

$$\int d\bar{1}\,G_F^{-1}(1\bar{1})C_u(\bar{1}2)=-\int d\bar{2}\,G_F(2\bar{2})2\pi_0(1\bar{2})\,, \tag{2.48}$$

where

$$2\pi_0(12)=2\pi(12)+\delta(t_1-t_0)\delta(12)\epsilon_I+\Pi(12)+\Pi(21)\,. \tag{2.49}$$

The only choices of interest here for Π are those which are local in time and symmetric in $\mathbf{R}_1$ and $\mathbf{R}_2$:

$$\Pi(12)=\delta(t_1-t_2)\Pi(\mathbf{R}_1-\mathbf{R}_2,t_1)\,. \tag{2.50}$$

Using the result

$$G_F(\mathbf{R}_1,\mathbf{R}_2,t_1t_1)=-\frac{i}{2}\delta(\mathbf{R}_1,\mathbf{R}_2) \tag{2.51}$$

discussed in MVZ, it is easy to show that, at equal times, (2.47) reduces to

$$\begin{aligned}\left[\frac{\partial}{\partial t}+2\hat{\Gamma}(\mathbf{R})(-\nabla_R^2)\right]&C(\mathbf{R},t)+\hat{\Gamma}(\mathbf{R})C_{20}(\mathbf{R},t)\\&=\delta(t-t_0)2C(\mathbf{R},t_0)-\Pi(\mathbf{R},t)\end{aligned} \tag{2.52}$$

and (2.48) reduces to

$$\left[\frac{\partial}{\partial t}+2\hat{\Gamma}(\mathbf{R})[-\nabla_R^2+q_0^2(t)]\right]C_u(\mathbf{R},t)=2T\hat{\Gamma}(\mathbf{R})\delta(\mathbf{R})+\delta(t-t_0)2\delta(\mathbf{R})\epsilon_I+\Pi(\mathbf{R},t)\,. \tag{2.53}$$

E. Choice for Π

Consider now the appropriate choice for Π. From Eqs. (2.52) and (2.53) one sees that Π has the physical interpretation of a source term transferring material between the interfaces represented by the σ variables to the bulk represented by the u variable. Physically one expects this source to be operative in the scaling regime and serves to speed up the ordering.

Looking at the equation of motion for $C(\mathbf{R},t)$ given by (2.39), it is apparent that all of the terms except the time derivative are of $O(L^{-4})$ for large t and it is the time derivative term which is of undetermined order. In the scaling regime, where (2.38) holds, one has

$$\frac{\partial}{\partial t}C(\mathbf{R},t)=-\frac{\dot{L}}{L}\mathbf{R}\cdot\nabla_R C(\mathbf{R},t)\,, \tag{2.54}$$

and if the flow into the bulk represented by Π is to match with this term and speed up the ordering, then one has the simple choice

$$\Pi(\mathbf{R},t)=z(t)\mathbf{R}\cdot\nabla_R C(\mathbf{R},t)\,, \tag{2.55}$$

where $z(t)$ is a time-dependent factor to be determined. If one assumes, for large L, that

$$z(t)=\frac{z_0}{L^y} \tag{2.56}$$

and looks for a scaling solution for (2.52), one easily finds that the growth law is given by

$$\frac{\dot{L}}{L}=\frac{z_0}{L^y} \tag{2.57}$$

or

$$L=(yz_0t)^{1/y}+\cdots\,. \tag{2.58}$$

If $y=4$, one has the previous result obtained without coupling the ordering and fluctuating fields. For $y=3$ one obtains the LSW result and for $y=2$ the NCOP result. One must be aware, however, that the choice of y has strong effects on the long-time and distance behavior for C_u. In the scaling regime where, from (2.16), $q_0^2(\infty)=q_0^2>0$, (2.53), using (2.55) and (2.56), reduces to

$$-q_0^2\nabla_R^2 C_u=\frac{z_0}{L^y}\mathbf{R}\cdot\nabla_R C(\mathbf{R},t)\ . \tag{2.59}$$

Inserting the scaling form (2.38) for $C(\mathbf{R},t)$ into (2.59) and assuming

$$C_u(\mathbf{R},t)=\frac{\psi_0^2}{L^{y_1}}F_u(\mathbf{R}/L) \tag{2.60}$$

leads to the result

$$\nabla_x^2 F_u(\mathbf{x})=-\frac{z_0}{q_0^2}L^{2+y_1-y}\mathbf{x}\cdot\nabla_x F(\mathbf{x})\ . \tag{2.61}$$

One obtains a coupling which survives on the longest-distance scales only if $y_1=y-2$. If $y=2$, $y_1=0$, and, from (2.60), there is no separation between the ordering and fluctuating components to the structure factor. For this case the system does not properly order to $\langle\psi^2\rangle\to\psi_0^2$ since $C_u(\mathbf{0},\infty)=F_u(0)$ is not zero. For $y=3$, $y_1=1$ and $C_u(\sim L^{-1})$ does not contribute to the asymptotic scaling function. However, it is significant, as discussed below, that C_u does contribute to the leading term giving the approach of the local order parameter $\langle\psi^2(\mathbf{R},t)\rangle$ to its asymptotic value. Thus, both $C_u(0)$ and $[1-\tilde{C}(0)]$ go as L^{-1} for large L. If $y=4$ then $y_1=2$ and there is a strong separation between the ordering and fluctuating components and one returns essentially to the uncoupled situation described by (2.35). Clearly the choice $y=3$ leads to the most rapid growth for the system where it properly orders.

Given that $y=3$, the parameter z_0 can also be chosen using the criterion that the system order as rapidly as is compatible with the constraints on the system. Let us assume that the local order parameter

$$S_T(t)=\langle\psi^2(\mathbf{R},t)\rangle \tag{2.62}$$

serves as a good measure of the degree of order in the system. Then, at zeroth order for long times, the leading contributions to S_T are the first and last terms on the right-hand side of (2.46). As discussed below, the other terms decay to zero faster than L^{-1} as $L\to\infty$. Therefore,

$$S_T(t)=C(\mathbf{0},t)+C_u(\mathbf{0},t) \tag{2.63}$$

and, in the long-time limit using (2.37),

$$C(\mathbf{0},t)=\psi_0^2\left[1-\frac{\xi}{\sqrt{2}L(t)}+\cdots\right], \tag{2.64}$$

while, with $y=3$ and $y_1=1$,

$$C_u(\mathbf{0},t)=\frac{\psi_0^2}{L(t)}F_u(\mathbf{0})\ . \tag{2.65}$$

From (2.61), as will be shown in detail below, one sees that

$$F_u(0)=-\frac{z_0}{q_0^2}A_d\ , \tag{2.66}$$

where A_d depends only on the spatial dimensionality of the system. Using (2.58) with $y=3$ to express L in terms of z_0 and t, one can rewrite (2.63), using (2.64)–(2.66), in the form

$$S_T(t)=\psi_0^2\left[1-\frac{\xi}{\sqrt{2}(3z_0t)^{1/3}}-\frac{A_d z_0}{q_0^2(3z_0t)^{1/3}}\right]\ . \tag{2.67}$$

If, in direct analogy with (2.37), one defines the overall growth length

$$L_T=\frac{\xi}{\sqrt{2}}[1-S_T(t)/\psi_0^2]^{-1}\ , \tag{2.68}$$

then

$$L_T=(3z_0t)^{1/3}(1+z_0/z_1)^{-1}\ , \tag{2.69}$$

where $z_1=q_0^2\xi/\sqrt{2}A_d$. The condition for the most rapid growth is that $z_0^{1/3}(1+z_0/z_1)^{-1}$ be a maximum. This maximum for L_T is easily found to correspond to z_0 given by

$$z_0=\frac{z_1}{2}=\frac{q_0^2\xi}{2\sqrt{2}A_d}\ . \tag{2.70}$$

The net ordering length is related to the length L associated only with the interfacial component by

$$L_T=\tfrac{2}{3}L\ . \tag{2.71}$$

Note also, as a consequence of (2.60) and (2.64), the value at the origin of F_u is given by

$$F_u(0)=-\frac{\xi}{2\sqrt{2}}\ . \tag{2.72}$$

For the ψ^4 potential, $\xi=2$ and $F_u(0)=-1/\sqrt{2}$.

The key result of this section, which will be crucial in the development below, is that different interfaces can communicate via the terms proportional to Π on the right-hand sides of (2.52) and (2.53). Such terms speed up the growth law from $t^{1/4}$ to $t^{1/3}$ if Π is chosen to be of the form (2.55) with $z(t)$ given by (2.56) with $y=3$. The fluctuation field u corresponds to a diffusion field which couples to the interfacial field at long times and short distances only for $y=3$. The amplitude of the coupling z_0 can be chosen to give the most rapid net growth. The chosen value for z_0 is given by (2.70) where ξ is defined by (I.2.42) and q_0^2 by (2.16). The parameter A_d, determined using (2.67), is defined by

$$A_d=\lim_{L\to\infty}\frac{q_0^2L}{z_0}C_u(\mathbf{R}=\mathbf{0},t) \tag{2.73}$$

and depends only on the dimensionality of the system.

The asymptotic results of the theory developed here do not depend on the time dependence of $z(t)$ as long as

$$z(t)\xrightarrow[t\to\infty]{}z_0$$

faster than L^{-1}. A simple choice which vanishes as $t\to 0$ and goes exponentially to z_0 as $t\to\infty$ is

$$z(t)=z_0\{1-\exp[-(t/\tau)]\}\ , \tag{2.74}$$

where various choices of the parameter τ will be discussed in Sec. V.

F. Basic theory

Once the intermediate-time behavior of $z(t)$ is specified, then the zeroth-order approximation is completely determined. As discussed in detail in MVZ and I, one does not expect the higher-order terms in perturbation theory to contribute to the asymptotic scaling behavior other than to generate nonzero-temperature corrections to $\langle\psi\rangle$ and to the other static equilibrium properties (such as the correlation length ξ). For the rest of this paper the analysis is restricted to the zeroth-order approximation given by (2.46).

The quantity $C_B(12)$ appearing in (2.46) and defined by (2.30) is rather complicated. The term in (2.46) proportional to C_B vanishes as t^2 as $t\to 0$. One can also use arguments similar to those used below Eq. (I.4.9) in I, to show that $C_B(12)$ does not contribute to the long-distance scaling behavior ($C_B\sim L^{-3}$). It seems reasonable therefore to drop this term and it will not be discussed further here.

For very short times, the overlap terms in (2.46), like

$$-i\int d\bar{2}\, G_F(2\bar{2})\delta(t_{\bar{2}}-t_0)C(1\bar{2})\ ,$$

are important. It is just these terms which guarantee that C^0_ψ satisfy the appropriate initial condition (2.6). This term just cancels an equivalent contribution from C to avoid the problem of double counting. Since $C(R_1-R_{\bar{2}},t,t_0)$ remains of $O(\epsilon_I)$ for all times, and $G_F(R_2-R_{\bar{2}},t_2t_0)$ decays exponentially to zero for large t_2, one can drop these overlap terms with the understanding that there will be double counting for

$$C^0_\psi(12)=C(12)+C_u(12) \tag{2.75}$$

for very short times where C^0_ψ is $O(\epsilon_I)$. It will be assumed for the rest of the paper that C^0_ψ is given by (2.75) where C satisfies[35]

$$\frac{\partial}{\partial t}C(\mathbf{R},t)=-\nabla^2[-\tfrac{1}{2}C_{20}(\mathbf{R},t)+\nabla^2C(\mathbf{R},t)] -\frac{z(t)}{L^3}\mathbf{R}\cdot\nabla_R C(\mathbf{R},t) \tag{2.76}$$

and C_u, satisfies

$$\frac{\partial}{\partial t}C_u(\mathbf{R},t)=-\nabla^2[-q_0^2(t)+\nabla^2]C_u(\mathbf{R},t) +\frac{z(t)}{L^3}\mathbf{R}\cdot\nabla_R C(\mathbf{R},t)\ . \tag{2.77}$$

We assume here that the quench is to $T=0$ and these equations are supplemented with the initial conditions

$$C(\mathbf{R},0)=C_u(\mathbf{R},0)=\epsilon_I\delta_{\mathbf{R},0}\ . \tag{2.78}$$

III. LONG-TIME SOLUTION FOR THE ORDERING CONTRIBUTION

In this section the long-time solution $L\gg\xi$ for the ordering contribution to the structure factor governed by (2.76) is analyzed in detail. As discussed in detail in I, (2.76) is complete only after expressing $C_{20}(\mathbf{R},t)$ as a function of $C(\mathbf{R},t)$ or, equivalently, expressing C and C_{20} in terms of the intermediary

$$C_0(\mathbf{R},t)=\langle m(\mathbf{R},t)m(\mathbf{0},t)\rangle$$

and solving the associated equation for $C_0(\mathbf{R},t)$.

A. Near-field results ($R\ll L$)

Consider first the near field $R\ll L$. For $R=0$, as shown in I, any even function of the peak variable $g(\sigma(m))$ can be written as

$$g(\sigma(x))=g(\sigma(\infty))+\Delta g(x) \tag{3.1}$$

and the long-time average is given by

$$\langle g(\sigma)\rangle=g(\sigma(\infty))+\frac{\sqrt{2}}{L}\int_{-\infty}^{+\infty}dx\ \Delta g(x)+O\left[\frac{1}{L^3}\right]\ . \tag{3.2}$$

This result is independent of the driving kinetics. Therefore, one still has

$$S(1)=\langle\sigma^2\rangle_0=\psi_0^2\left[1-\frac{\xi}{\sqrt{2}L(t)}+\cdots\right] \tag{3.3}$$

as assumed in (2.64) where ξ is defined by (I.2.42).

For fields separated by distances $R\ll L$, it was shown in I that, to $O(1/L)$,

$$C(\mathbf{R},t)=\psi_0^2[1-W(b)/L] \tag{3.4}$$

and

$$C_{20}(\mathbf{R},t)=-\frac{\psi_0^2}{L}\frac{2}{b}\frac{\partial W}{\partial b}\ , \tag{3.5}$$

where

$$W(b)=\int_{-\infty}^{+\infty}\frac{dx\,e^{-x^2}}{\sqrt{2\pi}}J(bx)/\psi_0^2\ , \tag{3.6}$$

$$J(r)=\int_{-\infty}^{+\infty}dy[\psi_0^2-\sigma(y)\sigma(y+r)]\ , \tag{3.7}$$

and

$$b^2=2\left[1+\frac{C_0}{S_0}\right](S_0-C_0)\ , \tag{3.8}$$

where $S_0(t)=C_0(\mathbf{R}=0,t)$. For small b it was shown in I that

$$W=\frac{\xi}{\sqrt{2}}\left[1+\frac{\kappa_1 b^2}{4}+\cdots\right]\ , \tag{3.9}$$

where κ_1 is a constant that depends on the local structure. For large b,

$$W=\sqrt{2/\pi}\,b\ . \tag{3.10}$$

Inserting (3.4) and (3.5) into (2.76) and keeping terms of $O(L^{-1})$, one obtains

$$\nabla^2\left[\frac{1}{b}\frac{\partial W(b)}{\partial b}-\nabla^2 W(b)\right]=0\ . \tag{3.11}$$

The dependence of W on b is given by (3.6) and (3.7). Equation (3.11) has the partial solution

$$D(R)=\frac{1}{b}\frac{\partial W(b)}{\partial b}-\nabla^2 W(b)=D_1+D_2 S_d(R)\ , \tag{3.12}$$

where D_1 and D_2 are constants and the $S_d(R)$ are the singular solutions in d dimensions for the isotropic form of Laplace's equation as $R\to 0$. One easily finds that $S_2(R)=\ln R$ and $S_d(R)=R^{-(d-2)}$ for $d>2$. From (3.9) it is clear that $D(R)$ is regular as $R\to 0$ and one must choose $D_2=0$. Similarly, from (3.10) one finds for large $b\sim R$, $D(R)\sim R^{-1}$ and one must choose $D_1=0$. Together these results show that W satisfies exactly the same equation found in the NCOP case [see (I.5.2)]:

$$\frac{1}{b}\frac{\partial W(b)}{\partial b}=\nabla^2 W(b)\ . \tag{3.13}$$

Thus, the short-distance behavior is *identical* to that for the NCOP case and is described in detail in Sec. III C 3 of I. In particular, one regains Porod's law given by (1.3) and (1.4) or (1.6).

One therefore obtains the important result that the interfacial structure is unaffected by the nature of the dynamics leading to the final equilibrium state. Instead it depends on establishing local equilibrium and then the solution of the associated static problem given by (3.13). This static problem does depend on the details of the potential except in the limit of large R where one regains Porod's law.

B. The scaling regime ($R\gg\xi$)

It was shown in I that, in the scaling regime ($R\gg\xi$, $L\gg 1$), $C_{20}(\mathbf{R},t)$ can be expressed in terms of $C(\mathbf{R},t)$ in the form

$$C_{20}(\mathbf{R},t)=-2\frac{\psi_0^2}{L^2(t)}\tan\left[\frac{\pi}{2}C(\mathbf{R},t)/\psi_0^2\right] \tag{3.14}$$

and the equation satisfied by C given (2.76) can be written as

$$\frac{\partial C}{\partial t}=-\nabla^2\left[\frac{\psi_0^2}{L^2}\tan\left[\frac{\pi}{2}\frac{C}{\psi_0^2}\right]+\nabla^2 C\right]-\frac{z_0}{L^3}\mathbf{R}\cdot\nabla_R C\ . \tag{3.15}$$

If one now looks for a scaling solution of the form

$$C(\mathbf{R},t)=\psi_0^2 F[R/L(t)]\ , \tag{3.16}$$

one finds a situation somewhat different than in the NCOP case. Using (3.16), the time derivative in (3.15) can be expressed as

$$\frac{\partial C(\mathbf{R},t)}{\partial t}=-\psi_0^2\frac{\dot L}{L}\mathbf{x}\cdot\nabla_x F(x)\ , \tag{3.17}$$

where $\mathbf{x}=\mathbf{R}/L(t)$. This term must cancel the leading term of $[O(L^{-3})]$ on the right-hand side of (3.15) which is proportional to z_0. The terms contributing to the scaling equation enter at $O(L^{-4})$. If one defines a parameter μ by

$$\mu=\lim_{L\to\infty} L^4\left[\frac{\dot L}{L}-\frac{z_0}{L^3}\right]\ , \tag{3.18}$$

(3.15) reduces, in this case, to the scaling equation

$$-\mu\mathbf{x}\cdot\nabla_x F=-\nabla_x^2\left[\tan\left[\frac{\pi}{2}F\right]+\nabla_x^2 F\right]\ . \tag{3.19}$$

This is the same scaling equation found for the COP case in Ref. 3. The form of the scaling law in the COP case differs from the NCOP case only by the Laplacian operator acting on the right-hand side of (3.19). It also differs in that the definition of μ differs qualitatively from the NCOP case where $\mu=\lim_{L\to\infty} L\dot L$.

C. Small-x solution

One can learn quite a bit about the scaling function $F(x)$ by looking at the solution of (3.19) in the small- and large-x limits. Consider first the small-x expansion of the form

$$F=1-\alpha\sum_{n=1}^{\infty}\beta_n x^n\ , \tag{3.20}$$

where things are normalized such that $\beta_1=1$. Then, to first order in x,

$$F=1-\alpha x\ , \tag{3.21}$$

$$\tan\frac{\pi}{2}F\approx\frac{1}{(\pi/2)\alpha x}\ , \tag{3.22}$$

and (3.19) has the leading contributions of $O(x^{-3})$ as $x\to 0$ which must cancel and

$$-\nabla^2\left[\frac{1}{(\pi/2)\alpha x}+\nabla^2(-\alpha x)\right]=0 \tag{3.23}$$

gives the small result, as in the NCOP case

$$\alpha=\sqrt{2/[\pi(d-1)]}\ , \tag{3.24}$$

since $\nabla^2 x=(d-1)/x$. The next order contributions are of the form $-\nabla^2$ (constant) and give the result that β_2 is undetermined. This result will be important in the subsequent development. It is a matter of brute force to determine the higher-order coefficients in (3.20). One finds

$$\beta_3=\frac{1}{2(2d+1)}\left[(d-1)\beta_2^2-\frac{\pi}{6}\right]\ , \tag{3.25}$$

$$\beta_4=\frac{1}{(2d+1)(5d+7)}(-\beta_2)\left[(d-1)(d+2)\beta_2^2+\frac{\pi d}{2}\right]\ , \tag{3.26}$$

$$\beta_5=\frac{1}{2(3d+7)}\left[\frac{\mu}{3(d+1)}+(d-1)(C_0+C_2\beta_2^2+C_4\beta_2^4)\right]\ , \tag{3.27}$$

where

$$C_0=\frac{\pi^2}{80(d-1)^2(2d+1)^2}(d^2-4d-1)\ , \tag{3.28}$$

$$C_2=\frac{\pi}{4(2d+1)^2(5d+7)}(-3d^2+8d+7)\ , \tag{3.29}$$

$$C_4=\frac{3}{4(5d+7)(2d+1)^2}(3d^3+37d^2+73d+31)\ . \tag{3.30}$$

D. Large-x regime

In the large-x regime one can assume that F is small and replace (3.19) by the linearized equation

$$\mu x\frac{\partial F}{\partial x}=\nabla^2\left[\frac{\pi}{2}F+\nabla^2 F\right]\ . \tag{3.31}$$

The four linearly independent solutions of (3.31) are of the form

$$F=F_0 x^{y_2}\exp[-(\Gamma x^{\gamma}+\Gamma_1 x^{\nu})]\ . \tag{3.32}$$

One solution is the trivial constant solution $y_2=\gamma=\nu=0$. The other three solutions require

$$\gamma=\tfrac{4}{3}\ , \tag{3.33}$$

$$y_2=-2d/3\ , \tag{3.34}$$

$$\Gamma^3=-\frac{\mu}{\gamma^3}\ , \tag{3.35}$$

$$\nu=\tfrac{2}{3}\ , \tag{3.36}$$

and

$$\Gamma_1=\frac{\pi}{6\mu\nu}(\Gamma\gamma)^2\ . \tag{3.37}$$

The cubic equation for Γ, (3.35), has one unstable solution ($\mathrm{Re}\Gamma<0$),

$$\Gamma=-\mu^{1/3}/\gamma \tag{3.38}$$

and two stable solutions

$$\Gamma=\tfrac{3}{8}\mu^{1/3}(1\pm\sqrt{3}i)\ . \tag{3.39}$$

Since F is real, one is left with the nontrivial asymptotic solution

$$F=F_0 x^{-2d/3}\times\exp\left[-\tfrac{3}{8}\mu^{1/3}\left[x^{4/3}-\frac{\pi}{3\mu^{2/3}}x^{2/3}\right]\right]\times\cos\left[\sqrt{3}\tfrac{3}{8}\mu^{1/3}\left[x^{4/3}+\frac{\pi}{3\mu^{2/3}}x^{2/3}+\phi\right]\right]\ , \tag{3.40}$$

where F_0 and ϕ are constants which cannot be determined by the linear analysis. Thus, the nontrivial solution for $\mu>0$ decays to zero exponentially as $x\to\infty$.

E. Fourier transform

One can also make some analytical statements concerning the Fourier transform of the structure factor defined as usual by

$$C(\mathbf{q},t)=\int d^d\mathrm{R}e^{+i\mathbf{q}\cdot\mathbf{R}}C(\mathbf{R},t)\ . \tag{3.41}$$

Changing to scaling variables $\mathbf{x}=\mathbf{R}/L$ and $\mathbf{Q}=\mathbf{q}L$ and introducing (3.16) into (3.41), one obtains

$$C(\mathbf{q},t)=L^d(t)\psi_0^2 f(\mathbf{Q})\ , \tag{3.42}$$

where

$$f(\mathbf{Q})=\int d^d x\ e^{+i\mathbf{Q}\cdot\mathbf{x}}F(x)\ . \tag{3.43}$$

One can then analyze $f(Q)$ in the small- and large-Q limits. In the small-Q limit one immediately has that

$$f(\mathbf{Q})=\int d^d x\ F(x)-\frac{1}{2d}Q^2\int d^d x\ x^2F(x)+O(Q^4)\ . \tag{3.44}$$

Using the basic differential equation (3.19), one easily obtains, for $\mu\neq 0$, that

$$\int d^d x\ F(x)=0\ . \tag{3.45}$$

This is a very nice result because it means that the area under the scaling function is zero and therefore, in the scaling regime, using (3.15), (3.41), and (3.45), one establishes the conservation law

$$\lim_{q\to 0}\frac{\partial}{\partial t}C(\mathbf{q},t)=0\ . \tag{3.46}$$

Taking the second moment of (3.19), one finds

$$\int d^d x\ x^2F(x)=-\frac{2d}{\mu(2+d)}\int d^d x\tan\left[\frac{\pi}{2}F\right]\ . \tag{3.47}$$

In practice, the integral on the right-hand side of (3.47) is much easier to determine numerically than the second moment of F since $\tan[(\pi/2)F]$, falls off much faster with x than $x^2F(x)$. If one defines

$$B_d=\int d^d x\tan\left[\frac{\pi}{2}F\right]\ , \tag{3.48}$$

then[36]

$$f(\mathbf{Q})=\frac{Q^2}{\mu(2+d)}B_d\ . \tag{3.49}$$

$\mu(d)$ and B_d will be determined numerically later in this section.

In the large-Q limit it is necessary to first carry out the angular integrations in the Fourier integral (3.43). For $d=2$ one finds

$$f_2(Q)=2\pi\int_0^{+\infty}dx\ xJ_0(Qx)F(x)\ , \tag{3.50}$$

where J_0 is the standard Bessel function while, for $d=3$, one obtains

$$f_3(Q)=4\pi\int_0^{+\infty}dx\ x\frac{\sin qx}{q}F(x)\ . \tag{3.51}$$

In the large-Q limit for $d=2$ one can extract the leading behavior in (3.50) by (i) changing variables to $y=Qx$, (ii) using the identity

$$yJ_0(y)=\frac{d}{dy}[yJ_1(y)] ,$$

(iii) integrating by parts with respect to y, and (iv) replacing $(d/dy)F(y/Q)$ by $-\alpha/Q$ for large Q. Then,

$$f_2(Q)=\frac{2\pi\alpha}{Q^3}\int_0^{+\infty}dy\, yJ_1(y) =\frac{4\sqrt{\pi}\alpha}{Q^3} , \tag{3.52}$$

where the integral over $J_1(y)$ is standard. In three dimensions the large-Q behavior is extracted using the standard integrations by the parts procedure to obtain

$$f_3(Q)=\frac{8\pi\alpha}{Q^4} . \tag{3.53}$$

Clearly one has the Fourier representation of Porod's law with

$$f_d(Q)=\frac{C_d\alpha}{Q^{d+1}} \tag{3.54}$$

for large Q where the C_d are listed in Table I for $d=2$ and 3.

F. Numerical determination of $F(x)$

Since it is not obvious that one can solve (3.19) analytically, it is sensible to construct a numerical solution. Since it is clear that, in the scaling regime, F is isotropic, one can rewrite (3.19) as

$$\mu x\frac{\partial F}{\partial x}=\left[\frac{\partial}{\partial x}+\frac{(d-1)}{x}\right]\frac{\partial}{\partial x} \times\left[\tan\left[\frac{\pi}{2}F\right]+\left[\frac{\partial}{\partial x}+\frac{(d-1)}{x}\right]\frac{\partial F}{\partial x}\right] . \tag{3.55}$$

The method for the solution for this fourth-order differential equation is worth a bit of discussion. If one starts at $x=0$, one can integrate this equation forward in x using, for example, the second-order Runga-Kutta method with a very small Δx ($=0.000\,01$ in these calculations). Of course, this requires specifying $F(0)$, $F'(0)$, $F''(0)$, and $F'''(0)$. These are known from the small-x expansion (3.20) up to the undetermined parameter β_2. Thus, at this point, β_2 and μ are undetermined. In close analogy to the NCOP case where the parameter μ had to be chosen so that one obtains the proper large-x solution, here one also has a nonlinear eigenvalue problem. The difference is that, in the present case, there are two eigenvalues (μ and β_2) which must be fixed to eliminate the constant $F(\infty)\neq 0$ and the exponentially growing solutions.

The numerical method used to determine $\mu^*(d)$ and $\beta_2^*(d)$ was converging, but rather time consuming. A description of the procedure may be useful for developing a faster algorithm. The first step is to choose values for $\beta_2^{(1)}$ (a bit of trial and error shows that the appropriate range for $\beta_2^{(1)}$ is small and negative) and $\mu^{(1)}>0$. As soon as one tries to forward step (3.55) for small x, one finds singular pieces for small x which must cancel. One can avoid this numerical problem by matching the known power-series expansion for $F(x)$ for small x to the numerical solution starting at some $x=x_0$, where $F(x_0)$, $F'(x_0)$, $F'''(x_0)$ are known. This procedure is stable and rather insensitive to the choices of $\beta_2^{(1)}$ and $\mu^{(1)}$. As one forward steps the differential equation in x, there will be a rather definite range of x (>1) where $F(x)$ will become unstable and rapidly cross over to an exponentially unstable solution of positive or negative sign. Assume for simplicity that $F(x)$ is exponentially increasing, then one should, keeping $\beta_2^{(1)}$ fixed, increase μ to $\mu^{(2)}$. By taking $\mu^{(2)}$ large enough, one finds that the instability will become exponentially large and negative. Thus, for this value of $\beta_2(=\beta_2^{(1)})$ the values $\mu^{(1)}$ and $\mu^{(2)}$ bracket the value $\bar{\mu}(\beta_2^{(1)})$ for which $F(x)$ converges to a constant $F[\infty,\bar{\mu}(\beta_2^{(1)}]$ as $x\to\infty$. By choosing $\mu^{(3)}$ such that $\mu^{(1)}<\mu^{(3)}<\mu^{(2)}$, one can extend the convergent solution for $F(x)$ to larger values of x. Clearly one can continue this sequence by observing whether, for $\mu^{(3)}$, the solution is eventually unstable positively or negatively. Carrying out this sequence, one converges to a definite value $\bar{\mu}(\beta_2^{(1)})$ with a specific form for $F(x,\beta_2^{(1)})$. For this choice of $\beta_2^{(1)}$, $F(x,\beta_2^{(1)})$ will not, in general, go to zero for large x. One must repeat the procedure for another choice $\beta_2^{(2)}$ and obtain $\bar{\mu}(\beta_2^{(2)})$ and $F(x,\beta_2^{(2)})$. Fortunately, given $F(\infty,\beta_2^{(1)})$ and $F(\infty,\beta_2^{(2)})$, one can make an estimate of what $\beta_2^{(3)}$ is necessary to obtain $F(\infty,\beta_2^{(3)})=0$. Using this value of β_2, one can again obtain $\bar{\mu}(\beta_2^{(3)})$ and $F(x,\beta_2^{(3)})$ and, as was found in the calculations here, a $F(\infty,\beta_2^{(3)})$ which is very near zero.

Convergent results out to $x=20$ which give $|F(\infty)|<10^{-6}$ have been obtained for $d=2$ and 3. The values for $\beta_2^*(d)$ and $\mu^*(d)$ are given in Table I. The corresponding plots for $F(x)$ are shown for $d=2$ and 3 in Fig. 1. The specific special values corresponding zeros, maxima, and minima are given in Table II.

In Fig. 2 the numerically determined $F(x)$ is plotted versus the best available direct numerical simulations for this problem. In Fig. 2(a) the results for two dimensions are plotted versus the results of Rogers *et al.*[28] and Gawlinski *et al.*[29] The x scale has been adjusted such that the first zero comes at the same value of x in the theory and simulations. The small-x agreement is excellent as well as the position of the first maximum and second minimum. The overall amplitudes of the theory are less than those in the simulations, but the results in Ref. 29 are at considerably longer times than in Ref. 28 so the simulations do seem to be approaching the theory. In Fig. 2(b) the theoretical results for three dimensions are plotted versus the results of Chakrabarti *et al.*[30] These numerical results are for an earlier relative time when compared to two dimensions. The theory does capture the qualitative feature that the amplitude of $F(x)$ is smaller in three than two dimensions. In summary, the overall agreement between theory and numerical experiment is good.

TABLE II. Features associated with the scaling function $F(x)$ for $d=2$ and 3.

	$d=2$		$d=3$	
	x	F	x	F
First zero	1.68	0	2.47	0
First minimum	2.60	−0.134 6	3.41	−0.073 22
Second zero	4.13	0	5.04	0
Second maximum	4.98	0.033 9	5.97	0.018 6
Third zero	6.37	0	7.45	0
Second minimum	7.17	−0.008 9	8.34	−0.005 19

One can fit the large-x behavior of the numerically determined $F(x)$ to the asymptotic form (3.40). For both $d=2$ and 3, the fit is excellent in the range $10\leq x\leq 20$ and the fitted values for Γ and Γ_1 lead to values of μ, via (3.35) and (3.37), in good agreement with the selected values of $\mu^*(d)$ given in Table I. Given the numerically determined $F(x)$ one can then evaluate the integrals B_d given by (3.48). The values for the B_d from the numerical integration are listed in Table I. The coefficient of Q^2 in the Fourier transform is given then by $B_d/\mu_d^*(2+d)$.

Finally, one can compute the Fourier transform $f(Q)$ using (3.50) and (3.51). The results for $d=2$ and 3 are shown in Fig. 3. The small- and large-Q behaviors are, as expected, in agreement with the analytic treatments.

IV. DIFFUSIVE FIELD

In this section $C_u(\mathbf{R},t)$, the diffusion contribution to the structure factor governed by (2.77), is evaluated. Since the short-time and short-distance behavior of C_u is rather nonuniversal, the discussion here will focus on the long-time and long-distance scaling regime. The approach to this regime will be discussed as part of the numerical analysis in the next section.

In the long-time limit one can replace $q_0^2(t)$ with $q_0^2>0$ and $z(t)$ with z_0 in (2.77). q_0^2, defined by (2.16), depends on the form of the potential. For the ψ^4 potential, $q_0^2=2$. The parameter z_0, discussed in Sec. II E, can be explicitly determined once the parameter A_d, defined by (2.73), is determined. As discussed in Sec. II, in the scaling regime $C_u(\mathbf{R},t)$ has a solution of the form

$$C_u(\mathbf{R},t)=\frac{\psi_0^2 F_u(\mathbf{R}/L)}{L} \tag{4.1}$$

and, inserting (4.1) into (2.77) and neglecting higher-order terms in powers of L^{-1}, one obtains

$$q_0^2\nabla_x^2 F_u(x)=-z_0\mathbf{x}\cdot\nabla_x F(x)\ , \tag{4.2}$$

which is equivalent to (2.61) with $2+y_1-y=0$. Thus, $F_u(x)$ satisfies Poisson's equation and the term $(z_0/4\pi)\mathbf{x}\cdot\nabla_x F(x)$ serves as the charge density for the system. It is at this stage that one can appreciate the importance of the result (3.45) which guarantees over all charge neutrality since

$$\int d^dx\ \mathbf{x}\cdot\nabla_x F(x)=d\int d^dx\ F(x)=0\ . \tag{4.3}$$

Equation (4.2) can be solved as usual by constructing the appropriate Green's function which is regular at the origin. The case of $d=2$ is slightly different than for $d>2$. For $d=2$ one easily finds the solution

$$F_u(x)=-\frac{z_0}{q_0^2}\int\frac{d^2y}{2\pi}[\,\Theta(x-y)\ln x\ \mathbf{y}\cdot\nabla_y F(y) +\Theta(y-x)\ln y\ \mathbf{y}\cdot\nabla_y F(y)]\ . \tag{4.4}$$

Integrating in (4.4) by parts, one obtains

$$F_u(x)=\frac{z_0}{q_0^2}\int\frac{d^2y}{2\pi}F(y)[\,2\ln x\,\Theta(x-y) +(2\ln y+1)\Theta(y-x)]\ . \tag{4.5}$$

Setting $x=0$ in (4.5) gives

$$F_u(0)=\frac{z_0}{q_0^2}\int\frac{d^2y}{2\pi}F(y)(2\ln y+1)\ . \tag{4.6}$$

Since the volume integral over $F(y)$ is zero, (4.6) reduces to

$$F_u(0)=\frac{z_0}{q_0^2}\int\frac{d^2y}{2\pi}F(y)2\ln y\ . \tag{4.7}$$

Using (4.6), one can rewrite (4.5) in the form

$$F_u(x)=F_u(0)+\frac{z_0}{q_0^2}\int_0^x y dy\ F(y)[2\ln(x/y)-1] \tag{4.8}$$

which is convenient for explicit computation.

For $d>2$, one has the solution to (4.2) given by

$$F_u(x)=\frac{z_0}{q_0^2}\int_0^{+\infty}dy\ y^{(d-1)}y\frac{\partial}{\partial y}F(y)\frac{1}{(d-2)} \times\left[\frac{\Theta(x-y)}{x^{d-2}}+\frac{\Theta(y-x)}{y^{d-2}}\right]\ . \tag{4.9}$$

Doing an integration by parts, this can be put into the form

$$F_u(x)=-\frac{z_0}{(d-2)q_0^2}\left[\frac{d}{x^{d-2}}\int_0^x dy\ y^{d-1}F(y) +2\int_x^\infty dy\ yF(y)\right]\ . \tag{4.10}$$

The value of $F_u(x)$ at the origin is given by

$$F_u(0)=-\frac{2z_0}{q_0^2(d-2)}\int_0^{+\infty}dy\ yF(y) \tag{4.11}$$

and one can rewrite $F_u(x)$ in the convenient form

$$F_u(x)=F_u(0)-\frac{z_0}{q_0^2(d-2)}\int_0^x dy\, F(y)\left[\frac{dy^{d-1}}{x^{d-2}}-2y\right]. \tag{4.12}$$

At this stage one can explicitly determine the parameter A_d defined by (2.73) and, in turn, z_0 given by (2.70) and $C_u(0)$ given by (4.1), (4.6), and (4.11). First, using (2.73), (4.1), and (4.7) for $d=2$, one finds that

$$A_2=-\int_0^{+\infty} dy\, 2y \ln y\, F(y)\ , \tag{4.13}$$

while from (2.73), (4.1), and (4.11) for $d>2$,

$$A_d=\frac{2}{(d-2)}\int_0^{+\infty} dy\, yF(y)\ . \tag{4.14}$$

Clearly A_d depends only on the dimensionality of the system. One finds numerically the values of A_d listed in Table I, along with the selected values of $z_0=\xi q_0^2/2\sqrt{2}A_d$ for the ψ^4 potential.

Using (4.8) and (4.12), one can rather easily determine $F_u(x)$ numerically given the solution for $F(x)$ determined in the last section. The results for $d=2$ and 3 are shown in Fig. 4.

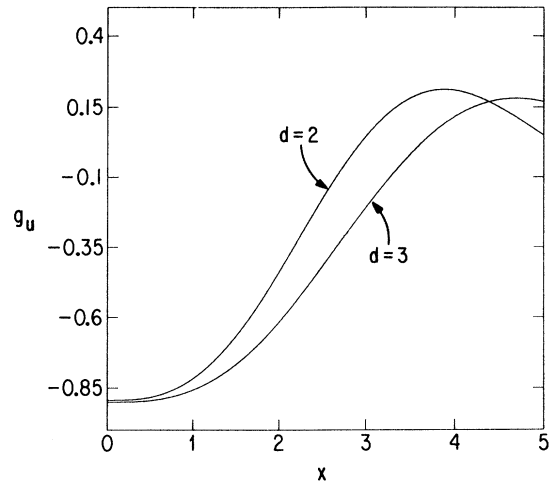

FIG. 4. The scaling function $g_u(x)=q_0^2F_u(x)/z_0$ for $d=2$ and 3.

V. NUMERICAL ANALYSIS OF ZEROTH-ORDER THEORY

The numerical solution of the complete zeroth-order theory for the ψ^4 potential is given in this section. In the complete theory one expresses C and C_{20} in terms of the intermediate function C_0 and then integrates this equation for C_0 forward in time. In close analogy with Eqs. (I.4.11) and (I.4.12), one has, in this case, for $R=0$, that $C_0(\mathbf{R}=0,t)=S_0(t)$ satisfies

$$[C_{11}(0)+C_{20}(0)]\frac{dS_0(t)}{dt}=\left[(-\nabla_R^2)[-\tfrac{1}{2}C_{20}(\mathbf{R})+\nabla_R^2 C(\mathbf{R})]\right]_{\mathbf{R}=0}, \tag{5.1a}$$

while, for $R\neq 0$,

$$\frac{\partial C_0(\mathbf{R})}{\partial t}=-\frac{\partial S_0}{\partial t}\frac{C_{20}(\mathbf{R})}{C_{11}(\mathbf{R})}-\frac{1}{C_{11}(\mathbf{R})}(-\nabla_R^2)[\tfrac{1}{2}C_{20}(\mathbf{R})-\nabla_R^2 C(\mathbf{R})]+\frac{1}{C_{11}(\mathbf{R})}\frac{z(t)}{L^3(t)}\mathbf{R}\cdot\nabla_R C(\mathbf{R})\ , \tag{5.1b}$$

where C, C_{11}, and C_{20} are given in terms of $C_0(\mathbf{R},t)$ and $S_0(t)$ by (I.4.14)–(I.4.16), respectively. C_u satisfies (2.77) with, for the ψ^4 potential (2.5) with $\lambda=0$, $q_0^2=2$.

The set of equations (5.1) was studied in detail in the isotropic limit where

$$\nabla_R^2=\frac{d^2}{dR^2}+\frac{(d-1)}{R}\frac{d}{dR} \tag{5.2}$$

and the magnitude of $\mathbf{R}$ was discretized in the form $R=h(n+1)$ with n an integer ranging from 0 to N. Notice that a short-distance cutoff of size h has been introduced. For the present calculations, a lattice spacing $h=1.73$ was chosen since it leads to smooth early growth in the problem. Periodic boundary conditions were used $C(R+N)=C(R)$ and N was taken to be 200 for the calculations presented here. The role of finite-size effects will be discussed below. Equation (5.1) is then integrated forward in time with a time step $\Delta t=0.01$ assuming an initial condition $\epsilon_I=0.01$.

At this stage the only undetermined parameter in the theory is the time τ which characterizes the buildup of the parameter $z(t)$ given by (2.74). It was found that, for small values of τ (<300), the set of equations (5.1) does not equilibrate. In particular, this set of equations with $z_0=0$ does not equilibrate. This is strictly a function of the way in which the isotropic limit is introduced. If one solves (5.1) on a two-dimensional square lattice, one finds that the equations properly equilibrate. Thus, the equations (5.1), at short times, are sensitive to how one treats the spatial gradients in the problem. For $\tau>300$ one finds that the z_0 term stabilizes the growth and one can investigate the approach to the asymptotic limit.

A detailed discussion of the nature of the short- and intermediate-time behavior will not be attempted here.

TABLE III. Numerical values for the fits of the curves for L and L_0 shown in Fig. 6 to the form (5.3).

$L^{(1/3)}$	$L^{(0)}$	$L^{(-1/3)}$	$L_0^{(1/3)}$	$L_0^{(0)}$	$L_0^{(-1/3)}$
1.62	4.43	−55.0	3.38	17.2	−141.2

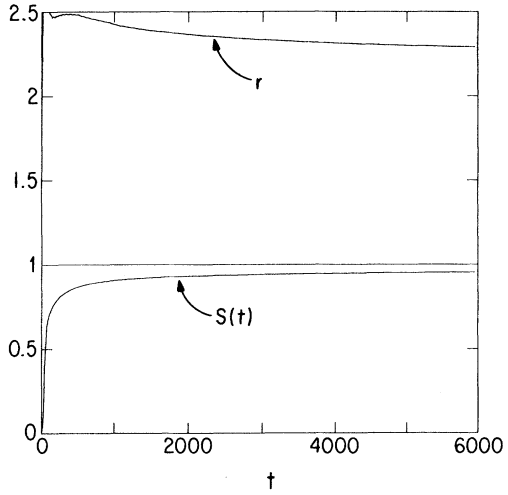

FIG. 5. The local order parameter $S(t)=\langle\sigma^2(\mathbf{R},t)\rangle$, and the ratio of lengths $r=L_0/L$ vs time for the calculation described in Sec. V.

The key point is that, when the system equilibrates, it does so via the asymptotic fixed point discussed in Sec. III. The short- and intermediate-time behavior depends on various nonuniversal components of the problem such as the value of ϵ_I, the nature of the space (on a lattice or in the continuum), and the other zeroth-order contributions neglected in (2.63).

The results of the calculation described above for the local order parameter $S(t)=\langle\sigma^2\rangle$ are shown in Fig. 5. One sees that the system does properly order. The associated length L, defined by (2.37) with $\xi=2$, is shown in Fig. 6 and is fit to the form

$$L=L^{(1/3)}t^{1/3}+L^{(0)}+L^{(-1/3)}t^{-1/3}\ , \tag{5.3}$$

where the coefficients are given in Table III. Note that the value of $L^{(1/3)}=1.62$ is in good agreement with the theoretical result $(3z_0)^{1/3}=1.67$. Also shown in Fig. 6 is the position of the first zero in the structure factor as a function of time $L_0(t)$, which is also fit to the form (5.3) with L replaced by L_0. These coefficients are also listed in Table III. In the scaling limit the ratio $L_0/L=r$ should approach the fixed universal value $r_*=2.47$ for $d=3$ as can be obtained from Table II. The numerically determined r is plotted in Fig. 5. The large-time plateau is indicative of scaling behavior in the system. The numerically determined value of r comes from the ratio of coefficients $L^{(1/3)}$ and $L_0^{(1/3)}$ given in Table II with the result

$$r=L_0^{(1/3)}/L^{(1/3)}=2.09\ .$$

It is not clear why this differs as much as it does from the asymptotic result of 2.47. It may be due to finite-size effects.

Consider next the behavior of the diffusion field as governed by the quantity

$$C_u(\mathbf{R}=0,t)\equiv S_u(t)\ .$$

This quantity and LS_u are plotted in Fig. 7. Note the rather slow convergence to the asymptotic value $LS_u=-1/\sqrt{2}$.

Turning now to the spatial structure associated with the ordering, $C(\mathbf{R},t)$ is plotted versus n for various times t after the quench in Fig. 8. One can see that the depth of the minima is established rather early in the evolution. The scaling function C versus $x=R/L$ is plotted in Fig. 9. Comparing Fig. 9 with Fig. 1, it is clear that the numerically determined structure factor has a deeper first maximum for the time presented than the asymptotic structure factor. As shown in Fig. 10, the first minimum is a slowly increasing function of time, but it will take a very long time to approach the asymptotic limit. One finds a similar development in comparing the numerically determined

$$F_u(x,t)=LC_u(\mathbf{R},t)$$

as a function of time given in Fig. 11 with the asymptotic

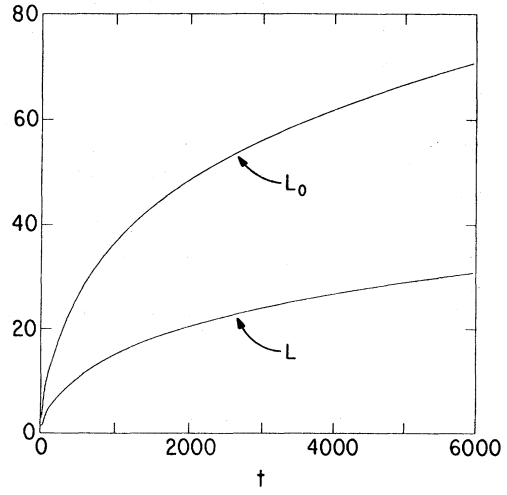

FIG. 6. Numerical determination of the lengths L and L_0 as a function of time.

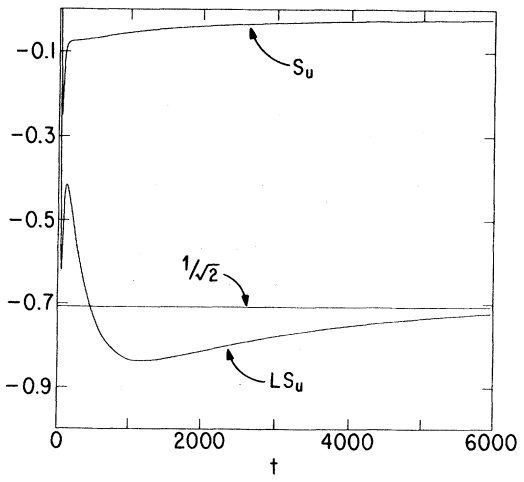

FIG. 7. $S_u=C_u(\mathbf{R}=0,t)$ and LS_u vs time.

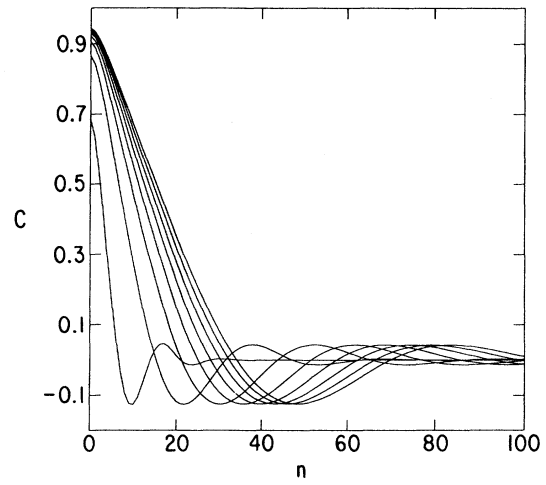

FIG. 8. The ordering component of the structure factor $C(R,t)$ vs $n=R/h$ for various times after the quench. As the curves move out from near $R=0$ as a function of time, one has $t=100$, 500, 1000, 1500, 2000, 2500, 3000, and 3500.

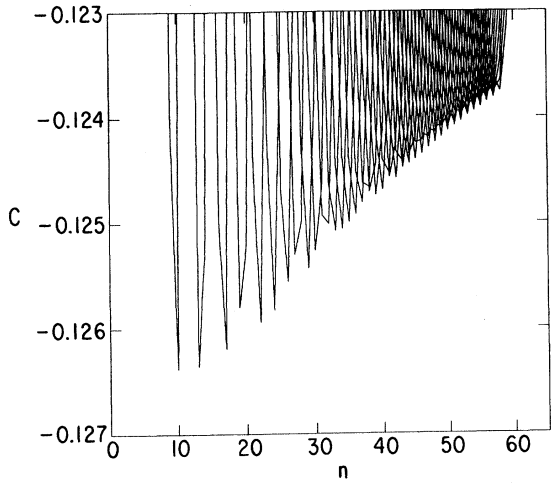

FIG. 10. Same as Fig. 8 except the position of the depth of the first minima is accentuated. The times here run out to $t=3900$ in steps of 100.

rm $F_u(x,\infty)$ shown in Fig. 4. The curves are qualitatively the same but the height of the first maximum is slowly decreasing as a function of time.

It is found here, as in I, that the theoretical amplitude in the growth law is larger than that found in simulations. In I this was reflected in the parameter $\mu=L\dot{L}$, which was theoretically predicted to have a value of 1.104, but in the simulations described in I, went to an apparent long-time limit of ~ 0.6 (see Fig. 9 in I). It was suggested in I that this is a finite-size effect which influences the nature of the eigenvalue problem which determines μ. In the present case, the coefficient of $t^{1/3}$ for the first zero is given theoretically by

$$L_0^{(1/3)}=L^{(1/3)}r=(3z_0)^{1/3}r\ ,$$

where $r=L_0/L=1.68$ as given in Table II for the position of the first zero in the scaling function for $d=2$. One has then, using $z_0=1.58$, that $L_0^{(1/3)}=2.82$. This amplitude is determined by Rogers *et al.*[28] and they find $L_0^{(1/3)}=0.84\pm 0.02$ (see their Table II). There are several points to be made.

(i) Rogers *et al.* point out that this coefficient is rather sensitive to the choice of the lattice spacing h. As shown in their Fig. 9, the amplitude of L_0 varies rapidly as a function of h in the region of h for which $L_0^{(1/3)}$ was determined ($h=1.7$). Smaller values of h give larger values of $L_0^{(1/3)}$. The authors believe, however, that $L_0^{(1/3)}$ will saturate at some final value for $h\leq 1$.

(ii) z_0 was chosen to give maximum growth and it indeed bounds the simulational result from above. Possibly some other choice for z_0 is physically relevant.

(iii) Again the finite-size question and the role of

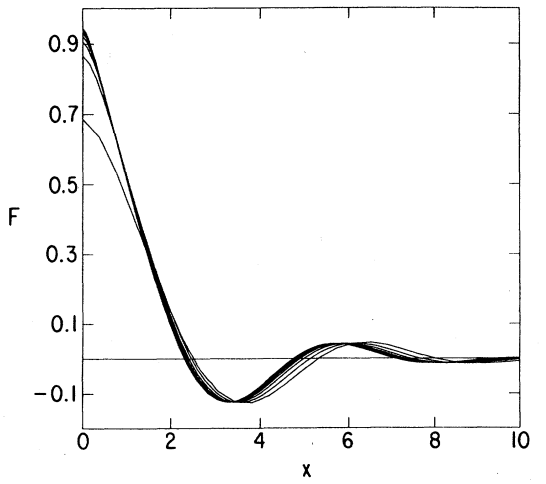

FIG. 9. Same as Fig. 8 except $C(\mathbf{R},t)=F(R/L,t)$ is plotted vs $x=R/L$.

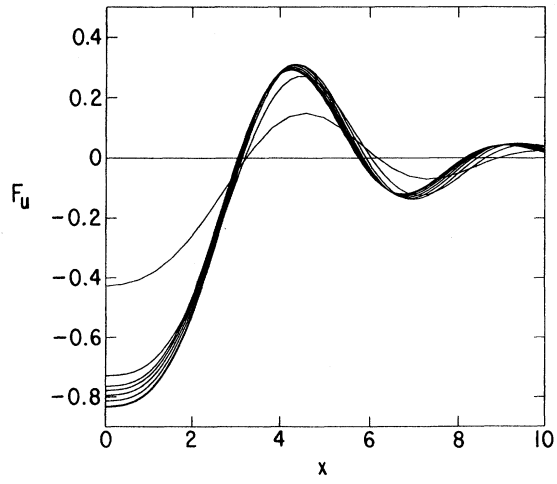

FIG. 11. Same as Fig. 8 except $g_u(R/L,t)=q_0^2LC_u(\mathbf{R},t)/z_0$ is plotted vs $x=R/L$.

periodic boundary conditions in determining the asymptotic $F(x)$ must be analyzed in more detail since the choice of z_0 is related to the solution for $F_u(x)$ in terms of $F(x)$. The physics selecting z_0 is operative only in the regime where $F_u(x)$ is determined by the diffusion equation (4.2). It is not clear how one approaches this regime for systems on a small lattice. For systems on a lattice, it is not required that $F(\infty)=0$ since there is a natural cutoff for $x=N/L$. Eventually the growth will be influence by the presence of the periodic boundary conditions at $x=N/L$ and the system may not reach the asymptotic fixed point discussed in Sec. III. The question is how large N has to be in order to approach the asymptotic fixed-point solution.

VI. CONCLUSIONS

The theory developed in Refs. 1 and 3 has been extended to the case of a conserved order parameter. A key formal and physical component of this extension is the inclusion of a nontrivial coupling between the ordering component and the fluctuating component of the order-parameter field. In MVZ the case was made for separating the order-parameter field into these distinct components. In Refs. 1 and 3 it was shown how the theory could be constructed so that these fields became decoupled in the shortest amount of time. In the NCOP context this seems to be the most natural way to organize the theory. In that case the growth is controlled by the local curvature and one expects the development in I to be correct. In the COP case the role of bulk diffusion is missing in this description and one is led to the $t^{1/4}$ growth law and growth via surface diffusion only. The development in this paper shows how bulk diffusion can be incorporated into the theory via some simple functional transformations. The final form of the theory then leads trivially to the LSW $t^{1/3}$ growth law and ordering via bulk diffusion.

Even though the coupling of the ordering field to the fluctuating field leads to a qualitative change in the growth law, it does not lead to a substantial change in the scaling properties of the system. The double eigenvalue problem solved here for the scaling function $F(x)$ is the same problem proposed in Ref. 3 (although it was not appreciated that this led to an eigenvalue problem there). The coupling of the ordering and fluctuating fields has led basically to the reinterpretation of the parameter μ entering the scaling equation. In the uncoupled theory $\mu=L\dot{L}$, while in the coupled theory μ has a more complicated form given by (3.18). Given the good agreement with the simulation results shown in Fig. 2, one believes that the theory is in good shape.

Where are there loose ends in this development? One assumption which should be scrutinized is the assumption that z_0 be chosen such that the growth rate be maximized. Other choices are possible. Clearly the approach of the theory to the long-time scaling limit given in the last section is strongly dependent on a number of assumptions and nonuniversal ingredients. Clearly the choice for $z(t)$ given by (2.70) and (2.74), including the choice for τ, is relatively arbitrary. Detailed calculations for short times and short distances will depend strongly on the choice for the realization of the gradients in the equations of motion. They also require including the other terms contributing to the zeroth-order approximation (2.28). It is important to remember, however, that all of the long-time scaling results should be independent of all of these ingredients.

The analysis in this paper has been limited to the case of critical quenches. Thus, it has been assumed that $\langle\psi\rangle=0$ for all finite times. Off-critical quenches also lead to long-time scaling behavior in the case of a COP with the added feature of compact morphologies for minorities phases. The original LSW theory was developed for the case of a system quenched to near the coexistence curve where there is a small relative concentration fraction of the minority phase. It will be shown in the next paper in this series that this case can be treated within the context of the theory developed here with only the modest additional complication of treating the case where there is no up-down symmetry associated with the fields ψ, m, and σ. In this case it will be found that the scaling function $F(x)$ does depend on the degree of broken symmetry $M=\langle\psi\rangle$.

ACKNOWLEDGMENTS

This work was supported by National Science Foundation (NSF) Grant No. DMR 88-19860 at The University of Chicago.

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[1]G. F. Mazenko, Phys. Rev. B **42**, 4487 (1990). This paper will be referred to as I and, for example, Eq. (2.3) in I will be listed here as (I.2.3).

[2]General reviews include J. D. Gunton, M. San Miguel, and P. S. Sahni, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, London, 1983), Vol. 8; H. Furukawa, Adv. Phys. **34**, 703 (1985); K. Binder, Physica A **140**, 35 (1986).

[3]A preliminary version of both cases was given in G. F. Mazenko, Phys. Rev. Lett. **63**, 1605 (1989).

[4]For a recent review see K. Binder, in *Materials Science and Technology,* Vol. 5 of *Phase Transitions in Materials,* edited by P. Haasen (VCH Verlagsgesellschaft Weinheim, Germany, in press). For discussions of the early theory see K. Binder, in *Spinodal Decomposition,* Vol. 132 of *Lecture Notes in Physics,* edited by L. Garrido (Springer-Verlag, Berlin, 1980); J. D. Gunton and M. Droz, in *Introduction to the Dynamics of Metastable and Unstable States,* Vol. 183 of *Lecture Notes in Physics,* edited by J. Zittartz (Springer-Verlag, Berlin, 1983).

[5]In, for example, the theory of K. Kawasaki and T. Ohta, Prog. Theor. **68**, 129 (1982); Physica (Amsterdam) A **118**, 175 (1983), the COP case has been treated only at the level of dimensional anlaysis. No detailed analysis has been worked out as for the NCOP case in T. Ohta, D. Jasnow, and K.

Kawasaki, Phys. Rev. Lett. **49**, 1223 (1983).
[6]This statement is from the point of view of developing a first-principles theory.
[7]J. S. Langer, M. Bar-on, and H. D. Miller, Phys. Rev. A **11**, 1417 (1975).
[8]K. Binder, C. Billotet, and P. Mirold, Z. Phys. B **30**, 183 (1978); **30**, 183 (1978).
[9]H. Horner and K. Jüngling, Z. Phys. B **36**, 97 (1979).
[10]G. F. Mazenko, O. T. Valls, and M. Zannetti, Phys. Rev. B **38**, 520 (1988), referred to here as MVZ.
[11]I. M. Lifshitz and V. V. Slyozov, J. Phys. Chem. Solids **19**, 35 (1961); C. Wagner, Z. Elektrochem. **65**, 581 (1961).
[12]J. S. Langer, Ann. Phys. **65**, 53 (1971).
[13]J. J. Weins and J. W. Cahn, in *Sintering and Related Phenomena,* edited by G. C. Kuczynskie (Plenum, London, 1973), p. 1.
[14]M. Tokuyama and K. Kawasaki, Physica A **123**, 386 (1984).
[15]D. Huse, Phys. Rev. B **34**, 7845 (1986).
[16]A. Bray, Phys. Rev. Lett. **62**, 2841 (1989).
[17]H. Furukawa, Adv. Phys. **34**, 703 (1985); Y. Oono and S. Puri, Phys. Rev. A **38** (1988).
[18]See P. Voorhees, J. Stat. Phys. **38**, 231 (1985), for a recent treatment.
[19]To be precise, ζ is related to the diffusion field u by (2.40).
[20]The results did not depend on the form of the potential beyond the fact that it was bounded with two degenerate quadratic minima. They also did not depend on the initial conditions or the spatial structure (lattice or continuum).
[21]This question of nonzero-temperature effects was partially addressed in Ref. 10. A more complete treatment in the context of the present theory will be given elsewhere.
[22]In the NCOP case the result (1.1), with μ^* given by the associated nonlinear eigenvalue problem, is well satisfied when making direct comparisons with simulations. The result (1.2) is not well satisfied and there are various comments in I concerning the possible reasons for the discrepancy.
[23]G. Porod, in *Small Angle X-Ray Scattering,* edited by O. Glatter and L. Kratky (Academic, New York, 1983).
[24]Of course, the L used in defining $x=R/L$ has a different time dependence in the two cases but it can be defined by (2.37) in both cases.
[25]H. Tomita, Prog. Theor. Phys. **72**, 656 (1984); **75**, 482 (1986).
[26]The term eigenvalue is being used somewhat loosely here. The meaning is simply that of a parameter in the differential equation or boundary conditions which must take on specific values if one is to obtain an acceptable solution.
[27]This means that μ is the coefficient of the term $\mathbf{x}\cdot\nabla_x F(x)$ in the determining equation for $F(x)$.
[28]T. M. Rogers, K. R. Elder, and R. C. Desai, Phys. Rev. B **37**, 9638 (1988).
[29]E. T. Gawlinski, J. Viñals, and J. D. Gunton, Phys. Rev. B **39**, 7266 (1989).
[30]A. Chakrabarti, R. Toral, and J. D. Gunton, Phys. Rev. B **39**, 4386 (1989).
[31]While there appears to be a small discrepancy in the depth of the first minima, this may be a finite-time effect. The numerical results discussed in Sec. V show a very slow tendency for the magnitude of the minimum to decrease with time. Indeed the numerically determined $F(x,t)$ found in Sec. V has a minima in very good agreement with the simulations over similar time regions. There may also, as in the NCOP case, be finite-size effects which make it difficult to access the final asymptotic fixed-point solution.
[32]C. de Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).
[33]The field ϕ is related to the field ζ discussed in Sec. II by (2.26). Except for early times, one expects ϕ and ζ to have similar physical interpretations.
[34]One could also think of this in terms of shifting part of the initial condition if II (12) is proportional to $\delta(t_1-t_0)$.
[35]It is assumed here that the quench is to zero temperature and the initial condition is treated as part of an initial value problem rather than by adding a δ-function term as in (2.52). Clearly the two approaches are equivalent. The initial value approach is preferable when treating the equal-time case. For unequal times, the formulation in terms of δ functions, as in (2.47), (2.52), and (2.53), is preferable.
[36]This result does not seem to agree with the results of H. Furukawa (unpublished) who finds a universal Q^4 behavior for the structure factor at small Q.