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Charge and vortex dynamics in arrays of tunnel junctions

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Arrays of tunnel junctions can be fabricated with small nearest-neighbor capacitances between the islands and with even smaller self-capacitances of the islands. It has been suggested by Mooij *et al.* that electric charges on the islands of such an array interact logarithmically over large distances, and as a consequence, a Kosterlitz-Thouless-Berezinskii transition can occur where charge-anticharge pairs dissociate. Furthermore, in superconducting junction arrays the more familiar vortex-unbinding transition can occur. We investigate the effect of single-electron and Cooper-pair tunneling on these transitions and discuss the competition between the charge and the vortex unbinding.

I. INTRODUCTION

In recent years much attention has been devoted to the study of tunnel-junction arrays.[1] Even if at low temperatures the individual islands of the array are superconducting, it may require still lower temperatures for *global* phase coherence to establish across the whole array. In classical, two-dimensional (2D) arrays this transition is of the Kosterlitz-Thouless-Berezinskii (KTB) type.[2] It separates a superconducting low-temperature phase, where vortices and antivortices are bound in pairs, from a resistive high-temperature phase with free vortices. The transition temperature T_v is of the order of the Josephson coupling energy E_J of the junctions.

When the dimensions of the islands and the capacitances involved are small, the charging energy with characteristic scale E_C is large, and quantum fluctuations of the phase gain importance. They suppress the vortex-unbinding transition temperature;[3] and for small values of $E_J/E_C \lesssim 1$ even at $T=0$ only a disordered phase exists. The tunneling of quasiparticles or Ohmic shunts introduces dissipation. The most striking consequence,[4] observed in experiments involving granular films[5] and in regular 2D networks,[6] is the existence of a critical value of the normal state conductance $\alpha_t \equiv h/(4e^2R_n) \approx 1$ above which the superconductivity is recovered even for small values of E_J/E_C.

Recently Mooij *et al.*[7] pointed out a different collective effect involving the charge on the islands of a 2D array. If the junctions are of high quality the charges of the islands can change only due to tunneling, and the total charge on each island is an integer multiple of the elementary charge e. Furthermore, it is possible to fabricate junction arrays where the capacitance C between the islands, i.e., the one associated with the junctions, is significantly larger than the capacitance C_0 of the islands to the ground. In this case the electrostatic interaction of two charges $\pm e$ separated by r is characterized by the potential energy $U(r)=(2E_C/\pi)\ln r$, up to distances of the order $\Lambda=\sqrt{C/C_0}$. The unit of the length is the lattice spacing, the charging-energy scale is defined as $E_C \equiv e^2/2C$. In the ideal situation where $C_0=0$ the logarithmic interaction extends to infinity. The system is thus a physical realization of a two-dimensional Coulomb gas. This implies that a KTB transition[2] can occur at a temperature T_c of the order E_C where charge-anticharge pairs dissociate. This charge-unbinding KTB transition differs from the mentioned vortex-unbinding KTB transition: If the charges are bound in pairs, the array is insulating; if the charges are free, the array has a finite conductance. In contrast, the vortex-unbinding transition separates a superconducting from a resistive phase.

In superconducting arrays the charge-unbinding transition can involve either single electrons (charge e) or Cooper pairs (charge $2e$). Since the charging energy differs by a factor of 4 in these two cases, we expect a corresponding difference in the transition temperature.[7] Furthermore, in superconducting arrays both the charge- and the vortex-unbinding transition can occur, depending on the ratio between the energies E_J and E_C. The experiments reported in Ref. 7 show a transition of normal junction arrays from insulating to conducting, consistent with the picture of the charge-KTB transition. Furthermore, in superconducting arrays a similar transition, but at a higher temperature, is observed. Also, arrays with larger E_J/E_C show the vortex-unbinding transition between a superconducting and a resistive state. It is clear that in a real experiment the charge-unbinding transition is washed out due to the finite range of the logarithmic interaction (for $C_0 \neq 0$),[8] but if C_0/C is small enough a fairly sharp crossover remains observable. The finite size of the array has a similar effect.

The junction array is equivalent to a classical 2D Coulomb gas as long as the single-electron and Cooper-pair tunneling is weak. The tunneling allows the relaxation to thermal equilibrium and determines the response of the system. We will discuss the weak-tunneling limit in Sec. II. If the tunneling is strong, the quantum fluctuations associated with it further affect the charge dynamics and the charge-unbinding transition. In Secs. II and IV we will discuss this influence for normal and superconducting arrays, extending our previous work.[7,9]

The superconducting case is even more interesting due to the properties of the vortices. In Sec. V we study the dynamics of vortices and their interaction with the charges. We derive a "coupled-Coulomb-gas" description for the two types "charges," i.e., the electric charges and the vortices. In the appropriate range of parameters either one can behave as a particle, characterized by a mass. There exists a near duality between charges and vortices.[9,10] We will describe the competition between the charge- and vortex-unbinding transitions in Sec. VI and establish the phase diagram for all values of E_J/E_C. Finally, the complete phase diagram in the superconducting case, which depends on E_J/E_C and the normal-state tunnel conductance, is presented in Sec. VII.

II. COULOMB ENERGY AND WEAK TUNNELING IN NORMAL ARRAYS

Consider a square 2D array, as shown in Fig. 1, of small tunnel junctions with capacitance C connecting the islands $j=(x_j,y_j)$ with their nearest neighbors. (All lengths are discrete and measured in units of the lattice spacing.) In addition, each island has a capacitance C_0 to the ground; other capacitances are ignored. Let us assume that on the island j there is the total charge Q_j, which is an integer multiple of the single-electron charge e. Depending on the ratio between the two capacitances, this charge polarizes the islands such that charge density accumulates on the junction capacitances. Notice that the integrated density on each junction electrode can take any continuous value. The charge distribution for the case of two charged islands in the limit where C dominates $C \gg C_0$ is indicated schematically in Fig. 1. The resulting electrical potential of the islands is denoted as $V_j=V(x_j,y_j)$. It is fixed by the charges by the discrete, two-dimensional form of Poisson's equation

$$C_0V(x,y)+C[4V(x,y)-V(x-1,y)-V(x+1,y) -V(x,y-1)-V(x,y+1)]=Q(x,y) .$$

For large distances $r=(x^2+y^2)^{1/2}\gg 1$ we can use the continuous version $C_0V-C\nabla^2V=Q(x,y)$. The potential

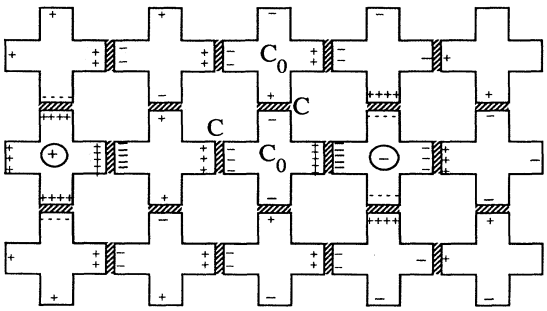

FIG. 1. A two-dimensional array of tunnel junction is shown. The capacitance is a nearest-neighbor capacitance located at the junction interfaces C and a capacitance to the ground C_0. Also shown is the polarization produced by discrete charges on two islands for $C_0 \ll C$.

due to one test charge at the origin then is

$$V(r)=(e/2\pi C)K_0(r/\Lambda), \quad \Lambda\equiv(C/C_0)^{1/2} . \tag{1}$$

The ratio of the capacitances determines the screening length Λ. For $r/\Lambda \gg 1$ the modified Bessel function $K_0(r/\Lambda)$ falls off exponentially. For shorter distances $r/\Lambda \ll 1$ it varies logarithmically, and $V(r) = -(e/2\pi C)\ln(r/\Lambda)$.

In the following we will study the charging energy of the array. Expressed in terms of the voltages of the islands it is

$$H_{\rm ch}=\sum_i \frac{C_0}{2}V_i^2+\sum_{\langle i,j\rangle}\frac{C}{2}(V_i-V_j)^2=\sum_{i,j}\tfrac{1}{2}V_iC_{ij}V_j , \tag{2a}$$

where the capacitance matrix contains a diagonal part and one connecting nearest neighbors $C_{ij} =(C_0+4C)\delta_{ij}-C\delta_{i,j\pm1}$. It is merely a matrix inversion problem to rewrite this energy in terms of the charges Q_i on the islands

$$H_{\rm ch}(\{Q_i\})=\sum_{i,j}\tfrac{1}{2}Q_iC_{ij}^{-1}q_j . \tag{2b}$$

The inverse capacitance matrix C_{ij}^{-1} is long range, with the distance dependence as that of the potential $V(r)$ given in (1). In the limit $C_0 \ll C$ it varies logarithmically with the distance.

The long-range interaction of the charges in the limit $C_0 \ll C$ is the same as the Coulomb interaction in a two-dimensional world. Furthermore, the charges are discrete, e.g., in a normal junction array $Q_i=0,\pm e,\pm 2e,\ldots$. This means that the junction array is a physical realization of the so-called 2D Coulomb gas and should show the Kosterlitz-Thouless-Berezinskii phase transition. There are several detailed predictions of the KTB theory: The transition temperature is

$$k_BT_c=k_BT_{cn}^{(0)}\equiv\frac{E_C}{4\pi\epsilon_c}, \quad E_C\equiv\frac{e^2}{2C} . \tag{3}$$

(The dielectric constant ϵ_c depends on details on short length scales. For the lattice considered here it is only slightly larger than 1, and we will ignore it in most of the following.) Above T_c, free charges are present; their density near T_c is given by the square-root cusp expression

$$n_e=K\exp[-2b(T/T_c-1)^{-1/2}] , \tag{4}$$

where K and b are constants of order 1. The conductance in this regime scales with this dependence. Below T_c the charges are bound in dipoles and the system is insulating. By applying a voltage across the array the dipoles can be broken, which leads to a nonlinear conductance

$$I\propto V^{a(T)} . \tag{5}$$

The exponent $a(T)$ depends on the temperature. It is proportional to $1/T$ for low temperatures and shows a universal jump at T_c from the value $a(T=T_c)=3$ to the value $a(t>T_c)=1$.

Several of the predictions of the response of the Coulomb gas are based on model assumptions about the

dynamics of the system.[11] In the present case the dynamics of the single-electron tunneling is known from the microscopic theory. It has been studied extensively in the recent years (see, e.g., Refs. 12 and 13), and many of the predictions derived from it, such as the Coulomb blockade, the Coulomb gap in the current-voltage characteristics, and the single-electron-tunneling (SET) oscillations of the voltage have been confirmed by experiments. The rate of tunneling of an electron (one out of many electrons on the mesoscopic island) from an island to a neighboring one, changing the charge configuration from $\{Q_i\}$ to $\{Q_i'\}$, depends on the energy difference ΔE and hence on the configuration of charges before and after the tunneling process

$$\Gamma^{\pm}(\{Q_i\}\rightarrow\{Q_i'\},T)=\frac{\Delta E}{e^2R_t}\left[\exp\left[\frac{\Delta E}{k_BT}\right]-1\right]^{-1}. \quad (6)$$

The question whether the relevant energy change is that of the junction only where the tunneling occurs (local rule) or whether it refers to a larger system, possibly the whole array (global rule), has been discussed.[14] Only the part of the array within a "relativistic horizon" $\Delta r=c\Delta\tau$, where $\Delta\tau$ is one of the "tunneling times," can influence the tunneling.[15] The work of Nazarov[16] demonstrated that the longest time, the inverse of the energy difference $\Delta\tau=\hbar/\Delta E$, is the relevant time scale. The energy change is of order E_C or smaller. Hence typically the whole array is probed, and it is the change in the equilibrium energy $\Delta E=H_{\rm ch}(\{Q_i\})-H_{\rm ch}(\{Q_i'\})$ given by Eq. (2), that enters into the tunneling rate (6).

It is straightforward to simulate the stochastic charge dynamics of a junction array based on the rate (6).[14] The dc I-V characteristic of an array with N_s junctions in series shows a "Coulomb gap" $V_g\sim N_se/C$, below which the current vanishes (Coulomb blockade). Beyond, the current sets in reaching a differential conductance given by the single junction value $1/R_t$. At finite temperature the Coulomb blockade is not perfect, a current flows already at small voltages, although for $T\ll E_C$ it is still much smaller than the classical value. These features have been known for some time and have been confirmed in experiments. We recently extended our simulations to large systems[17] and find that the low-voltage part of the I-V characteristic shows a nonlinear conductance as given by Eq. (5), the exponent $a(T)$ shows the temperature dependence predicted for a KTB transition (with finite-size smearing), the transition temperature is given by Eq. (3), and also the conductance above T_c fits to the prediction (4). At larger voltages the I-V characteristic shows the Coulomb gap. The analysis shows that the single-electron tunneling governed by the rate (6), which had been derived from the microscopic theory, yield both features, the KTB properties and the Coulomb gap. Both are fully consistent with one another. Further results of the numerical work including extensions to disordered systems will be published in a separate paper.

We would like to conclude this section with a comment: In equilibrium the state of the junction array is characterized by the number of (total) charges Q_i on the islands. Each charge produces a (in general long-range) voltage profile. Due to the linearity of Poisson's equation the voltages add. In good quality junctions the charges change only by tunneling; hence they are conserved except that they can be created or annihilated in pairs. This picture resembles very much the concept of "solitons." Ben-Jacob, Mullen, and Amman[18] and Averin and Likharev[12] drew this picture of their analysis of linear chains. It is clear, however, that in the junction arrays the properties of the solitons are merely a consequence of the discreteness of the charges on the islands.

III. ARRAYS OF NORMAL JUNCTIONS WITH ARBITRARY STRENGTH OF THE TUNNELING

The KTB transition described above is based on electrostatic energy considerations. We assumed that the tunneling of single electrons is weak. The tunneling is needed to establish an equilibrium distribution of the charges Q_i; moreover, it determines the response of the system. On the other hand, if the tunneling is strong, it may itself influence the KTB transition. In order to study this question we have to go beyond the simple stochastic tunneling picture based on the rate (6). Indeed, we will find modifications if the dimensionless tunneling conductance

$$\alpha_t\equiv h/(4e^2R_t)\approx 6.45k\Omega/R_t\ , \quad (7)$$

is of order 1 or larger.

In order to analyze the problem for general strength of the tunneling we start from a microscopic Hamiltonian which contains the charging energies but also accounts for the tunneling of electrons across the tunnel barriers. We have shown[19-20]—for a single junction, but the extensions to arrays is straightforward—that the partition function of an array of normal tunnel junctions can be expressed as a path integral over a macroscopic field ϕ_i, defined as the integral of the voltage

$$\phi_i=\int^t dt'2eV_i(t') \quad (8)$$

on the island i. (The factor 2 is introduced for a unified description of the normal and superconducting case.) The partition function reads (for $\hbar=1$)

$$Z=\prod_i\int D\phi_i\exp(-S_n[\phi])\ . \quad (9)$$

It involves the action $S_n[\phi]=S_{\rm ch}[\phi]+S_t[\phi]$,

$$S_{\rm ch}[\phi]=\frac{1}{16E_0}\sum_i\int_0^\beta d\tau\left[\frac{d\phi_i}{d\tau}\right]^2+\frac{1}{16E_C}\sum_{\langle i,j\rangle}\int_0^\beta d\tau\left[\frac{d\phi_{ij}}{d\tau}\right]^2, \quad (10)$$

$$S_t[\phi]=\sum_{\langle i,j\rangle}\int_0^\beta d\tau\int_0^\beta d\tau'\alpha(\tau-\tau')\left[1-\cos\left[\frac{\phi_{ij}(\tau)-\phi_{ij}(\tau')}{2}\right]\right]. \quad (11)$$

The two terms in S_{ch} represent the charging energy due to the self-capacitance $E_0=e^2/2C_0$ and the nearest-neighbor capacitance $E_C=e^2/2C$, respectively. The action S_t describes the dissipation due to the single-electron tunneling. The subscripts i label the islands on the lattice, and $\phi_{ij}=\phi_i-\phi_j$ refers to nearest neighbors. The dissipative kernel for normal junctions is

$$\alpha(\tau)=\alpha_t\frac{(1/\beta)^2}{\sin^2(\pi\tau/\beta)}\ , \tag{12}$$

where the dimensionless tunneling conductance has been defined in (7).

We have not yet specified the limits of the path integrals over the fields ϕ_i in (9). They depend on the allowed charge states of the system.[13,20,21] We consider junction arrays without Ohmic shunts. This allows us to assume that the total charges Q_i in the islands are integer multiples of e. (Actually, in a real experiment, trapped charged impurities can polarize the islands. This makes Q_i effectively noninteger, but the changes of Q_i are integer. The disorder washes out the transition;[17] here we ignore the effect of disorder.) The discreteness of the charge implies that the conjugate phases ϕ_i have to be defined on a unit circle. Values that differ by 4π are equivalent. (The 4π, rather than 2π, is again a consequence of the factor 2 in the definition of ϕ_i.) As a consequence, the integral in Eq. (9) includes a summation over winding numbers $\phi_i(\beta)=\phi_i(0)+4\pi n_i$.

For the present problem it is convenient to formulate the problem in terms of the winding numbers. For this purpose we decompose the phases as

$$\phi_i(\tau)=\phi_{i0}+\frac{4\pi n_i}{\beta}\tau+\theta_i(\tau),\quad \theta_i(0)=\theta_i(\beta)=0\ . \tag{13}$$

Accordingly, the charging part of the action is decomposed into two terms, one $S_{ch}[\theta]$ depending on $\phi_i(\tau)$, the other depending on the winding numbers

$$S_{ch}[n]=\frac{\pi^2}{\beta E_0}\sum_i n_i^2+\frac{\pi^2}{\beta E_C}\sum_{\langle i,j\rangle}(n_i-n_j)^2\ . \tag{14}$$

This action (14) coincides with the "discrete Gaussian model" (DGM), which has been studied in connection with the roughening transition of solid-solid interfaces.[22] For $1/E_0=0$ it is known to be analog to the Coulomb gas problem and to have a KTB transition at a critical temperature, which, of course, coincides with the result (3).

Next we study the effect of the dissipative tunneling. As long as it is weak, i.e., α_t is small, we can treat it perturbatively. To first order in α_t the system is described by

$$Z\approx Z_{ch,\theta}\sum_{\{n\}}\exp(-S_1[n])\ .$$

The winding-number part of the effective action is

$$S_1[n]=S_{ch}[n]-\sum_{\langle i,j\rangle}\int_0^\beta d\tau\int_0^\beta d\tau'\alpha(\tau-\tau')g_{ij}(\tau-\tau')\times\cos[2\pi n_{ij}(\tau-\tau')/\beta]\ , \tag{15}$$

with $n_{ij}=n_i-n_j$, and the correlation function is

$$g_{ij}(\tau)=\exp\{-\tfrac{1}{8}\langle[\theta_{ij}(\tau)-\theta_{ij}(0)]^2\rangle_{ch,\theta}\}=\exp[-2E_C|\tau|(1-|\tau|/\beta)]\ . \tag{16}$$

The average in $\langle\cdots\rangle_{ch,\theta}$ and the partition function $Z_{ch,\theta}$ are taken with the charging action $S_{ch}[\theta]$.

The correlation function $g_{ij}(\tau)$ decays exponentially in time, which allows us to expand the second term of Eq. (15) in n_{ij}. As a result we recover the model (14), but with a renormalized nearest-neighbor capacitance C. This implies that in the limit of vanishing C_0 the transition is again of the KTB type, but with a renormalized transition temperature[7]

$$T_{cn}/T_{cn}^{(0)}=1-0.11\alpha_t,\quad \text{for } \alpha_t\ll 1\ . \tag{17}$$

For weak tunneling the picture of charges on the island is still meaningful, but the tunneling leads to a "dynamic screening." This screening effectively decreases the strength of the Coulomb interaction and lowers the KTB transition temperature.

For stronger dissipation the fluctuations of θ_{ij} are suppressed, and we can expand in these fields. In this limit our parametrization (13) can be viewed as a description in terms of instantons plus fluctuations. It is related to the description presented by Korshunov,[23] simplified for the normal junction and generalized to an array. In second order the dissipative term of the action reduces to

$$S_t[\theta,n]\approx 2\alpha_t\sum_{\langle i,j\rangle}|n_{ij}|+\tfrac{1}{8}\sum_{\langle i,j\rangle}\int_0^\beta d\tau\int_0^\beta d\tau'\alpha(\tau-\tau')[\theta_{ij}(\tau)-\theta_{ij}(\tau')]^2\cos[2\pi n_{ij}(\tau-\tau')/\beta]\ . \tag{18}$$

The combination $\alpha_t|n_{ij}|$ arises from Fourier-transforming $\alpha(\tau)$. For very strong dissipation it is even sufficient to consider only the first term in (18), and we arrive at

$$S[n]=\frac{\pi^2}{\beta E_0}\sum_i n_i^2+\frac{\pi^2}{\beta E_C}\sum_{\langle i,j\rangle}(n_i-n_j)^2+2\alpha_t\sum_{\langle i,j\rangle}|n_i-n_j|\ . \tag{19}$$

The first two parts have been already recognized as the DGM, the third term is known as the "absolute solid on solid" (ASOS) model, which has also been studied in connection with the roughening transition of at interfaces. As the temperature approaches zero only this third term survives. Monte Carlo simulations[24] show that the ASOS model also undergoes a transition of the KTB type at the critical value of coupling constant

$$\alpha_{t,c}\approx 0.45\ \text{ at } T=0\ . \tag{20}$$

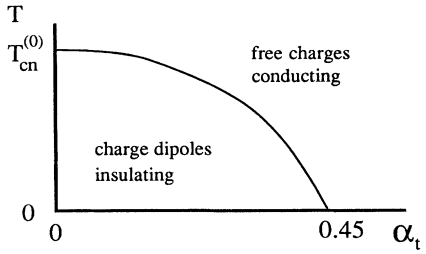

FIG. 2. Phase diagram for the charge-unbinding transition of normal junction array is plotted as a function of the tunneling conductance $\alpha_t = h/4e^2R_t$.

A comment is appropriate concerning the validity of the result (20). The effective action (19) was obtained under the assumption of strong dissipation. This implies, strictly speaking, that we can use it only to discuss the plasma phase of the system. However, all the considered step models belong to the same universality class, hence the transition, if it exists, is of the KTB type. From the fact that at $T=0$ in the two limits, weak or strong dissipation, we are in the ordered or disordered phase demonstrates the existence of a KTB transition at a critical value of α_t. We therefore expect that the expansion (19), even when applied to small values of α_t, only leads to quantitative errors, e.g., the numerical value of $\alpha_{t,c}$ may change, but the existence of the phase transition and its properties are not affected.

The effect of the temperature is to decrease the critical value of the dissipation. We can evaluate the phase boundary by mapping (19) into a generalized XY model. The details of this calculation are presented in Appendix A. The models are dual in the sense that the weak-coupling region is mapped into the strong coupling region of the XY action. We learn from it that at low temperatures the shift of the critical value of dissipation is

$$\alpha_{t,c} \approx 0.45(1-\pi T/4T_{cn}^{(0)}) . \tag{21}$$

The phase diagram for an array of normal junctions is shown in Fig. 2. In the region below the transition temperature the charges are bound in dipoles so that the array is insulating. Above, the free charges form a plasma phase and give a finite conductance.

IV. THE CHARGE-UNBINDING TRANSITION IN JOSEPHSON JUNCTION ARRAYS

In the case of superconducting arrays the action contains the Josephson term proportional to the coupling energy E_J,

$$S_s[\phi] = S_{\rm ch}[\phi] + S_t[\phi] - E_J \sum_{\langle i,j \rangle} \int_0^\beta d\tau \cos\phi_{ij}(\tau) . \tag{22}$$

The charging part and the dissipative part $S_{\rm ch}+S_t$ are the same as in Eqs. (10) and (11); however, the kernel $\alpha(\tau)$ is modified by the superconducting gap.[19] We will specify this further below. For the moment, we disregard the quasiparticle tunneling ($\alpha_t=0$). In this case only Cooper pairs with charge $2e$ can tunnel. Correspondingly, the boundary condition for the phases in the partition function (9) are $\phi_i(\beta)=\phi_i(0)+2\pi n_i$.

Since the unit of the charge is increased by a factor of 2 and the scale of the charging energy by a factor of 4, it is immediately clear that in the limit $C_0 \approx 0$ the KTB transition temperature is increased by the factor of 4 compared to the normal-state result[7,25,26]

$$T_{cs}^{(0)} = E_C/(\pi\epsilon_Q) . \tag{23}$$

This result may be surprising. There is a temperature range where the normal array is in the plasma phase, i.e., conducting, whereas the Josephson array is still in the charge ordered phase, i.e., insulating. The experiments reported in Ref. 7 are consistent with this picture.

The effect of weak Cooper-pair tunneling can be treated in the winding-number representation, similar to the weak single-electron tunneling analysis presented in Sec. III. The charging action is still given by (14), except that the coupling constants differ by factors of 4. If we expand in the Josephson coupling energy we again reproduce a theory with charging energy only, but with renormalized nearest-neighbor capacitance. This shifts the transition temperature as

$$T_{cs}/T_c^{(0)} = 1 - 0.98(E_J/E_C)^2 . \tag{24}$$

This reduction of T_{cs} can be viewed again as the result of a dynamic screening, in the present case due to the tunneling of Cooper pairs.

V. DYNAMICS OF CHARGES AND VORTICES IN JOSEPHSON JUNCTION ARRAYS

It is well known that the properties of classical, 2D Josephson arrays (with $E_J \gg E_C$) are described by vortices, which are topological excitations in the configuration of phases, and by the spin waves, which are the small-amplitude oscillations of the phases. The vortices have attracted further interest since they undergo a KTB phase transition, where vortex-antivortex pairs dissociate. The transition separates a superconducting low-temperature phase from a resistive high-temperature phase. Charging effects lower the vortex-unbinding transition temperature. (This has been studied extensively for the case where the self-capacitance dominates.[3,4,27] A smaller number of papers dealt with the nearest-neighbor capacitance model,[28,29] but missed the charge-unbinding transition.) On the other hand, in Josephson junction arrays with small capacitances ($E_C \gg E_J$) the charges are the important degrees of freedom, and, as described above, in the limit $C_0 \ll C$ the charges unbind at a critical temperature T_{cs}. Thus the question arises as to which type of order is present in a Josephson array for general parameters. In particular, does the system undergo a charge-KTB transition between an insulating and a conducting state or a vortex-KTB transition between a superconducting and resistive state? Below we will derive from the action (22) in the phase representation an action for two coupled Coulomb gases (CCG), namely the elec-

tric charges and the vortices. This allows us to answer the question raised. At the same time we learn more about the physical properties of the charges and the vortices and their interaction. There exists a near-duality relation between both. Later, in Sec. VI we will draw conclusions about the phase diagram for general ratios of E_J/E_C.

A. Coupled Coulomb gas

We consider an array of Josephson junctions without quasiparticle tunneling and other sources of dissipation. Its partition function can be rewritten in a representation involving charge trajectories $Q_i(\tau)$ instead of the velocities $\dot{\phi}_i(\tau)$ (Ref. 30),

$$Z=\prod_j \int_{Q_{j0}}^{Q_{j0}} DQ_j \sum_{\{n_i\}} \prod_i \int_{\phi_{i0}}^{\phi_{i0}+2\pi n_i} D\phi_i \exp\left[-\frac{4e^2}{2}\int_0^\beta d\tau \left[\sum_{i,j} Q_i(\tau)C_{ij}^{-1}Q_j(\tau)+i\sum_i Q_i(\tau)\dot{\phi}_i(\tau)+\sum_{\langle i,j\rangle} E_J\cos\phi_{ij}(\tau)\right]\right] . \tag{25}$$

The summation over the winding numbers n_i fixes the charges to be integer $Q_i=0,\pm1,\pm2,\ldots$. (The dimensions $2e$ are written explicitly in the prefactor.) Vortex degrees of freedom can be introduced by means of the Villain transformation.[31] It allows us to integrate out the phases at the expense of introducing at each (dual) space-time lattice point an integer-valued two-dimensional vector field $\mathbf{m}_i(\tau)=(m_i^{(x)},m_i^{(y)})$. (For this purpose we introduce a lattice, with spacing ϵ, also in the time direction, but we keep denoting it by τ.) In the Villain approximation one replaces

$$\exp\left[\epsilon\sum_{\langle i,j\rangle,\tau} E_J\cos\phi_{ij,\tau}\right] \rightarrow \text{const}\sum_{\{\mathbf{m}_{i,\tau}\}}\exp\left[-\frac{\epsilon E_J}{2}\sum_{i,\tau}|\nabla\phi_{i,\tau}-2\pi\mathbf{m}_{i,\tau}|^2\right] . \tag{26}$$

The right-hand side of (26) can be rewritten in terms of another integer 2D vector field $\mathbf{J}_{i,\tau}$ as

$$\text{const}\sum_{\{\mathbf{J}_{i,\tau}\}}\exp\left[\sum_{i,\tau}\left[-\frac{1}{2\epsilon E_J}|\mathbf{J}_{i,\tau}|^2-i\mathbf{J}_{i,\tau}\cdot\nabla\phi_{i,\tau}\right]\right] .$$

After integration over the phase variables the partition function becomes

$$Z=\sum_{\{Q_{i,\tau}\}}{\sum_{\{\mathbf{J}_{i,\tau}\}}}'\exp\left[-\epsilon 2e^2\sum_{i,j,\tau}Q_{i,\tau}C_{ij}^{-1}Q_{j,\tau}-\frac{1}{2\epsilon E_J}\sum_{i,\tau}|\mathbf{J}_{i,\tau}|^2\right] , \tag{27}$$

where the summations are constrained by the "continuity" equations at each lattice point

$$\nabla\cdot\mathbf{J}_{i,\tau}-\partial_\tau Q_{i,\tau}=0 . \tag{28}$$

Here $\partial_\tau Q_{i,\tau}=Q_{i,\tau+\epsilon}-Q_{i,\tau}$. This constraint can be solved by[32]

$$J_{i,\tau}^{(\mu)}=n^{(\mu)}(\mathbf{n}\cdot\nabla)^{-1}\partial_\tau Q_{i,\tau}+\epsilon^{(\mu\nu)}\nabla_\nu A_{i,\tau} . \tag{29}$$

The operator $(\mathbf{n}\cdot\nabla)^{-1}$ is the line integral on the lattice, $\epsilon^{(\mu\nu)}$ is the antisymmetric tensor, and $A_{i,\tau}$ is a discrete scalar field to be summed over without further constraints. There are other ways to solve the constraint,[33] but the form (29), which corresponds to the axial gauge, allows us to express the partition function in terms of the real charges and the vortices.

After employing the Poisson resummation identity, which replaces A by the integer V, we can write the partition function (details of the derivation are discussed in Appendix B) as

$$Z=\sum_{\{Q_{i,\tau}\}}\sum_{\{V_{i,\tau}\}}\exp[-S_{\text{CCG}}(\{Q_{i,\tau}\},\{V_{j,\tau}\})] . \tag{30}$$

Here $Q_{i,\tau}$ and $v_{i,\tau}$ are integer-valued fields representing the charge and vorticity in space-time lattice. The action of the coupled Coulomb gas is

$$S_{\text{CCG}}(\{Q_{i,\tau}\},\{V_{j,\tau}\})=\sum_{i,j,\tau}\left[\frac{2\epsilon E_C}{\pi}Q_{i,\tau}G_{ij}Q_{j,\tau}+\pi\epsilon E_J V_{i,\tau}G_{ij}V_{j,\tau}+iQ_{i,\tau}\Theta_{ij}\partial_\tau V_{j,\tau}+\frac{1}{4\pi\epsilon E_J}\partial_\tau Q_{i,\tau}G_{ij}\partial_\tau Q_{j,\tau}\right] , \tag{31}$$

with kernels

$$G_{ij}=-\ln|\mathbf{r}_i-\mathbf{r}_j|-\pi/2 , \tag{32}$$

$$\Theta_{ij}=\arctan\left[\frac{y_i-y_i}{x_i-x_j}\right] . \tag{33}$$

It should be noted that the vortices and the charges are defined on dual lattices.

Obviously for $E_J=0$ or $E_C=0$ the fields must be constant in time τ, and the classical Coulomb gases of charges or vortices are recovered, respectively. In general, the two "charges" interact as described by the ker-

nel Θ_{ij}. The action (31) shows a high degree of symmetry between the vortex and the charge degrees of freedom. However, the duality is broken by the last term $\partial_\tau Q_{i,\tau} G_{ij} \partial_\tau Q_{j,\tau}$. This nonlocal kinetic contribution for the charge is not an artifact. It arises as the spin-wave contribution to the charge correlation function, and can be easily obtained by combining the term $\exp(i\dot{Q}_i \phi_i)$ in Eq. (25) and the $\mathbf{m}=0$ term in (26). The corresponding excitations in the charge gas are absent in the model defined by (22), so that an equivalent term $\partial_\tau V_i G_{ij} \partial_\tau V_j$ does not arise.

B. Vortex dynamics

The representation (31) displays the coupling between the charges and the vortices. In the extreme limits, $E_J=0$ or $E_C=0$, only one or the other persists. In this subsection we focus on the limit $E_C \ll E_J$, where we obtain an effective action for the vortices, but they are influenced by the charging effects. In the considered limit the charges are fluctuating strongly and can be treated as continuous variable. Hence they can be integrated out from the partition function, which produces a kinetic-energy term for the vortices. Due to the spin-wave contribution $\partial_\tau Q_{i,\tau} G_{ij} \partial_\tau Q_{j,\tau}$ the kinetic term is actually nonlocal in time, however, it remains short range. Here we can ignore this modification. After the Gaussian integration we get

$$S_{\text{eff}}[V]=\frac{\pi}{8E_C}\sum_{i,j,k,l}\int_0^\beta d\tau\, \dot{V}_{i,\tau}\Theta_{ik}G_{kl}^{-1}\Theta_{lj}\dot{V}_{j,\tau} + \pi E_J \sum_{i,j}\int_0^\beta d\tau\, V_{i,\tau}G_{ij}V_{j,\tau} .$$

The convolution in the kinetic term can be simplified to the property $(\nabla\Theta)^2=(\nabla G)^2$. After an integration (summation) by parts (the steps are discussed in the appendix of Ref. 34) the result is

$$S_{\text{eff}}[V]=\frac{\pi}{8E_c}\sum_{i,j}\int_0^\beta d\tau\, \dot{V}_{i,\tau}G_{ij}\dot{V}_{j,\tau} + \pi E_J \sum_{i,j}\int_0^\beta d\tau\, V_{i,\tau}G_{ij}V_{j,\tau} . \quad (34)$$

The sum over the configurations in the partition function is constrained by a "charge neutrality" condition $\sum_i V_i=0$, which also implies $\sum_i \dot{V}_i=0$.

The result (34) has further interesting consequences for the quantum dynamics of the vortices in the limit where the charges are strongly fluctuating. In general, all vortex trajectories shown in Fig. 3 are allowed; those labeled (b) and (c) become relevant as the temperature is lowered. In the semiclassical region, however, we can limit ourselves to the trajectories of the type (a), which thread all time slices. If we label the vortices by their center coordinate $\mathbf{r}_n(\tau)$ and their sign $v_n=\pm 1$, the vortex density can be represented as

$$V(\mathbf{r},\tau)=\sum_n v_n \delta(\mathbf{r}-\mathbf{r}_n(\tau)) .$$

Substituting this expression into (34) we obtain the effective action

$$S_{\text{eff}}=\sum_{n,m} v_n v_m \int_0^\beta d\tau \left[\frac{\pi}{8E_C}\sum_{\alpha,\beta=x,y}\dot{r}_n^{(\alpha)} M_{nm}^{(\alpha\beta)} \dot{r}_m^{(\beta)} + \pi E_J G(\mathbf{r}_n(\tau)-\mathbf{r}_m(\tau))\right] , \quad (35)$$

which contains the "vortex mass" tensor

$$M_{nm}^{(\alpha\beta)}=\frac{\partial}{\partial r_n^{(\alpha)}}\frac{\partial}{\partial r_m^{(\beta)}} G(\mathbf{r}_n-\mathbf{r}_m) . \quad (36)$$

This expression for the vortex mass has been obtained before by Eckern and Schmid[35] within a semiclassical description. The action (34) is valid more generally, the only requirement is that the charges are in the disordered phase.

C. Charge dynamics

A similar analysis can be performed to obtain the effective quantum action for the charges. In the limit $E_J \ll E_C$ the vortices are integrated out, with the result

$$S_{\text{eff}}[Q]=\frac{1}{2\pi E_J}\sum_{i,j}\int_0^\beta d\tau\, \dot{Q}_{i,\tau}G_{ij}\dot{Q}_{j,\tau} + \frac{2E_C}{\pi}\sum_{i,j}\int_0^\beta d\tau\, Q_{i,\tau}G_{ij}Q_{j,\tau} . \quad (37)$$

Notice that the "charge-mass" in (37) is actually composed of two terms, one from the integration of the vortex field, the other from the duality-breaking term explicit in (31). Both are equal in magnitude. Now the "mass" of the charge is the band mass of a particle moving in a lattice with the matrix element of strength E_J.

VI. COMPETITION BETWEEN THE CHARGE- AND THE VORTEX-UNBINDING TRANSITION

A. Corrections to the classical results

For $E_J=0$ or $E_C=0$ the charge and vortex fields are constant in time τ, and the classical Coulomb gases of charges or vortices are recovered, respectively. Both show a KTB transition where dipoles unbind. The transi-

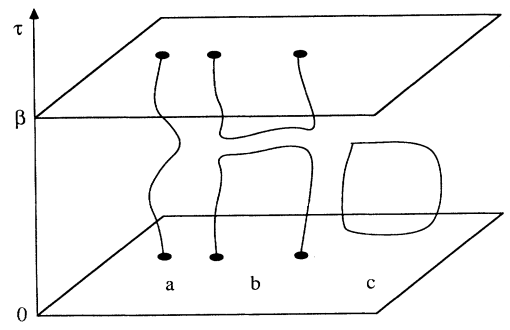

FIG. 3. Examples of vortex configurations considered at finite temperatures.

tion temperatures are given by (23) for the charge-unbinding transition and by

$$T_{cv}=\frac{\pi}{2\epsilon_v}E_J \tag{38}$$

for the vortex-unbinding transition. The dielectric constant ϵ_v in general differs from that of the charge transition, but it is again of order 1.

In Sec. V we had seen that the charges, if they are treated in a continuum approximation and are integrated out, provide a kinetic energy in the action for the vorticity (34). This approach is justified if $E_C \ll E_J$. Following, e.g., the arguments of Feynman and Hibbs,[36] we can immediately conclude that the transition temperature of the vortex-unbinding KTB transition is lowered below the classical value (38). Similarly we had seen that the vortices, which in the limit $E_J \ll E_C$ can be treated in a continuum approximation, provide a kinetic energy for the charges (37). Hence the charge-unbinding transition temperature is reduced by effect of the vortices, consistent with the result (24) (which had been obtained within a different approach). These results, the classical limits and the corrections for small and large ratios of E_J/E_c, are shown qualitatively in Fig. 4. We denote the disordered phase, where free charges and free vortices exist, as conducting or resistive, although such a state requires a dissipation mechanism, which we have not included in the present section. What we mean is that as soon as a weak dissipation mechanism is turned on, e.g., the quasiparticle tunneling to be discussed in Sec. VII, the state is resistive, whereas it remains insulating or superconducting in the other two phases.

The question remains as to what happens for $E_J \approx E_C$. If the duality between charges and vortices would be perfect, i.e., if the duality breaking last term in (31) would be absent, the transition temperatures would be symmetric around the self-dual point

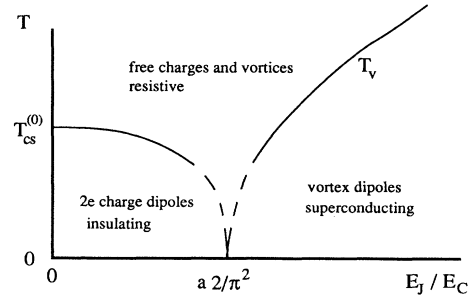

FIG. 4. Phase diagram for the charge-unbinding transition and of the vortex-unbinding transition of a superconducting junction array is shown. Quasiparticle tunneling is ignored. The two transitions meet at $T=0$ at a critical value of $E_J/E_C=a2/\pi^2$. The numerical coefficient a is larger but close to 1.

$$(E_J/E_c)_{\text{self-dual}}=2/\pi^2 . \tag{39}$$

Assuming that, e.g., at $T=0$ there exists only one transition (below we will demonstrate this), we could immediately conclude that the critical value of E_J/E_C, separating the charge- from the vortex-ordered phase, at $T=0$ is given by (39). van Wees[10] conjectured that (39) is the critical value. But the duality-breaking term, even if it becomes irrelevant at the fixed point, can lead to a shift of the critical value away from (39). This shift can be interpreted as the influence of the dielectric constants of the two transitions, which may differ since the short length scale properties of the two subsystems differ.

Coupled Coulomb gas models similar to (31) were discussed in Refs. 34 and 37–39. They refer to classical problems, which means they do not contain a time dependence. The coupling between different time slices in (31) is somewhat similar to the coupling of different replicas which arise in connection with the study of transitions in frustrated XY systems.[39] However, there the coupling is the same between all the replicas, whereas in our problem the interaction has a finite range. This means we cannot take over the results obtained there.

B. $T=0$ limit

Above we showed how to obtain the effective action for the vortices and charges, respectively. This approach is useful at finite temperatures where the system undergoes two different transitions of the KTB type in the charge and the vortex system, respectively. At $T=0$ the system is effectively three dimensional and the character of the phase transitions may change. In this subsection we will study the transition in the $T=0$ limit in more detail.

It is known that in the self-charging limit ($C_0 \gg C$) the system described by the action (22) (without dissipation) is isomorphic to an isotropic 3D XY model (see, e.g., Ref. 3), which, in contrast to the 2D case, has a "ferromagnetic" phase transition to true long-range order. In the limit where the nearest-neighbor capacitance dominates ($C \gg C_0$), the long-range Coulomb interaction introduces a strong anisotropy between the space and time directions. This limit was discussed so far by means of mean-field[29] and variational[40] approaches. In the following we want to consider the problem in terms of the relevant topological excitations. At $T=0$ it turns out that these are vortex loops.[41,42]

We start from the Villain approximation (26) for the Josephson term. In contrast to (25) we do not introduce charge trajectories, but use the fact at $T=0$ only the contributions with winding numbers $n_i=0$ survive. The phases can again be integrated out with the result

$$Z_V=\sum_{\{\mathbf{m}_{i,\tau}\}} \exp(-S_{\text{eff}}[\mathbf{m}_{i,\tau}]) , \tag{40}$$

where $\mathbf{m}$ is a 2D vector in the 3D space-time lattice $\mathbf{r}=(i_x,i_y,\tau)$ and

$$S_{\text{eff}}[\mathbf{m}_{i,\tau}]=2\pi^2 E_J \sum_{\mathbf{k},\omega}\sum_{\mu,\nu=x,y}\left[\delta_{\mu,\nu}-E_J\frac{k_\mu k_\nu}{\frac{\omega^2}{8E_0}+\frac{\omega^2}{8E_C}k^2+E_J k^2}\right]m^{(\mu)}(\mathbf{k},\omega)m^{(\nu)}(-\mathbf{k},-\omega)\ . \tag{41}$$

It is convenient to view $\mathbf{m}$ as a 3D vector in a gauge where $m^{(\tau)}=0$. Then we can rewrite (41) as

$$S_{\text{eff}}[\mathbf{m}]=\tfrac{1}{2}\sum_{\mathbf{r},\mathbf{r}'}\sum_{\mu=x,y,\tau}[\nabla\times\mathbf{m}(\mathbf{r})]_\mu G_\mu(\mathbf{r}-\mathbf{r}')\times[\nabla\times\mathbf{m}(\mathbf{r}')]_\mu\ . \tag{42}$$

The integer-valued "current" $\mathbf{J}=\nabla\times\mathbf{m}$ describes the topological excitations of the original ϕ field. It is an alternative to the representation by charges and (phase) vortices. The present current field is divergence free, i.e., it forms closed loops. (Frequently these loops are called "vortex loops." However, they are not the "physical" phase vortices introduced in Sec. V.) The general expression for the kernel G_μ, when both self-capacitance and mutual capacitance are present, is given in Appendix C; here we quote the two limiting cases. Introducing $k^2=2(1-\cos k_x)+2(1-\cos k_y)$ and the notation $G_\parallel$ for the x and y components and G_τ for the τ component of the interaction, we find the following.

In the self-charging limit ($C=0$)

$$G_\parallel(\mathbf{k},\omega)=G_\tau(\mathbf{k},\omega)=4\pi^2 J_0\frac{1}{k^2+\omega^2}\ , \tag{43}$$

where $J_0=\sqrt{E_J/8E_0}$ and the frequencies are rescaled as $\omega/\sqrt{8E_JE_0}\to\omega$.

In the nearest-neighbor capacitance limit ($C_0=0$)

$$G_\parallel(\mathbf{k},\omega)=4\pi^2 J_C\frac{1}{1+\omega^2}\ ,\qquad G_\tau(\mathbf{k},\omega)=4\pi^2 J_C\frac{1}{1+\omega^2}\frac{1}{k^2}\ , \tag{44}$$

where $J_C C=\sqrt{E_J/8E_C}$ and the frequencies are rescaled as $\omega/\sqrt{8E_JE_C}\to\omega$.

From Eq. (43) we see that in the self-charging limit the interaction is isotropic and coincides with the Green's function appearing in the dual representation of the 3D XY model.[42] Therefore the $T=0$ transition leads to long-range order, whereas at finite T it belongs to the KTB universality class.

In the nearest-neighbor capacitance limit the propagators are strongly anisotropic and lead to a *qualitatively* different behavior. The problem strongly resembles the 3D layered superconductors studied in Refs. 43 and 44 as a model for high-T_c materials. In this work it was shown that the subsystem formed by the vortex loops oriented parallel to the x-y plane show critical behavior. For these loops $[\nabla\times\mathbf{m}(\mathbf{r})]_\tau=\mathbf{0}$, they can be parametrized by a scalar integer $n(\mathbf{r})$ chosen such that

$$\mathbf{m}(\mathbf{r})=\nabla n(\mathbf{r})\ .$$

After a further transformation $\partial_\tau n\to p$ the action of the loops oriented parallel to the space-plane becomes

$$S_{\text{eff}}[p]=\tfrac{1}{2}\sum_{\mathbf{k},\omega}p(\mathbf{k},\omega)W(\mathbf{k},\omega)p^*(\mathbf{k},\omega)\ , \tag{45}$$

where the interaction is given by

$$W(\mathbf{k},\omega)=4\pi^2 J_C\frac{k^2}{1+\omega^2}\ .$$

The action (45) represents a roughening model defined in space-time. From this one can proceed in the standard way to a Coulomb gas representation[22] with interaction potential $\tilde{W}(\mathbf{k},\omega)=4\pi^2/W(\mathbf{k},\omega)$. This means the 2D Coulomb gas interacts logarithmically within the same and the nearest-neighbor layers. Korshunov[44] studied this problem and found in the limit of vanishing fugacity a phase transition of the KTB type at a critical value of the coupling constant, which translates to

$$(E_J/E_C)_c=a\,2/\pi^2\ . \tag{46}$$

Korshunov obtained the value of the coefficient $a=\frac{9}{16}$. However, the duality arguments described above led to a numerically slightly larger value, namely $a=1$, and comparing the kinetic terms in (34) and (37) we expect that the duality-breaking term shifts the critical value of E_J/E_C to even larger values. At least in the semiclassical regime it increases the "mass" of the charges and reduces the effect of their fluctuations as compared to the situation for the vortices. These modifications appear not to be properly accounted for in the present approach. On the other hand, we learn here that in the nearest-neighbor capacitance model the transition belongs to the KTB universality class even at $T=0$, so that the dimensional crossover which occurs in the self-charging model as T approaches zero is not present. The critical value of E_J/E_C (46) also differs from those obtained in mean-field approximations in Refs. 27–29 for $C_0=0$. Actually different results have been published for this limit. This may be related to subtleties, whether or not a dimensional crossover occurs in these calculations in the $T=0$ limit. The combination of (46) and the expansions about the classical limits (the charge- and vortex-unbinding transitions) leads to the qualitative phase diagram of a Josephson array (without single-electron tunneling) shown in Fig. 4.

VII. INFLUENCE OF QUASIPARTICLE TUNNELING ON THE CHARGE- AND VORTEX-UNBINDING TRANSITIONS

We turn now to the question of how the quasiparticle tunneling influences the properties of the charge and vortex gases and their phase transitions. A superconducting

array is described by the action (22); the quasiparticle tunneling is accounted for by S_t in Eq. (11). But the kernel $\alpha(\tau)$ depends on the superconducting gap and is modified as compared to the form (12). General expressions are given in Refs. 13 and 19. For a qualitative understanding of its influence it is here sufficient to consider three limits.

(i) If the typical frequencies and energy scales are large compared to the superconducting gap Δ, the quasiparticle tunneling is the same as the single-electron tunneling in the normal state, $\alpha(\tau)$ is given by Eq. (12), and the conductance entering is the normal-state conductance

$$\alpha_t = h/(4e^2 R_n) . \tag{47}$$

(For definiteness we always denote by α_t the normal-state conductance.)

(ii) If the typical frequencies and energy scales are small compared to the superconducting gap Δ, the tunneling is characterized by the subgap conductance. In good quality junctions this subgap conductance decreases exponentially,

$$\alpha_{t,\text{eff}} \approx \alpha_t \exp(-\Delta/k_B T) , \tag{48}$$

as the temperature is lowered and vanishes for $T=0$. Nevertheless, unless the voltage is constant everywhere a mixing of different frequency components occurs and the normal-state conductance is felt. For example, a moving vortex is associated with anharmonic effects.[35]

(iii) In ideal BCS tunnel junctions at low temperatures, if all relevant frequencies and energy scales are small compared to Δ, no real quasiparticle tunneling occurs. Nevertheless, virtual tunneling does take place. It can be accounted for by a renormalization of the nearest-neighbor capacitance,[19,45] which changes the associated charging-energy scale by an amount depending on the normal-state conductance

$$E_C^{\text{ren}}/E_C = [1+\alpha_t 3E_C/(8\Delta)]^{-1} . \tag{49}$$

The fabricated junction arrays of Ref. 7 have a rather ideal tunneling I-V characteristic, and the three limits (i)–(iii) characterize them well. In the simplest case the bare Josephson coupling energy is given by the Ambegaokar-Baratoff[46] relation

$$E_J = \alpha_t \Delta/2 . \tag{50}$$

On the other hand, E_J can be reduced below this value by the application of a magnetic field. For this reason we consider E_J, E_C, and α_t (i.e., the normal-state conductance) as independent parameters and study the phase diagram in the T/E_C versus E_J/E_C-α_t space. It is important to notice that the space is divided, as shown for the $T=0$ plane in Fig. 5, into a regime where $E_C > 2\Delta$ (close to the α_t axis) and a regime where $E_C < 2\Delta$ (close to the E_J/E_C axis).

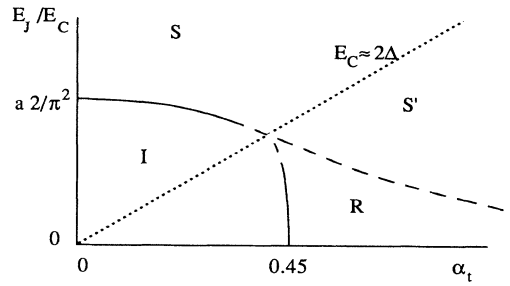

FIG. 5. $T=0$ phase diagram of a superconducting junction array with quasiparticle tunneling is shown. In the insulating phase (I) single electrons and Cooper pairs are bound in dipoles. In the resistive phase (R) we have no superconducting order (free vortices), but the quasiparticles are free. In the phases S and S' the vortices are bound and the array is superconducting. In S the quasiparticles are frozen or bound in dipoles, in S' they are free.

A. $T=0$

We first consider the zero-temperature limit. We can obtain a feeling for the emerging phase diagram by combining the information we have obtained so far. We have seen that at $E_J/E_C = 2/\pi^2$ (setting the coefficient $a=1$) there occurs a transition between a state with bound Cooper pairs and a superconducting ordered state. On the other hand, at a critical strength of the single-electron tunneling conductance we found a transition between a state with bound single electrons and a state with strong tunneling and free charges. Notice that the typical energy scale for the single-electron tunneling is E_C. Hence it freezes out when 2Δ is larger or near E_C.

We thus find four regimes (see Fig. 5).

(i) At small E_J/E_C and α_t both Cooper pairs and single electrons are frozen. This state is insulating (I).

(ii) At small E_J/E_C but large α_t Cooper pairs remain frozen (no long-range phase order), but the single electrons are free. The single-electron dynamics is the same as in the normal state. This corresponds to a resistive state (R). Notice that at $T=0$ the resistance may be infinite (see Ref. 47). However, at any finite temperature it is finite. In contrast to the "insulating" phase the array remains insulating up to some finite temperature.

(iii) At large E_J/E_C and small α_t we have superconducting long-range phase order (of the KTB type) and the single electrons are frozen. This is a superconducting state (S).

(iv) At large E_J/E_C and large α_t we have superconducting long-range phase oder (of the KTB type) and tunneling of single electrons. The latter process yields a parallel dissipative channel, but it is shortened by the superconducting channel. Therefore we denote the phase as S'. The finite-frequency response in this phase may differ from the one in S.

The four phases will persist when we include the mutual influence of the tunneling of Cooper pairs and single electrons. However, there will be quantitative modifications.

(a) Above the line $E_C = 2\Delta$ (i.e., close to the E_J/E_C axis) only virtual tunneling processes occur (the characteristic energy scale is E_C) resulting in the capacitance renormalization (49). From this it is immediately clear that

the critical value of E_J/E_C decreases[48] as

$$(E_J/E_C)_c = 2/\pi^2 - 3\alpha_t^2/16 , \quad (51)$$

until it approaches the line $E_C = 2\Delta$. Below this line real dissipative tunneling processes occur. They decrease the critical value of E_J/E_C even stronger, but a quantitative analysis of this limit has not yet been obtained. We plot in Fig. 5 the result (51) and our qualitative understanding of the transition curve beyond.

(b) Below the line $E_C = 2\Delta$ (i.e., small values of E_J/E_C) there is a regime where the single-electron tunneling process dominates and Cooper pair tunneling can be taken into account perturbatively. The Cooper-pair tunneling again leads to a dynamic enhancement of screening. It lowers the critical value of $\alpha_{t,c}$ below the normal-state value 0.45 (see Fig. 5). When we approach the line $E_C = 2\Delta$, the single-electron tunneling begin to freeze out. Hence we might observe a reentrant-type curve. On the other hand, the numerical coefficients just happen to be such that this crossover is close to where also the line (51) meets and the reentrant features may remain unobservable.

B. $T \neq 0$

If we combine the limits discussed so far, i.e., the normal junction result (Sec. III), the $\alpha_t = 0$ limit (Secs. IV and VI) and the $T = 0$ diagram, we obtain a rather complete phase diagram as shown in Fig. 6. An interesting question is how the charge-unbinding transition dominated by single-electron tunneling with a transition temperature T_{cn} crosses over to the charge-unbinding transition dominated by Cooper-pair tunneling with a transition temperature $T_{cs} = 4T_{cn}$. Again the transition should occur at $E_C \approx 2\Delta$. For smaller values of Δ single-electron tunneling dominates since the activation energy is of order E_C and hence smaller by a factor of 1/4 than that of Cooper-pair tunneling. For larger values of Δ the quasiparticle tunneling rate is suppressed by the exponential factor $\exp(-\Delta/k_B T)$ and the Cooper-pair tunneling becomes dominant. As a further quantitative detail we can mention that the capacitance renormalization due to virtual quasiparticle tunneling (in the region where it is the relevant description) lowers T_{cs} since the charging-energy scale is reduced. We combine our understanding, which in part is still qualitative, in the diagram of Fig. 6.

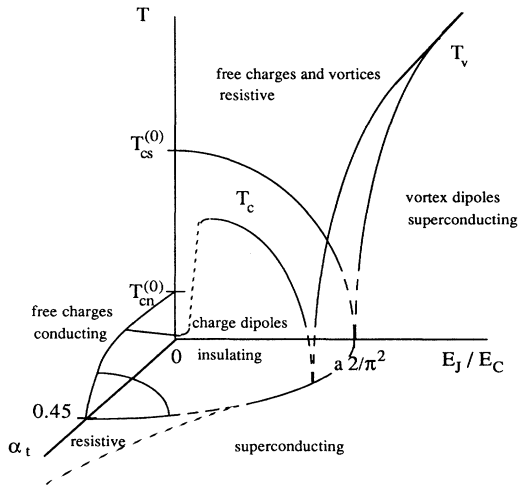

FIG. 6. KTB transition temperatures for the charge- and the vortex-unbinding transitions are plotted (in part qualitatively) as a function of the normal-state tunneling conductance $\alpha_t = h/4e^2 r_n$ and E_J/E_C. They separate an insulating, a superconducting, and a resistive region.

VIII. CONCLUSIONS AND OPEN PROBLEMS

Two-dimensional array of tunnel junctions can undergo (if the nearest-neighbor capacitance is much larger than the capacitance to ground) a charge-unbinding KTB transition. In the ordered phase the charges on different islands are bound in dipoles and the array is insulating. Both the dissipation due to single-electron tunneling and the Josephson tunneling lower the transition temperature. In the superconducting case the additional possibility of phase ordering gives rises to a complex phase diagram in which charge unbinding and vortex unbinding compete.

We have not yet considered all aspects of the single-electron tunneling in the superconducting state. Even in situations close to ideal, where its dominant effect is a capacitance renormalization (49), rare real single-electron tunneling processes can occur. They lead to trapped single charges on the islands, which will also influence the dynamics of Cooper pairs (randomly distributed full frustration). Furthermore, charged impurities, e.g., in the substrate of the array, can polarize the islands, leading effectively to noninteger random trapped charges.[7] Random trapped charges will smear out the transition temperature. We will study these properties further.

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APPENDIX A

In this appendix we discuss the low-temperature phase diagram of the DG+ASOS model described in Eq. (19). A convenient way to explore the equivalence between the roughening and the XY model. The mapping was discussed in Ref. 49, but, in order to keep this appendix self-contained, we recall the main steps. Our starting point is

$$Z = \sum_{\{n_i\}} \exp(-S_{\text{eff}}[n]) , \quad (A1)$$

$$S_{\text{eff}}[n] = \frac{\pi^2}{\beta E_C} \sum_{\langle i,j \rangle} (n_i - n_j)^2 + 2\alpha_t \sum_{\langle i,j \rangle} |n_i - n_j| . \quad (A2)$$

Because the action depends only on the differences between nearest neighbors $n_{ij}=n_i-n_j$, it is convenient to consider the bond variables as independent. Accordingly the sum over configurations can be performed over the variables n_{ij} if one introduces N (N is the number of lattice sites) constraints which demand that the sum over each elementary plaquette is zero:

$$\sum_{\{n_i\}}{}' \rightarrow \sum_{\{n_{ij}\}} \delta\left[\sum_p n_{ij}\right]$$

$$\rightarrow (2\pi)^{-1}\int_0^{2\pi} d\theta_I \exp\left[i\theta_I \sum_p n_{ij}\right] . \tag{A3}$$

We identify each plaquette with the coordinate of its center (on the dual lattice), which we denote by a capital letter. The sum over the bond variables factorizes. Rearranging the terms, we obtain the partition function in terms of the "phase" variables introduced in (A3),

$$Z=(2\pi)^{-1}\int_0^{2\pi} d\theta_I \exp\left[-\sum_{\langle I,J\rangle} W(\theta_I-\theta_J)\right] , \tag{A4}$$

where the interaction energy $W(\theta)$ is given by

$$-W(\theta_I-\theta_J)=\ln\left[\sum_{n=-\infty}^{\infty}\exp\{-S_{\rm eff}[n]+in(\theta_I-\theta_J)\}\right] . \tag{A5}$$

For the model defined in (A1) we obtain

$$\exp[-W(\theta_I-\theta_J)] = \left[\frac{4\pi^3}{\beta E_C}\right]^{-1/2}\tanh(2\alpha_t) \times\int_{-\infty}^{\infty} dx \exp\left[-\frac{\beta E_C x^2}{4\pi^2}\right] \times\left[1-\frac{\cos(\theta_I-\theta_J-x)}{\cosh(2\alpha_t)}\right]^{-1} . \tag{A6}$$

The Boltzmann factor (A6) is of a generalized XY model; it contains terms proportional to $\cos k\theta$ ($k=1,2,\ldots$). Considering the first harmonic only, we get a contribution to the pure XY model (cosine interaction) with a coupling constant given by

$$W(\theta)=\text{const}-\exp\left[-\frac{\pi^2}{\beta E_C}\right]\frac{\cos\theta}{\cosh(2\alpha_t)} . \tag{A7}$$

At $T\approx 0$ we recover the standard result for the ASOS model; at finite small temperatures the coupling constant of the XY model is renormalized so that the critical line bends towards smaller values of the dissipation and to first order in temperature we obtain (21).

APPENDIX B

We outline here, in some more detail, the derivation of the coupled Coulomb gas action (31) for the charges and vortices. We solve the constraint (28) by the parametrization (29), which introduces the integer field A. We then use the Poisson resummation formula

$$\sum_{\{A\}} f(A)=\sum_{\{V\}}\int_{-\infty}^{\infty} dA\, f(A)\exp(i2\pi VA) \tag{B1}$$

and perform the Gaussian integrals to obtain

$$Z=\sum_{\{Q_{i,\tau}\}}\sum_{\{V_{i,\tau}\}}\int_{-\infty}^{\infty} dA_{i,\tau}\exp\left[-\frac{2\epsilon E_C}{\pi}\sum_{i,j,\tau} Q_{i,\tau}G_{ij}Q_{j,\tau}-\frac{1}{2\epsilon E_J}\sum_{i,\tau,\mu}[n^{(\mu)}(\mathbf{n}\cdot\nabla)^{-1}\partial_\tau Q_{i,\tau}]^2 -\frac{\epsilon E_J}{4\pi}\sum_{i,j,\tau}\left[2\pi V_{i,\tau}-\frac{i}{\epsilon E_J}\epsilon^{(\mu\nu)}\nabla_\nu n^{(\mu)}(\mathbf{n}\cdot\nabla)^{-1}\partial_\tau Q_{i,\tau}\right]G_{ij} \times\left[2\pi V_{j,\tau}-\frac{i}{\epsilon E_J}\epsilon^{(\mu'\nu')}\nabla_\nu n^{(\mu')}(\mathbf{n}\cdot\nabla)^{-1}\partial_\tau Q_{j,\tau}\right]\right] . \tag{B2}$$

Here we consider the limit of vanishing self-capacitance, but the generalization is obvious. In the form written the classical action of the charge gas and of the vortex Coulomb gas, that is, the first two terms in (31), are apparent. This shows that the integer fields $V_{i,\tau}$ introduced by (B1) represent the (physical) vortices in the configuration of the phases $\phi_{i,\tau}$. In the cross terms in (B2) we sum by parts

$$\sum_{i,j} V_{i,\tau}G_{ij}\epsilon^{(\mu\nu)}\nabla_\nu n^{(\mu)}(\mathbf{n}\cdot\nabla)^{-1}\partial_\tau Q_{j,\tau} = -\sum_{i,j} V_{i,\tau}\epsilon^{(\mu\nu)}\nabla_\nu n^{(\mu)}(\mathbf{n}\cdot\nabla)^{-1}G_{ij}\partial_\tau Q_{j,\tau} , \tag{B3}$$

which introduces the charge-vortex interaction potential

$$\Theta_{ij}=\sum_{\mu,\nu}\epsilon^{(\mu\nu)}\nabla_\nu n^{(\mu)}(\mathbf{n}\cdot\nabla)^{-1}G_{ij} . \tag{B4}$$

This expression has been studied in Refs. 32 and 34. It has been shown to reduce to the form (33). The remaining two terms in (B2) produce the symmetry-breaking, last contribution in Eq. (31).

APPENDIX C

In this appendix we consider arrays with both self-capacitance and nearest-neighbor capacitances and gen-

eralize the expressions (43) and (44) accordingly. In the present case we rescale the frequencies as

$$\omega\left[\frac{1}{8E_J}\left[\frac{1}{E_0}+\frac{1}{E_C}\right]^{1/2}\right]\rightarrow\omega\ , \tag{C1}$$

which reduces to the limits discussed in the text when one of the capacitances is set equal to zero. The current kernels are

$$G_{\parallel}(\mathbf{k},\omega)=4\pi^2 J\frac{1}{1+\lambda}\frac{1+\lambda^{-1}k^2}{\frac{\omega^2}{1+\lambda}+\frac{\omega^2}{1+\lambda^{-1}}k^2+k^2}\ , \tag{C2}$$

$$G_{\tau}(\mathbf{k},\omega)=4\pi^2 J\frac{1}{1+\lambda}\frac{1}{\frac{\omega^2}{1+\lambda}+\frac{\omega^2}{1+\lambda^{-1}}k^2+k^2}\ , \tag{C3}$$

where $\lambda=C/C_0$, $J=\sqrt{E_J/8E_C+E_J/8E_0}$, and $G_{\parallel}$ and G_{τ} refer to the x-y and the τ components of the interaction.

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