PHYSICAL REVIEW B VOLUME 43, NUMBER 7 1 MARCH 1991

Superfluidity of ^{4}He

Emilio Del Giudice
Sezione di Milano, Istituto Nazionale di Fisica Nucleare (INFN), via Celoria 16, I-20133, Milano, Italy

Matteo Giuffrida and Renata Mele
Dipartimento di Fisica, Università di Milano, Milano, Italy

Giuliano Preparata
Dipartimento di Fisica, Università di Milano, Milano, Italy and INFN, Sezione di Milano, Milano, Italy
(Received 31 May 1990)

We apply the recently proposed approach of quantum field theory of superradiance to liquid ^{4}He. We find natural, and input-free, explanations of (i) the temperature dependence of the roton gap $\Delta(T)$ and the normal-fluid fraction ρ_n/ρ, (ii) the microscopic two-fluid structure of superfluid ^{4}He; and (iii) the basic mechanism of vortex formation and the associated critical velocities.

I. INTRODUCTION

Quantum field theory (QFT) is the natural theoretical framework for dealing with quantum-mechanical systems comprising a very large (infinite in the limit) number of dynamical degrees of freedom, such as those occurring in condensed matter. The key notion in this fundamental approach to the dynamics of N elementary matter systems (atoms, molecules, ions, electrons), contained in a volume V, is the quantum wave field $\Psi(\mathbf{x},\alpha;t)$, which is a function of the translational variable $\mathbf{x}$, of a set of internal variables α (discrete or continuous) and of time. The procedure necessary for constructing the wave field $\Psi(\mathbf{x},\alpha;t)$ from the more familiar quantum-mechanical variables of the elementary matter systems can be found, for instance, in Ref. 1; here it suffices to recall that, to a system of pointlike nonrelativistic particles of mass m in interaction with a hard-core pair potential $v(r_{ij})$ (such as ^{4}He) described by the N-body short-range Hamiltonian

$$H_{\text{s.r.}} = -\frac{1}{2m}\sum_i \nabla_i^2 + \sum_{i<j} v(r_{ij}) , \tag{1.1}$$

one can associate a wave field $\Psi(\mathbf{x},t)$ obeying the canonical equal-time commutation relations

$$[\Psi(\mathbf{y},t),\Psi^\dagger(\mathbf{x},t)] = \delta^3(\mathbf{x}-\mathbf{y}) . \tag{1.2}$$

The Hamiltonian can then be given the simple expression

$$\begin{aligned} H_{\text{s.r.}} = & \int_V d^3\mathbf{x}\, \Psi^\dagger(\mathbf{x},t)\frac{-\nabla^2}{2m}\Psi(\mathbf{x},t) \\ & + \tfrac{1}{2}\int_V d^3\mathbf{x}\, d^3\mathbf{y}\, \Psi^\dagger(\mathbf{x},t)\Psi(\mathbf{x},t) \\ & \times v(\mathbf{x}-\mathbf{y})\Psi^\dagger(\mathbf{y},t)\Psi(\mathbf{y},t) , \end{aligned} \tag{1.3}$$

and from (1.2) one can obtain the Lagrangian of the quantum field $\Psi(\mathbf{x},t)$. Indeed, the equal-time commutator tells us that $i\Psi^\dagger(\mathbf{x},t)$ is just the canonical momentum operator, so that the Lagrangian of our system becomes

$$L(t) = \int_V d^3\mathbf{x}\, \Psi^\dagger(\mathbf{x},t) i\frac{\partial}{\partial t}\Psi(\mathbf{x},t) - H_{\text{s.r.}}(t) , \tag{1.4}$$

which, by Noether's theorem, implies the conservation of the number operator

$$\hat{N} = \int_V d^3\mathbf{x}\, \Psi^\dagger(\mathbf{x},t)\Psi(\mathbf{x},t) , \tag{1.5}$$

which, in our case, must equal the number of systems N present in our volume V. Note that it is here that the QFT formulation recovers the number of systems N that appears explicitly in the usual N-body formulation [Eq. (1.1)].

Coming to the ^{4}He problem, we could now try and solve the field theory given by (1.4) by applying the methods described in Ref. 1 that can be envisaged when N is very large, and make use of the saddle-point approximation to the appropriate path integral. However, as the solution can be described in terms of a "mean field," the "wave function" $\Psi_0(\mathbf{x};t)$, subject to a short-range pseudopotential $v(\mathbf{x})$ given by Eq. (1.3), we do not expect great novelty with respect to the excellent work carried out by Pines and his school.[2]

Instead, we wish to use such a QFT formulation to study the usually neglected problem of the long-range interaction between the ^{4}He atoms and the quantized electromagnetic radiation field.[3] But before we attack this problem in the sequel of this paper, it is perhaps appropriate that we pause to motivate why we need some new key element to describe the physics of superfluid ^{4}He that has occupied the minds of many physicists since its discovery by Kapitza in the 1930's.[4]

As it is well known, most of the very peculiar features of superfluid helium can be attributed to the existence of a macroscopically occupied quantum state of the N-body system. The wave function of such a "condensate" may be regarded as a "classical" coherent field, which is to be contrasted with the essentially "incoherent" motion of the particles in the noncondensed phase (the "rotons" of Landau). It was proposed by London[5] back in 1938 that this phenomenon of "condensation" in the superfluid phase of ^{4}He (also denoted He II) might be closely related to the well-known Bose condensation that can be proven to occur in a Bose gas of "pointlike" particles below a well-defined critical temperature. However, two important properties of superfluid helium—a finite critical velocity and a continuous phase transition—both show the superfluid phase transition to be very different from Bose condensation.

Later on, Bogoliubov[6] showed that, in a weakly interacting Bose gas, the interaction depletes the zero-momentum "condensate" and that only a fraction of the total number of particles is in such state even at zero temperature. However, one should notice that liquid helium is not a weakly interacting Bose gas, so that a perturbative treatment is not warranted. The notion of Bose condensation has been later generalized so as to apply to a system of interacting bosons;[7,8] one found that a statistical condensation occurs in systems without long-range configurational order, for the long-range correlation might destroy this kind of condensation. In such analysis the idea emerges that a statistical condensation may hold in spite of the long-range correlation, and such a correlation has the effect of depleting the condensed state, as it happens for the weakly interacting Bose gas.

Essentially all microscopic approaches start from the Hamiltonian (1.1), which completely neglects the internal structure of ^{4}He. The ground-state wave function and the density matrix are usually constructed so as to take into account both the short-range correlation arising from the strong repulsive interaction between particles, and the long-range correlation due to the density fluctuation. Variational techniques have been used to investigate the low-temperature behavior of ^{4}He and, in particular, the excitation spectrum, and they have greatly profited from the development of large-scale computational facilities.[9] This method has been rather successful, leading to a good understanding of the single-particle excitation spectrum and to a solid evidence for the existence of a condensed state at $T=0$.[10,11]

From a macroscopic standpoint, the properties of He II as a function of temperature are very well explained by the phenomenological two-fluid model proposed in the 1940's by Landau.[12] It is very interesting that such a two-fluid picture, normal and superfluid, is also borne out at a microscopic level by detailed studies of neutron scattering, as emphasized and thoroughly discussed by Woods and Svensson,[13] who also forcefully point out that the generally accepted theories of He II (of the type we have discussed above) do not seem to allow any room for such a behavior. The very complicated nature of the most popular approaches, together with their apparent failure to account for the very "*raison d'etre*" of superfluidity and its basic two-fluid nature, are, we believe, good motivations to develop a new way to look at this system based on QFT and its ability to take into account an aspect so far neglected of condensed-matter interactions, namely the many-body coupling of helium atoms through the quantized electromagnetic (em) radiation field. This is what we shall work out in detail in this paper, completing and extending our previous work.[3]

II. THE "WEAK" SUPERRADIANCE OF ^{4}He

Our starting point is the Hamiltonian of liquid ^{4}He which includes the interaction between the helium atoms and the radiative electromagnetic field. We work at $T=0$. Following Ref. 1, the Hamiltonian of the system is

$$H=H_{\text{s.r.}}+H_{\text{atom}}+H_{\text{int}}+H_{\text{em}}=H_{\text{mat}}+H_{\text{em}}\ , \tag{2.1}$$

where

$$H_{\text{int}}=\sum_k \left[\int_V d^3\mathbf{x}\ \mathbf{A}(\mathbf{x},t)\cdot\mathbf{j}_k\right] , \tag{2.2}$$

$$H_{\text{em}}=\tfrac{1}{2}\int_V d^3\mathbf{x}(|\mathbf{E}|^2+|\mathbf{B}|^2)\ . \tag{2.3}$$

$H_{\text{s.r.}}$ contains the kinetic energy and the short-range em interactions between helium atoms, which are not included in the field $\mathbf{A}(\mathbf{x},t)$. H_{atom}, describing the dynamics of the electrons of ^{4}He, accounts for the internal electromagnetic structure of the particles. H_{int} contains $\mathbf{j}_k$, the electromagnetic current operator of the kth single system, and it acts on the electron's coordinates. The last term represents the electromagnetic field Hamiltonian. The quantized radiation field $\mathbf{A}(\mathbf{x},t)$ is expanded in a plane-wave basis as

$$\mathbf{A}(\mathbf{x},t)=\sum_{\mathbf{k},\lambda}\frac{1}{\sqrt{2\omega_{\mathbf{k}}V}}[\epsilon_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}(t)e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})} +\epsilon^*_{\mathbf{k}\lambda}a^\dagger_{\mathbf{k}\lambda}(t)e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}]\ , \tag{2.4}$$

where the amplitudes $a_{\mathbf{k}\lambda}(t)$ obey the equal-time commutation relations

$$[a_{\mathbf{k}\lambda}(t),a^\dagger_{\mathbf{k}'\lambda'}(t)]=\delta_{(\mathbf{k}',\mathbf{k})}\delta_{(\lambda',\lambda)}\ . \tag{2.5}$$

Inserting this expression in the electromagnetic field Hamiltonian we obtain

$$\begin{aligned}H_{\text{em}}&=\tfrac{1}{2}\int_V d^3\mathbf{x}(|\mathbf{E}|^2+|\mathbf{B}|^2)\\&=\sum_{\mathbf{k},\lambda}\frac{1}{2\omega_{\mathbf{k}}}\{\dot a_{\mathbf{k}\lambda}(t)\dot a^\dagger_{\mathbf{k}\lambda}(t)\\&\qquad+i\omega_{\mathbf{k}}[a^\dagger_{\mathbf{k}\lambda}\dot a_{\mathbf{k}\lambda}(t)-\dot a^\dagger_{\mathbf{k}\lambda}(t)a_{\mathbf{k}\lambda}(t)]\\&\qquad+2\omega^2_{\mathbf{k}}a^\dagger_{\mathbf{k}\lambda}(t)a_{\mathbf{k}\lambda}(t)\}\ ,\end{aligned} \tag{2.6}$$

where the last term is the usual free field Hamiltonian and the others, in view of the potentially strong coupling with matter, cannot be neglected, as is usually done in the "slowly varying envelope approximation." The analysis of our problem will be performed in the QFT framework.[1] We introduce the quantum wave field of the atoms $\Psi(\mathbf{x},\alpha,t)$ where $\mathbf{x}$ is the center of-mass-coordinate, and α indicates the electrons' relative coordinates. For our purpose it is convenient to develop the wave field Ψ in terms of the one-particle eigenfunctions of $H_{\text{s.r.}}+H_{\text{atom}}$. The eigenfunctions of H_{atom} are very well known; the treatment of $H_{\text{s.r.}}$ is much more complicated. As well known, the difficulty arises from the hard-core interaction which cannot be dealt with in a mean-field scheme; it is for this reason that, in a "realistic" Hamiltonian, the variational ground states and the low-lying excited states are usually expressed in terms of the two-particle correlation functions.[10,11,14] As mentioned, a different theoretical approach has been developed[2] in which the consequences of the strong interactions in liquid ^{4}He are described in terms of self-consistent fields whose strengths are determined by physical arguments. The self-consistent field, or pseudopotential, describing the position correlation in ^{4}He is a soft-core two-body potential, whose Fourier transform is taken to be the Fourier transform of the bare-particle potential. In this framework we can apply a mean-field description to this interacting system and obtain a set of one-particle eigenfunctions $\phi_n(\mathbf{x},\alpha)$. Thus, one can write the wave field Ψ:

$$\Psi(\mathbf{x},\alpha,t)=\sum_n a_n(t)\phi_n(\mathbf{x},\alpha) . \tag{2.7}$$

As explained in the Introduction, in this paper we will not give an explicit form of this set of functions, for we are mainly interested in the effect of the radiative electromagnetic field on the ground state of the system and, as we will see, the $\mathbf{x}$-dependent part of the wave field is irrelevant.

We now write the Lagrangian of the system

$$L(t)=\int_V d^3\mathbf{x}\,\Psi^\dagger(\mathbf{x},\alpha,t)i\frac{\partial}{\partial t}\Psi(\mathbf{x},\alpha,t) -H_{\text{mat}}+\tfrac{1}{2}\int_V d^3\mathbf{x}(|\mathbf{E}|^2-|\mathbf{B}|^2) , \tag{2.8}$$

which still admits the conservation of the number operator

$$\hat{N}=\int_V d^3\mathbf{x}\,\Psi^\dagger(\mathbf{x},\alpha,t)\Psi(\mathbf{x},\alpha,t) . \tag{2.9}$$

Introducing the scaled fields

$$\Psi=\sqrt{N}\,\Psi_N,\quad a_{\mathbf{k}\lambda}=\sqrt{N}\,a_{\mathbf{k}\lambda N} , \tag{2.10}$$

the theory will be formulated in terms of these new fields. Following the theoretical development of Ref. 1, we can divide the fields in two components with different amplitude:

$$\Psi_N=\Psi_0+\delta\Psi , \tag{2.11}$$

$$a_{\mathbf{k}\lambda N}=a^0_{\mathbf{k}\lambda}+\delta a_{\mathbf{k}\lambda} , \tag{2.12}$$

where Ψ_0 and $a^0_{\mathbf{k}\lambda}$ are the vacuum-expectation values of Ψ_N and $a_{\mathbf{k}\lambda N}$, and $\delta\Psi$ and $\delta a_{\mathbf{k}\lambda}$ are the quantum fluctuations. The principle of stationary action applied to the Lagrangian (2.8) leads us to the classical field equations for the fields Ψ_0 and $a^0_{\mathbf{k}\lambda}$:

$$i\frac{\partial}{\partial t}\Psi_0(\mathbf{x},\alpha,t)=H_{\text{atom}}\Psi_0(\mathbf{x},\alpha,t)+e\sqrt{N/V}\sum_{\mathbf{k}\lambda}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}[\boldsymbol{\epsilon}_{\mathbf{k}\lambda}a^0_{\mathbf{k}\lambda}(t)e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}+\text{H.c.}]\cdot\mathbf{j}\Psi_0(\mathbf{x},\alpha,t) , \tag{2.13}$$

$$i\dot{a}^0_{\mathbf{k}\lambda}(t)-\frac{1}{2\omega_{\mathbf{k}}}\ddot{a}^0_{\mathbf{k}\lambda}(t)=e\sqrt{N/V}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}e^{i\omega_{\mathbf{k}}t}\boldsymbol{\epsilon}^*_{\mathbf{k}\lambda}\cdot\int_V d^3\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\Psi^*_0(\mathbf{x},\alpha,t)\mathbf{j}\Psi_0(\mathbf{x},\alpha,t) . \tag{2.14}$$

The coupling between the matter system and the radiative electromagnetic field establishes itself on the characteristic frequencies ω_n of the matter system. The electromagnetic modes $a^0_{\mathbf{k}\lambda}(t)$, whose frequencies $\omega_{\mathbf{k}}$ coincide with these frequencies ω_n, have a nontrivial "coherent" time evolution. In our case the frequencies are associated with the transition $1S$-nP of the atomic parastate. We note that the $1S$-$2P$ transition has a frequency of 21.2 eV while all the other transition frequencies do not differ from this value by more than 10%. Thus, it is reasonable that they are coupled to the same em mode with $\omega\simeq 20$ eV. Within a sphere of radius $R\simeq\pi/\omega=\lambda/2$ we can approximate

$$j_0(\omega r)=\int d\Omega_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\simeq\text{const} . \tag{2.15}$$

In this domain we can neglect the $\mathbf{x}$ dependence of the wave field while the atoms interact coherently with the appropriate mode of the electromagnetic field. In such a coherence domain the "classical field" Ψ_0 describes the part of the system which is involved in this collective behavior.

Limiting ourselves to a coherence domain, we can expand the wave field in terms of the atomic eigenfunctions. Substituting (the index n refers to the transition $1S$-nP)

$$a_0(t)=\beta_0(t),\quad a^{(n)}_m(t)=\beta^{(n)}_m(t)e^{-i\omega_n t} , \tag{2.16}$$

and

$$\alpha_m(t)=\frac{1}{4\pi}\sum_\lambda\int d\Omega_{\mathbf{k}}a^0_{\mathbf{k}\lambda}(t)\epsilon_{\mathbf{k}\lambda,m} , \tag{2.17}$$

in Eqs. (2.13) and (2.14) we get

$$\dot{\beta}_0(\tau)=\sum_{n,m}g_n\alpha^*_m(\tau)\beta^{(n)}_m(\tau) , \tag{2.18a}$$

$$\dot{\beta}^n_m(\tau)=-g_n\alpha_m(\tau)\beta_0(\tau) , \tag{2.18b}$$

$$\dot{\alpha}_m(\tau)+\frac{i}{2}\ddot{\alpha}_m(\tau)=\frac{1}{6\pi}\sum_n g_n\beta^*_0(\tau)\beta^{(n)}_m(\tau) , \tag{2.18c}$$

where the time derivatives are with respect to a dimensionless time $\tau=\omega t$ and the coupling constants are

$$g_n=2\pi\lambda_n e\sqrt{N/V}\frac{1}{\omega}\frac{1}{\sqrt{m_e}}=2\pi\lambda_n\left[\frac{\omega_p}{\omega}\right] , \tag{2.19}$$

where the plasma frequency ω_p

$$\omega_p=e\sqrt{N/V}\frac{1}{\sqrt{m_e}} \tag{2.20}$$

has been introduced. It is easy to see that the λ_n's, appearing in (2.19), obey the Thomas-Reiche-Kuhn sum rule

$$\sum_n \lambda_n^2 = 2 \tag{2.21}$$

that holds if $\{n\}$ is a complete set of quantum numbers. As usual, the system admits two constants of motion:

$$|\beta_0(\tau)|^2 + \sum_{n,m} |\beta_m^{(n)}(\tau)|^2 = 1 \ , \tag{2.22}$$

which represents the total number of systems described by the wave field, and

$$Q = \sum_{n,m} \left[|\alpha_m(\tau)|^2 + \frac{i}{2}[\alpha_m(\tau)^* \dot{\alpha}_m(\tau) - \dot{\alpha}_m(\tau)^* \alpha_m(\tau)] + \frac{1}{6\pi}|\beta_m^{(n)}(\tau)|^2 \right] \tag{2.23}$$

that tells us that the total momentum is conserved along the classical path.

Let us study the system so obtained in two different regimes: (1) short times, i.e., small field amplitudes, and (2) stable configurations, i.e., solutions for which the field amplitudes are constant in time.

(1) We differentiate Eq. (2.18c) and make use of Eqs. (2.18a) and (2.18b)

$$\ddot{\alpha}_m(\tau) + \frac{i}{2}\dddot{\alpha}_m = \frac{1}{6\pi}\sum_n g_n^2[\alpha_m(\tau)^* \beta_m^{(n)}(\tau)\beta_m^{(n)}(\tau)^* - \alpha_m(\tau)\beta_0(\tau)\beta_0(\tau)^*] \ . \tag{2.24}$$

Neglecting the term containing $\sum_{nm}|\beta_m^{(n)}(\tau)|^2$ which represents the fraction of population of the upper levels, we obtain a linearized equation

$$\ddot{\alpha}_m(\tau) + \frac{i}{2}\dddot{\alpha}_m(\tau) = -\frac{g^2}{6\pi}\alpha_m(\tau) \ . \tag{2.25}$$

With the substitution $\alpha_m(\tau) \propto e^{ipt}$ we get

$$p^3/2 - p^2 + \frac{g^2}{6\pi} = 0 \ . \tag{2.26}$$

For g^2 in the interval

$$0 < \frac{g^2}{6\pi} < \tfrac{16}{27} \ , \tag{2.27}$$

the solutions will be real, otherwise the system "runs away," i.e., evolves toward a completely different configuration. This is not the case in helium, for our coupling constant is

$$g^2 = \sum_n g_n^2 = (2\pi)^2 \left[\frac{\omega_p}{\omega_0}\right]^2 \sum_n \lambda_n^2 = 4.9 \ . \tag{2.28}$$

The system is thus in a "*weak-superradiant*" configuration where the interaction with the radiative em field, though amplified by the large number $\sqrt{N}$, does not radically modify the physics of liquid helium. The field amplitude will remain of the order $1/\sqrt{N}$, characteristic of the fluctuation above the quantum electrodynamics (QED) vacuum; the physical process which we are considering occurs as follows. The quantum fluctuations of some of the ^{4}He atoms induce a transition to the p state. These transitions are "real" and not "virtual" as in the standard analysis of the van der Waals forces. The electromagnetic interaction then couples the incoherent atomic fluctuations and, through the superradiant mechanism described above, renders them "coherent."

In order to analyze the contribution of the superradiant interaction to the ground state we now turn to the study of the stationary solutions of our system.

(2) Let us characterize these stable configurations as

$$\beta_0(\tau) = B_0 e^{i\psi_0(\tau)} \ , \tag{2.29a}$$

$$\beta_m^{(n)}(\tau) = B_m^{(n)} e^{i\psi_m(\tau)} \ , \tag{2.29b}$$

$$\alpha_m(\tau) = \frac{A}{\sqrt{3}} e^{i\phi_m(\tau)} \ . \tag{2.29c}$$

The constants of motion suggest the substitutions

$$B_0 = \cos\theta \simeq 1, \quad B_m^{(n)} = \frac{1}{\sqrt{3}}\theta_n \ , \tag{2.30}$$

by means of which the system (2.18) becomes

$$\dot{\psi}_0(\tau) = \sum_n g_n A \theta_n \ , \tag{2.31a}$$

$$\dot{\psi}_m(\tau) = g_n \frac{A}{\theta_n} \ , \tag{2.31b}$$

$$\dot{\phi}_m(\tau) - 1/2\dot{\phi}_m^2(\tau) = \sum_n \frac{g_n}{6\pi}\frac{\theta_n}{A} \ , \tag{2.31c}$$

with

$$Q = A^2[1 - \dot{\phi}_m(\tau)] + \frac{1}{6\pi}\theta^2 \ , \tag{2.32}$$

$$\psi_m - \psi_0 - \phi_m = \pi/2 \tag{2.33}$$

[we put $\theta_n = (g_n/g)\theta$]. We compute the constant Q by substituting in (2.32) the initial amplitudes

$$6\pi|A|^2(t=0) = \frac{4\pi \times 2}{2N} \ , \tag{2.34}$$

where 4π is the number of modes in the coherence domain and the factor of 2 stems from the photon polarization (from our normalization we assign $1/2N$ to each degree of freedom):

$$\sin^2\theta(t=0) \simeq \theta^2(t=0) = \frac{4\pi \times 3}{2N} \ , \tag{2.35}$$

where the factor of 3 comes from the degeneracy of the P state. Thus, we have the conditions

$$Q = \frac{5}{3N} \ , \tag{2.36}$$

$$\dot{\psi}_0(\tau) - \dot{\psi}_m(\tau) - \dot{\phi}_m(\tau) = 0 \ . \tag{2.37}$$

From the solution of (2.31), substituted in (2.36), we get, in the limit of small θ,

$$1 - \sqrt{1 - (g\theta/3\pi A)} = g\frac{A}{\theta} \ , \tag{2.38}$$

setting now $\xi = A/\theta$ the solution for (2.38) is

$$\xi \simeq 0.28 , \tag{2.39}$$

which leads to an em amplitude

$$|A|^2 \simeq \frac{1.6}{N} . \tag{2.40}$$

It is now possible to estimate the energy contribution of the superradiant interaction, i.e., the lowering of the ground-state energy due to "weak superradiance." We compute the expectation value of the interaction Hamiltonian on the stationary state. Setting

$$\frac{\Delta}{\omega} = \langle \Psi | H_{\text{int}} | \Psi \rangle , \tag{2.41}$$

where

$$\Psi = (\beta_0 \beta_1^{(n)} \beta_2^{(n)} \beta_3^{(n)}) , \tag{2.42}$$

we find

$$\Delta = -\omega 2 g A \theta . \tag{2.43}$$

In order to obtain ΔE_{tot}, the amount of energy gained per ^{4}He atom in the superradiant evolution, we must set

$$\Delta E_{\text{tot}} = E_{\text{mat}} + E_{\text{em}} + \Delta - E_{\text{z.p.}} , \tag{2.44}$$

where $E_{\text{mat}} = \omega \theta^2$ is the energy carried by the excited P state,

$$E_{\text{em}} = 6\pi\omega |A|^2 \left[1 - \frac{g\theta}{6\pi A} \right]$$

is the energy in the coherent em field, and

$$E_{\text{z.p.}} = \omega \theta_0^2 + \frac{4\pi}{N} \omega$$

is the zero-point energy.

The total number of particles interacting coherently in a coherence domain is

$$N \simeq \lambda^3 \rho = 4.4 \times 10^6 . \tag{2.45}$$

We evaluate

$$\Delta E_{\text{tot}} \simeq -1.1 \text{ K} . \tag{2.46}$$

This energy represents a "superradiant gap" for the excitation of quantum fluctuations upon the classical solution, i.e., the condensate. Put differently, the interaction with the radiative em field leads the system toward a stabler configuration whose extra binding energy is just ΔE_{tot}.

As for the em phase, it assumes the form

$$\dot{\phi}_m(\tau) = 1 - \sqrt{1 - (g\theta / 3\pi A)} , \tag{2.47}$$

$$\phi_m(\tau) = [1 - \sqrt{1 - (g / 3\pi\xi)}]\omega t . \tag{2.48}$$

The expression of the em phase inside a coherence domain is very interesting for it means that a frequency shift has occurred or, put differently, the photon has acquired a negative mass squared. As a result the em field generated by the weak superradiant process cannot propagate outside the coherence domain, thus rendering the coherence domain a "natural optical cavity."

We end this section by pointing out that we have found an explicit example that a superradiant behavior, contrary to expectations,[15] can occur spontaneously in condensed matter. Indeed, from (2.46) we find that the system is esothermic, so that one does not need a "pump," and from (2.47) the system develops spontaneously into an "optical cavity."

III. PHYSICAL CONSEQUENCES OF THE SUPERRADIANT INTERACTION ON THE SUPERFLUID HELIUM

All systems characterized by a phase transition into an ordered state, in which some kind of condensed phase is present, can be described by a two-fluid picture. Therefore, the two-fluid model is an essential aspect of superradiance and it emerges spontaneously if we consider the system at temperatures different from zero.[1] The superfluid component is the correlated phase while the normal component consists of thermal excitations. At $T=0$ the whole system is in the correlated states; this configuration is, as we have seen, energetically favorable. Increasing the temperature, the thermal fluctuations deplete the correlated phase and the normal component appears.

We introduce ρ_s, the density of the correlated system defined as $\rho - \rho_n$, where ρ is the helium density and ρ_n is the density of the normal fluid. At $T=0$, $\rho_s / \rho = 1$ and at higher temperature $\rho_s + \rho_n = \rho$. Thus, at finite temperature Eq. (2.22) becomes

$$|\beta_0(\tau)|^2 + \sum_{n,m} |\beta_m^{(n)}(\tau)|^2 = \frac{\rho_s}{\rho} . \tag{3.1}$$

On the other hand, the normalization condition on the full wave field is

$$1 = \frac{\rho_s}{\rho} + \frac{1}{N} \int_V d^3x \, \eta^*(\mathbf{x}) \eta(\mathbf{x}) , \tag{3.2}$$

where η is defined by the relationship

$$\delta\Psi = \frac{1}{N^{1/2}} \eta . \tag{3.3}$$

We can now look for the effects of the superradiant interaction on the collective excitation spectrum, i.e., the Landau dispersion curve, which describes the quanta of the field η. As mentioned, the Landau spectrum is determined mainly by the short-range Hamiltonian $H_{\text{s.r.}}$ and it has been studied in considerable detail during the last 50 years: in the long-wavelength portion of the excitation spectrum, the well-known Feynman relation holds:

$$\epsilon(\mathbf{k}) = \frac{\mathbf{k}^2}{2mS(\mathbf{k})} , \tag{3.4}$$

$S(\mathbf{k})$ being the liquid structure function. As is well known, this result fails when the system is probed at short distances. We shall now determine the effect of the superradiant state on the short-wavelength ($p > 1$ Å^{-1}) excitations of the wave field. In particular, we find that close to the "roton" minimum the spectrum is well approximated by a parabola

$$\epsilon=\frac{(p-p_0)^2}{2\mu}+\Delta(T)\ , \tag{3.5}$$

where

$$\Delta(T)=\delta\frac{\rho_n}{\rho}+\Delta(0)\ . \tag{3.6}$$

The meaning of this equation should be rather transparent, for it stipulates that the effect of the "superradiance binding" $\delta=\Delta E_{\rm tot}$ [see Eq. (2.46)] is to increase the "gap" Δ_n that characterizes the "roton" minimum in the normal fluid (above the critical temperature T_λ) by the quantity $-\delta(\rho_s/\rho)(T)$. Thus, at $T=0$, one has $\Delta(0)=\Delta_n-\delta$. In a simplified model we may describe the normal component of the fluid as formed essentially by roton excitations which are distributed according to the Boltzmann statistics. As a matter of fact, the temperature of the whole system is the temperature of the "gas" of elementary excitations, whereas the "coherent" phase has zero entropy and temperature. Since the superfluid helium is at $T\leq 2$ K and the total-energy gap Δ at the roton minimum is about 8 K, the "classical" Boltzmann distribution law is a very good approximation to the correct quantum Bose-Einstein function. The superradiant contribution to the roton energy has to also be taken into account in the determination of the T dependence of ρ_n:

$$\rho_n=\tfrac{2}{3}\sqrt{(\mu/kT)}p_0^4\frac{1}{(2\pi)^{3/2}}e^{-(\Delta(T)/kT)}\ . \tag{3.7}$$

Substituting the expression for $\Delta(T)$, we can calculate $\rho_n(T)$. A comparison with the experimental data is shown in Fig. 1, while Fig. 2 shows the T dependence of $\Delta(0)-\Delta(T)$. On imposing the condition $\rho_n/\rho=1$ one finds, without any extra input, the critical-temperature value $T_c\simeq 2.3$ K.

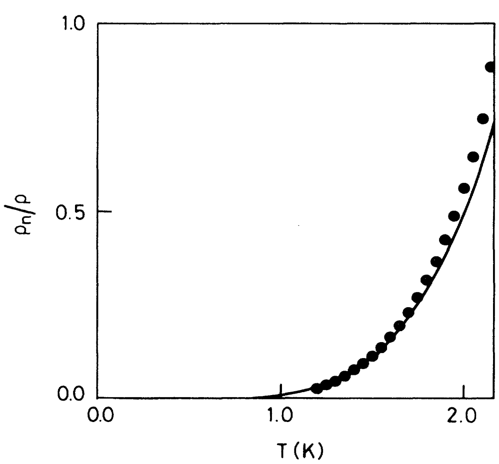

FIG. 1. Comparison of theory and experiment for the density of the normal-fluid fraction ρ_n/ρ as a function of temperature. The experimental points at 0.0 bar are due to Maynard (Ref. 16).

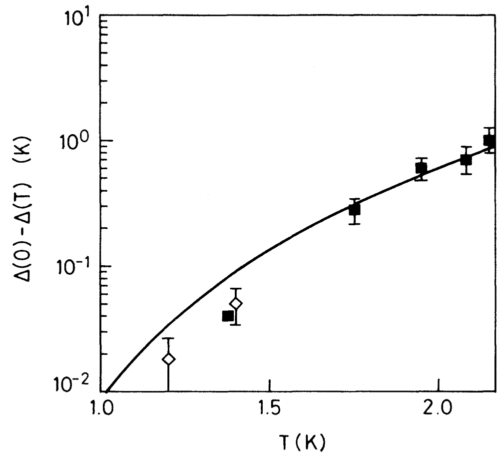

FIG. 2. The shift in the roton energy gap $\Delta(T)$ as a function of temperature at saturated vapor pressure (SVP). The experimental results are also shown and correspond to the work of Mezey (Ref. 17) and Wood and Svensson (Ref. 13).

The most striking consequence of the superradiant interaction is the emergence of the coherent macroscopic field Ψ_0 within a coherence domain. However, it appears very reasonable that, for $T=0$, Ψ_0 extends throughout the fluid. This is due to the fact that some tunneling mechanism, akin to the Josephson effect, should be able to lock the phases of adjacent coherence domains. We thus write the coherent field as

$$\Psi_0=\rho_c^{1/2}e^{i\phi(\mathbf{x},t)}\ , \tag{3.8}$$

where ρ_c is the density of the correlated fluid and it represents the order parameter of the superfluid transition.

Let us now estimate the fraction of zero-momentum particles in the condensed state. As explained above, as a result of some kind of Josephson tunneling the whole fluid is in a completely correlated state. However, due to the nontrivial space structure of the coherent em field, we must expect that Ψ_0 keeps some definite trace of the coherence domain's structure. Thus, a reliable estimate of the zero-momentum fraction of the condensate can be obtained by taking the average of the em field profile $|j_0(\omega r)|^2$ over a coherence domain, thus

$$\frac{\rho_c}{\rho}=\left[\frac{\omega}{\pi}\right]^3\frac{3}{4\pi}\int_V d^3\mathbf{x}|j_0(\omega r)|^2=\frac{3}{2\pi^2}\simeq 0.15\ , \tag{3.9}$$

in good agreement with the experimental data,[18] which give $\rho_c/\rho\simeq 0.13-0.14$.

We think that the appearance of large coherence domains (600 Å) is one of the most striking consequences of superradiant evolution and could provide an explanation of the nucleation process which is at the origin of vortex formation. It is well known that the hydrodynamic regime of liquid helium (both normal and superfluid) exhibits critical velocities, above which turbulence appears. We are interested here only in the critical velocities in the purely superfluid phase. In our approach the minimum vortex is produced when just one coherence domain is unlocked from the array of interconnected domains; this occurs when all the atoms belonging to such a domain are excited by perturbations that have wavelengths λ of the size of the coherence domain and that impart the atoms the related momentum transfer $p=(2\pi/\lambda)$. The Landau criterion

$$v_c=\frac{p}{2M} \tag{3.10}$$

then gives $v_c^{\max}\simeq 75$ cm/sec, which is in the range of the observed critical velocities (~ 50 cm/sec). We emphasize that, in our approach, Eq. (3.10) is a direct measurement of the size of coherence domains. The existence of such large domains allows us to understand the interesting phenomenon occurring at a small orifice (weak link) connecting two adjacent baths of superfluid helium. In that case, the difference $\phi_1-\phi_2$ of the phase across the orifice is governed by the difference $\Delta\mu=\mu_1-\mu_2$ of the chemical potential according to the Josephson relation

$$\frac{\partial}{\partial t}(\phi_1-\phi_2)=\mu_1-\mu_2=\Delta\mu \ . \tag{3.11}$$

The phase difference $\phi_1-\phi_2$ changes with time (phase slippage) when a nonvanishing $\Delta\mu$ is maintained across the orifice.[19] When the flow of superfluid helium through the hole reaches a critical velocity and vortices are being produced at a rate ν, the energy uptaken by vortices compensates the difference of chemical potential and the phase slippage is stopped. We have

$$\Delta\mu=nh\nu \ , \tag{3.12}$$

where n is an integer number. According to Eq. (3.12) the plot of $\Delta\mu$ exhibits a steplike behavior. The release of a quantum of energy corresponding to one step is expected when the velocity field is switched on by the onset of turbulence. Recently Avenel and Varoquaux[20] have reported that, by letting an oscillating flow of superfluid helium at $T=0$ K through a 0.3×5 μm^2 slit holed in a 0.2-μm thick wall, a discrete amount of energy $\Delta E\simeq 1.2\times 10^{-17}$ J is suddenly released when the flow velocity reaches a critical value $v_c\simeq 55$ cm/sec. Within our approach this result is understood as follows: the interaction between flowing helium and the wall unlocks the coherence domains sliding on the orifice walls. So we have

$$\Delta E=\frac{p^2}{2M}n_0 N \ , \tag{3.13}$$

where p is, as usual, ~ 20 eV, n_0 is the number of atoms enclosed in a domain, and $N\simeq 550$ is the number of coherence domains covering the orifice side walls. We thus get

$$\Delta E=1.9\times 10^{-17}\ \text{J} \tag{3.14}$$

in fair agreement with observations. It is interesting to observe that

$$\frac{\Delta^{\text{th}}}{\Delta^{\text{obs}}}=\frac{v_c^{\text{th}}}{v_c^{\text{obs}}}\simeq\frac{7}{5} \tag{3.15}$$

so that both discrepancies might be traced to the approximation used in the evaluation of the size of the coherence domain.

We think that the understanding of the Avenel-Varoquaux effect has been obscured just by the fact that conventional theories cannot easily produce a nucleation dynamics able to assemble vortices as large as several hundred Å. Superradiance can overcome this difficulty and furthermore can give some insight upon Josephson-like phenomena.

IV. CONCLUSIONS AND OUTLOOK

The aim of this paper, that completes and expand a previous work,[3] has been, on one hand, to provide some details about the application of the basic ideas of QFT of superradiance to ^{4}He, and on the other to utilize this simple and fascinating physical system to learn more about the potentialities of this new approach to condensed matter. Unlike in water and in simple plasmas,[1] superradiance in ^{4}He appears to manifests itself in a "weak," subtle way. The effective coupling of the atomic *S-P* transitions to the "resonating" coherent em mode turns out to be too weak [see Eqs. (2.27) and (2.28)] to induce a massive rearrangement of the ^{4}He atoms, but it is strong enough to produce a macroscopically ordered state of matter, protected against thermal fluctuations by a "gap" of about 1 K which, at the temperatures of superfluid ^{4}He, is a perfectly respectable gap.

Indeed, we believe that the most likely origin of the superfluid state of ^{4}He is to be looked for just in the superradiant evolution of the ^{4}He atoms that we have quantitatively determined in this work. This statement is quite strongly supported by the following results.

(i) The quantum fluctuations' spectrum (Landau spectrum) at $T=0$ (i.e., in the superradiant state) has been found to be modified very simply by a gap whose numerical value is in agreement with observations.

(ii) A microscopic two-fluid (superradiant and normal) picture emerges naturally in the superradiant state at $T\neq 0$ and explains the behavior with the temperature of both the roton gap $\Delta(T)$ and the normal fluid fraction ρ_n/ρ, and thus of the critical temperature $T_c\simeq 2.3$ K (experiment 2.17 K).

(iii) The notion, crucial in the superradiance framework, of coherence domain (whose size in ^{4}He is determined by the atomic transition dynamics to be ~ 600 Å) allows us to answer two different important questions: (a) the zero-momentum fraction of the superfluid, and (b) the formation of quantized vortices. The first is easily, and successfully, related to the space variation of the em field, and thus of the wave function Ψ_0 across adjacent "interlocked" coherence domains, while the second provides a natural nucleation mechanism for vortices of the right size, and with the right critical velocities.

We believe that, in spite of their rather preliminary character, these results look promising enough to recommend further work along this direction. For instance, one possible area of investigation might be the analysis of the process of excitation of fluctuations (by neutron and Raman scattering) below and above the critical temperature T_{λ}, in search for a quantitative test of the microscopic two-fluid picture. Another interesting problem to be investigated is a careful analysis of the space dependence of the superradiant equations (2.13) and (2.14), to clarify, on one hand, the role of the coherence domain structure in the macroscopic wave function Ψ, and on the other to derive the basic hydrodynamics of the superfluid.

[1]G. Preparata, in *Problems of Fundamental Modern Physics,* edited by R. Cherubini, P. Dal Piaz, and B. Minetti (World Scientific, Singapore, 1990).

[2]K. Bedell, D. Pines, and I. Fomin, J. Low. Temp. Phys. **48**, 417 (1982); C. H. Aldrich III and D. Pines, *ibid.* **25**, 677 (1976); D. Pines (unpublished).

[3]E. Del Giudice, M. Giuffrida, R. Mele, and G. Preparata, Europhys. Lett. (to be published).

[4]P. L. Kapitza, Nature **141**, 74 (1938).

[5]H. London, Proc. R. Soc. London **171**, 484 (1939).

[6]N. N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947).

[7]O. Penrose and L. Onsager, Phys. Rev. **104**, 412 (1956).

[8]S. T. Beliaev, Zh. Eksp. Teor. Fiz. **34**, 417 (1958) [Sov. Phys.—JETP **7**, 289 (1958)].

[9]R. P. Feynman, Phys. Rev. **94**, 262 (1954).

[10]R. Jastrow, Phys. Rev. **98**, 1479 (1955).

[11]L. Reatto and G. V. Chester, Phys. Rev. **155**, 88 (1967).

[12]L. D. Landau, J. Phys. (Moscow) **5**, 71 (1941).

[13]A. D. B. Wood and E. C. Svensson, Phys. Rev. Lett. **41**, 974 (1978).

[14]E. Feenberg, *Theory of Quantum Fluids* (Academic, New York, 1969).

[15]P. W. Anderson, *Basic Notions of Condensed Matter Physics* (Benjamin Cummings, Menlo Park, 1984).

[16]J. Maynard, Phys. Rev. B **14**, 3868 (1976).

[17]F. Mezei, Phys. Rev. Lett. **44**, 1601 (1980).

[18]V. F. Sears, Phys. Rev. B **28**, 5109 (1983).

[19]P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966).

[20]O. Avenel and E. Varoquaux, Phys. Rev. Lett. **55**, 2704 (1985).