Quantized voltage plateaus in Josephson-junction arrays: A numerical study

K. H. Lee and D. Stroud
Department of Physics, The Ohio State University, Columbus, Ohio 43210
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We study numerically the quantized voltage plateaus in an $N\times N$ array of resistively shunted Josephson junctions subjected to a combined dc and ac applied current $I_{dc}+I_{ac}\sin(2\pi\nu t)$, and a transverse magnetic field equal to $p/q\equiv f$ flux quanta per plaquette (p and q relatively prime integers). With periodic transverse boundary conditions, we find plateaus at *all* voltages satisfying $\langle V\rangle = nNh\nu/(2eq)$, where n is an integer, and the angular brackets $\langle\cdots\rangle$ denote a time average. With free transverse boundary conditions, additional steps at $\langle V\rangle = Nh\nu/(4eq)$ sometimes appear. For $f=\frac{1}{5}$ and $\frac{2}{5}$, we study the motion of the vortex lattice on the steps. At both fields, on every step, the lattice moves an integer number of array lattice constants per cycle of the ac field. For both zero and finite applied transverse magnetic field, the width of the steps varies sinusoidally with I_{ac}, in a manner reminiscent of that seen in single Josephson junctions. At a given field and current, the steps "melt" at a temperature no higher than the transition temperature of the underlying array at the same field and zero current. On the steps, the time-dependent voltage across the array has strong harmonics at multiples of the fundamental frequency. Off the steps, the power spectrum of the voltage has an apparently broad band with possible subharmonic structure.

I. INTRODUCTION

Superconducting arrays have been intensively studied in recent years and exhibit much complex behavior.[1] Such arrays can be artificially prepared by photolithographic techniques.[1] When one applies a transverse magnetic field B, which is a rational fraction of a flux quantum per plaquette, the vortices form a lattice at low temperature, which is commensurate with the array.[2] This lattice manifests itself experimentally in a strongly field-dependent superconducting transition temperature of the array $T_c(B)$. Weakly coupled Josephson systems have also been proposed as models for polycrystalline high-T_c superconductors.[3]

When a combined dc and ac external current $I_{dc}+I_{ac}\sin(2\pi\nu t)$ is applied, the current-voltage (I-V) characteristics of ordered two-dimensional arrays exhibit quantized voltage plateaus. In a square array of $N\times N$ plaquettes, with an applied transverse magnetic field of magnitude $f=p/q$ flux quanta per plaquette, where p and q are relatively prime integers, these plateaus occur at voltages $nNh\nu/(2eq)$, where n is an integer.[4] These plateaus are generalizations of similar phenomena, called Shapiro steps, long familiar in single resistively shunted Josephson junctions.[5] In the arrays at finite fields, the plateaus in a field are called fractional giant Shapiro steps;[4] at zero field, they are known as integer giant steps.[6]

Several theoretical studies of fractional giant Shapiro steps have appeared in the literature. Benz *et al.*, in their original experimental paper,[4] propose that the steps occur when the vortex lattice is able to lock onto the underlying periodic two-dimensional vortex potential formed by the Josephson-junction lattice. Lee *et al.*[7] showed numerically that the fractional steps emerged naturally from a model of coupled resistively shunted Josephson junctions (RSJ's). They also found (in a calculation with free transverse boundary conditions) additional steps beyond those reported experimentally by Benz *et al.* Free *et al.*[8] used a model of coupled RSJ's and, for the first time, periodic boundary conditions. They also found fractional giant steps, and by studying voltage drops across individual junctions, provided evidence that the steps arise from coherent motion of the entire vortex lattice. Halsey[9] considered special values of transverse magnetic field, at which his so-called staircase phase configuration[10] might be the ground state, and proposed that, for certain directions, amplitudes, and frequencies of the applied currents, there might be subharmonic steps beyond those reported in Ref. 4.

This paper reports detailed numerical studies of quantized voltage plateaus in RSJ arrays. We show that several "anomalous" steps found previously with free transverse boundary conditions disappear when periodic boundary conditions are used. We find that the step widths exhibit an oscillatory Besself-function-like dependence on ac current amplitude and frequency, similar to that of single junctions. By including finite-temperature noise in our simulations, we verify that the steps disappear at a critical temperature, which depends on magnetic field. The current motions on the steps are studied in detail for several values of the applied field. At both $f=\frac{1}{5}$ and $\frac{2}{5}$, we find that, on each step, the vortex pattern is always translated by an integer number of plaquettes per cycle of the ac field. Finally, we examine the power spectrum of the voltage on the steps. Not surprisingly, this voltage is not only periodic but has many higher harmonics. These could lead, in principle, to coherent radiation from the junction array at appropriate frequencies or, perhaps, also to coherent detection.[11]

II. MODEL

Our calculations proceed by directly solving the equations for a network of resistively coupled Josephson junctions in the limit of infinite shunt capacitance and negligible array self-inductance.[12–18]

$$I_{ij}=V_{ij}/R_{ij}+I_{c;ij}\sin(\phi_i-\phi_j-A_{ij}) , \tag{1}$$

$$V_{ij}=\frac{\hbar}{2e}\frac{d}{dt}(\phi_i-\phi_j) , \tag{2}$$

$$\sum_j I_{ij}=I_{i;\text{ext}} . \tag{3}$$

Equation (1) describes the current I_{ij} from grain i to grain j as the sum of a normal contribution V_{ij}/R_{ij} and a Josephson current. Equation (2) is the Josephson relation connecting the voltage difference V_{ij} between grains i and j and the phase difference $\phi_i-\phi_j$ between the phases of the order parameters. Finally, Eq. (3) is Kirchhoff's law, expressing current conservation at grain i ($I_{i;\text{ext}}$ being the external current injected into grain i). The given form of the Josephson current is appropriate in a transverse magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$. The factor $A_{ij}=(2\pi/\Phi_0)\int_{\mathbf{x}_i}^{\mathbf{x}_j}\mathbf{A}\cdot d\mathbf{x}$, where $\mathbf{x}_i$ is taken as the center of grain i. We assume a square array of $N\times N$ identical junctions. A current $I=I_{\text{dc}}+I_{\text{ac}}\sin(2\pi\nu t)$ is fed into each grain in the top row and extracted from each grain on the bottom row, with free or periodic boundary conditions on the two transverse boundaries. Combining Eqs. (1)–(3) yields coupled first-order nonlinear differential equations for the phases. We include finite temperature by adding to each junction a parallel Langevin noise current source $I_L(t)$, with a Gaussian distribution whose ensemble average satisfies

$$\langle I_L(t)\rangle_e=0 ;$$

$$\langle I_L(t)I_L(t')\rangle_e=(2k_BT/R)\delta(t-t')$$

and noise currents in different junctions assumed uncorrelated.[19]

The combined equations can be solved by a variety of algorithms.[12–18] We adopt a straightforward iterative approach described and used by several groups.[7,8,12–15] We have carried out this iteration using time steps usually of $0.05t_0$ but occasionally as small as $0.01t_0$ to $0.02t_0$, where $t_0=\hbar/(2eRI_c)$, R being the shunt resistance and I_c the critical current of each junction. We usually consider intervals of I_{dc}/I_c ranging from 0.01 to 0.05. To obtain average voltages, we perform a time average over an interval from $400t_0$ to $800t_0$. We have tested a variety of initial phase configurations (phases parallel, random initial phases, and ground-state phase configuration). Generally, these have little effect on the resulting current-voltage characteristics. We have also considered two different methods of ramping up or down the applied dc current: rerandomizing after each increase of dc current and using the final phase configuration of the previous current as the initial configuration of the new current. This choice also seems hardly to influence the resulting I-V characteristics.

III. RESULTS

Figure 1 shows the calculated current-voltage characteristics of a 10×10 array at $T=0$, field $f=\frac{1}{5}$, and with two types of transverse boundary conditions: periodic and free. In both cases, $I_{\text{ac}}/I_c=1.0$ and $\nu/\nu_0=0.1$, where I_c is the critical current of each individual Josephson junction, and $\nu_0=2eRI_c/\hbar$ is the natural unit of frequency. Changing the boundary conditions has a striking effect. With periodic boundary conditions, there are clear steps at $Nnh\nu/(2eq)$, for every integer $n\geq1$, and *no other steps.* [These steps correspond to $\langle V\rangle/(NRI_c)=(n/q)0.1(2\pi)=(n/q)0.628$ for our choice of frequency.] By contrast, when free transverse boundary conditions are used, one sees conspicuous "integer" and "half-integer" giant steps at $Nh\nu/(2e)$ and $(N/2)h\nu/(2e)$ but no other clear steps. We have seen such anomalous half-integer giant steps for most values of $f=p/q$ with q odd ("odd-denominator frustration") whenever we use free boundary conditions, but they disappear with periodic boundary conditions. Our results are thus consistent with the findings of Ref. 8 that the anomalous steps are absent at $f=\frac{1}{3}$ with periodic boundary conditions. The anomalous half-integer steps are always characterized by a periodic time-dependent voltage just as in Ref. 7, and they exist under a variety of initial conditions (phases parallel, random initial phases, and ground-state phase configuration).

The reasons for these anomalous steps are not understood. Most likely, they arise from an irregular motion of the vortex lattice near the free boundaries. This complicated vortex motion near the boundaries seems to be re-

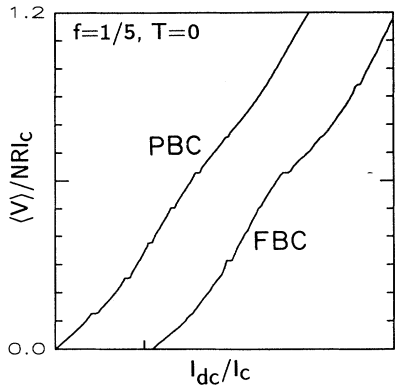

FIG. 1. Time-averaged voltage drop $\langle V\rangle$ across a 10×10 array, as a function of dc current I_{dc}, in the presence of a transverse magnetic field of strength $f=\frac{1}{5}$ flux quantum per plaquette, with both periodic (PBC) and free (FBC) boundary conditions in the transverse direction, at temperature $T=0$. In both cases, current is uniformly injected into nodes at the top of the array and removed from the bottom of the array. Amplitude of the ac current is $I_{\text{ac}}/I_c=1$, and its frequency is $\nu=0.1\nu_0$, where $\nu_0=2eRI_c/\hbar$, R being the shunt resistance and I_c the critical current of a single junction. $\langle V\rangle$ is shown in units of NRI_c, where $N=10$. The plot for free boundary conditions is displaced horizontally to the right by one unit (indicated by a vertical mark).

lated to the relatively lower periodic two-dimensional potential in which the vortices move near the boundaries. In an array with periodic boundary conditions, this potential is more nearly periodic, and the extra steps are absent.

Experiments are generally done in large arrays with free transverse boundary conditions. Because of the size of the experimental arrays (typically 300×300 or larger), it seems reasonable to model experiment with periodic rather than free boundary conditions. This would imply that the extra steps should be absent in experimental arrays, as they appear to be in the data so far published.[4] Further experimental studies may shed more light on this point.

To visualize the coherent vortex motion that produces the Shapiro steps, we have carried out detailed calculations for two fields: $f=\frac{1}{5}$ and $\frac{2}{5}$. These calculations (and all subsequent ones, unless otherwise stated) are carried out on a 10×10 array with periodic boundary conditions in the lateral directions and uniform current injection in the vertical direction. The case $f=\frac{2}{5}$ represents a particularly interesting test case: it is the simplest nontrivial fraction that cannot be reduced to the from $1/q$. (Because of the symmetry of the square lattice, the fractions f and $1-f$ are equivalent, so that one need not consider values of $f>\frac{1}{2}$.) It is, therefore, of interest whether or not *all* predicted voltage steps of the form $\langle V\rangle=nNh\nu/(2eq)$ actually appear in this case. Figure 2 shows that all such steps definitely do appear, at least up to $n=5$. The $n=1$ step is very weak at some (not all) values of I_{ac}, but every predicted step certainly occurs over some range of ac amplitudes.

The time-dependent variation of the phases of the Josephson junctions can be concisely represented in terms of "vortex motion," as is shown in Fig. 3(a) for $f=\frac{1}{5}$. A square marked with a plus sign holds a vortex (i.e., a plaquette of positive, or counterclockwise, "vorticity"). An open square contains an antivortex. The vorticity, in turn, is defined at the center of each plaquette as the sum of the supercurrents through the four junctions bounding

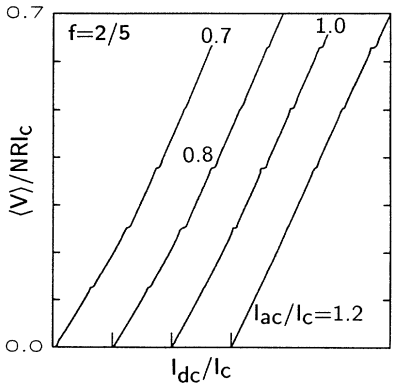

FIG. 2. Current-voltage characteristics of a 10×10 array ($N=10$) with $f=\frac{2}{5}$, periodic boundary conditions, $\nu/\nu_0=0.1$, and several values of I_{ac}/I_c ranging from 0.7 to 1.2. Graphs are horizontally displaced as in Fig. 1.

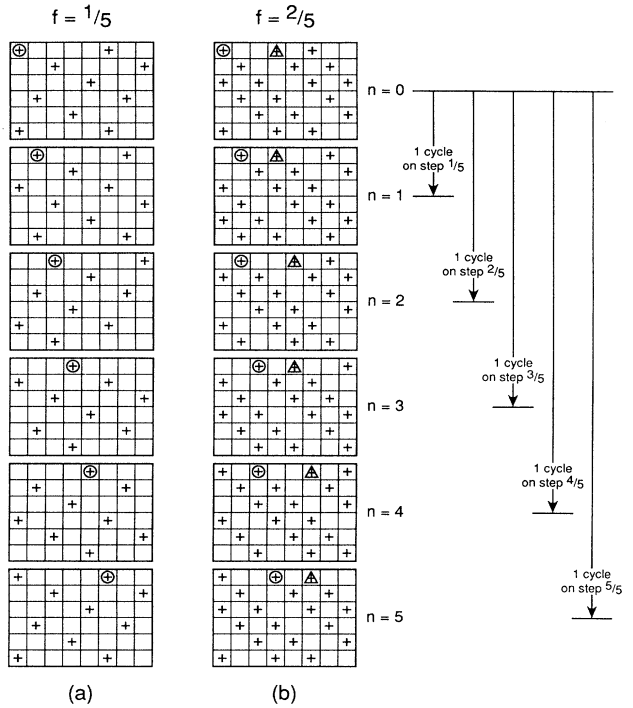

FIG. 3. A schematic diagram showing the motion of the vortex pattern at (a) $f=\frac{1}{5}$ and (b) $f=\frac{2}{5}$. A square containing a plus sign contains a vortex (i.e., a region of positive vorticity); an open square holds an antivortex. Each diagram of connected plaquettes represents a snapshot of the vorticity configuration at a particular time. The arrows represent the extent of motion of the vortex lattice during one cycle of the ac field, when the dc voltage corresponds to step n. In both pictures, we have tagged several vortices by enclosing them with circles or triangles.

the plaquette, the sum being taken in the counterclockwise direction. Each rectangle of plaquettes represents a "snapshot" of part of the array at the time shown. Time increases in the downward direction. In order to make the motion clearer, we have "tagged" one of the vortices with a circle. From examining many vorticity snapshots, we infer that the tagged vortex moves as shown. In general, on the nth step at $f=1/q$, the vortex pattern moves n times faster than for $n=1$. This is shown schematically in the diagrams of Fig. 3(a) for $f=\frac{1}{5}$, but we have obtained similar results for $f=\frac{1}{3}$.

Our interpretation of the $f=\frac{2}{5}$ steps is shown in Fig. 3(b). As in Fig. 3(a), time advances downwards. Once again, our simulations suggest that the motion of the vortex lattice on the nth step is roughly n times faster than on the $n=1$ step. On each of the steps, the vorticity pattern is identical at the beginning and end of an ac cycle, except for a uniform displacement to the right. This displacement is (i) by *three* array lattaice constants to the right on the $n=1$ step, (ii) by *one* lattice constant on the $n=2$ step, (iii) by *four* lattice constants at $n=3$, and (iv) by *two* lattice constants at $n=4$. On the $n=5$ step, the vorticity pattern is identical at the beginning and end of each ac cycle. The tagged vortices are meant to suggest

how the various parts of the pattern move during a cycle.

We turn next to a discussion of the step widths in these Josephson arrays. In a single Josephson junction, it is well known that some of the Shapiro steps disappear at certain amplitudes of the ac driving currents.[20] A similar effect occurs in arrays. Figures 4(a) and 4(b) show the step widths and critical current as functions of I_{ac} for two different fields and a frequency $\nu=0.1(2eRI_c/\hbar)$, obtained numerically. At $f=0$, the oscillating behavior of the step widths and the critical current is identical to that of a single junction, as calculated by Russer[20] (note that the variable ξ defined by Russer is related to our variable ν by $\xi=2\pi\nu/\nu_0$). The minima of the critical current occur roughly at the maxima in the widths of the first step. All step widths are zero at $I_{ac}=0$. A similar oscillatory behavior is also found at other frequencies, as shown in Figs. 4(c) and 4(d). At $f=\frac{1}{2}$, the oscillatory patterns of step widths are compressed relative to $f=0$. The critical current is about $0.34I_c$ at $I_{ac}=0$. At small values of I_{ac}, the step widths seem to increase quadratically with I_c for $n=2$ rather than varying linearly with I_{ac} as they do for $n=1$. Note that the plots shown are interpolations of our calculated points, which are spaced about $0.25I_c$ apart.

Figures 5(a) and 5(b) show the current-voltage characteristics of a 10×10 array with periodic boundary conditions as a function of temperature at $f=0$ and $\frac{1}{2}$. On both sets of curves, the Shapiro steps "melt" with increasing temperature, beginning with the higher-order steps and proceeding to the lowest steps. At $f=0$, the last step has disappeared by a temperature of

$$0.4\hbar I_c/(k_Be)=0.8J/k_B\ ,$$

where $J=\hbar I_c/(2e)$ is the Josephson coupling energy. At $f=\frac{1}{2}$, the corresponding temperature is

$$0.2\hbar I_c/(k_Be)=0.4J/k_B\ .$$

These temperatures are near or slightly lower than the melting temperatures at *zero* applied current (as discussed, e.g., by Teitel and Jayaprakash[2]), which are $\approx0.95J/k_B$ and $\approx0.4J/k_B$. Since we are, in effect, calculating the melting temperatures in the presence of *finite* ac and dc currents, these finite-current melting temperatures cannot be higher than the zero-current values. Extrapolating from these numerical results, we anticipate

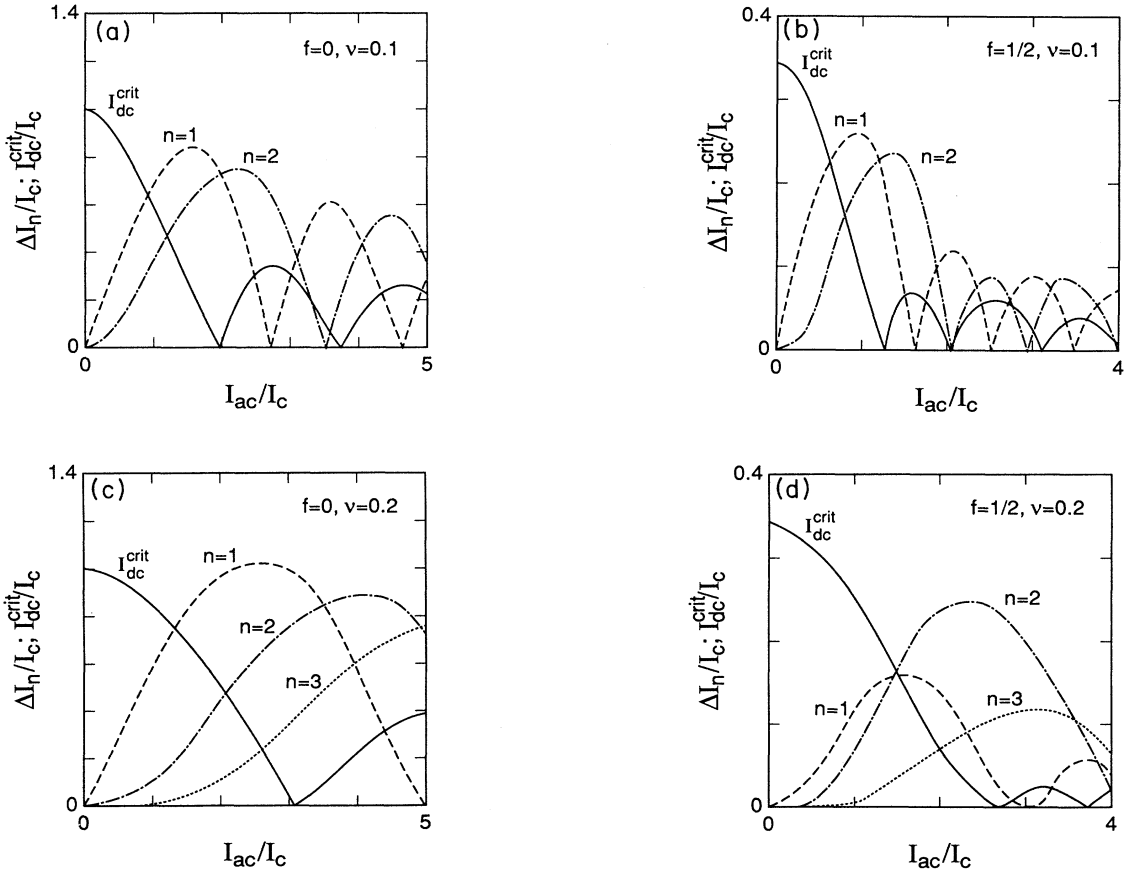

FIG. 4. Critical currents (I_{dc}^{crit}) and step widths ΔI_n computed as a function of I_{ac} at $\nu=0.1(2eRI_c/\hbar)$ and (a) $f=0$ and (b) $f=\frac{1}{2}$. (c) and (d) are the same as (a) and (b) but with $\nu=0.2(2eRI_c)/\hbar$.

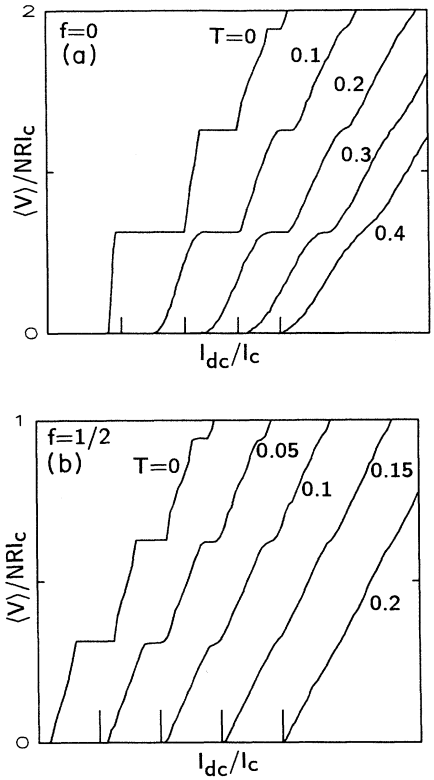

FIG. 5. Variation of step width with temperature in a 10×10 array at two different magnetic fields: (a) $f=0$ (no magnetic field) and (b) $f=\frac{1}{2}$ (one-half flux quantum per plaquette). All graphs are obtained using periodic boundary conditions in the lateral direction, uniform current injection at the top, uniform current removal at the bottom, and the ground-state phase configuration as the initial state. The ac driving current has amplitude $I_{ac}/I_c=1$ and frequency $\nu=0.1(2eRI_c/\hbar)$. The various plots of $\langle V\rangle$ are all displaced horizontally; the vertical marks on the horizontal axis denote the positions of zero current for the corresponding curve. Temperature in units of $\hbar I_c/(k_Be)$.

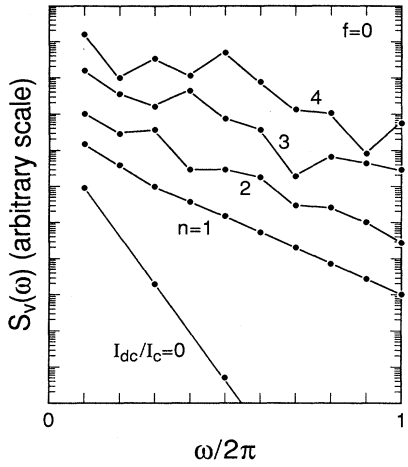

FIG. 6. Voltage power $S_V(\omega)$ in a 10×10 array at $f=0$ and several different values of I_{dc}. In all cases, $I_{ac}/I_c=1.0$, $\nu/\nu_0=0.1$, and temperature $T=0$. Shown are $I_{dc}=0$, $I_{dc}=I_c$ ($n=1$ step), $I_{dc}=1.5I_c$ ($n=2$ step), $I_{dc}=2.15I_c$ ($n=3$ step), and $I_{dc}=2.732I_c$ ($n=4$ step). The vertical scale is logarithmic but otherwise in arbitrary units; each large horizontal mark represents an increase of a factor of 10 from the one below it. The power spectra for successive values of n are displaced vertically by one decade relate to the previous step. Solid lines are merely to guide the eye. Only integer multiples of the fundamental appear in each case (for $n=0$, only odd integers appear). $\omega/(2\pi)$ is plotted in units of $2eRI_c/\hbar$.

that the fractional giant Shapiro steps at other values of the applied field will disappear at temperatures no higher than the corresponding vortex lattice melting temperature (i.e., the superconducting-normal transition temperature) in zero applied current.

We turn, finally, to a brief discussion of the power spectrum of the voltage on and off the Shapiro steps. As noted earlier and in previous papers, the voltage drop $V(t)$ across the array is periodic on the Shapiro steps, and aperiodic at other values of the current. This periodicity is reflected in the power spectrum $S_V(\omega)$ of the voltage, which is defined by the relation

$$S_V(\omega)=\left|\lim_{T_0\to\infty}\left[\frac{1}{T_0}\int_0^{+T_0}V(t)e^{i\omega t}dt\right]\right|^2. \tag{4}$$

Figures 6 and 7 show the power spectrum $S_V(\omega)$ for steps $n=0$, 1, 2, and 3 at $f=0$ and for $n=2-5$ at a representative finite field, $f=2/5$. We have studied a variety of *initial* conditions on the phases (phases parallel, phases in the ground-state configurations, and random initial phases), and, for the most part, obtain very similar power spectra after the initial transients have died out. Since the voltage is periodic on the steps, the power spectrum consists of a series of sharp harmonics that fall off roughly exponentially with increasing order m of the harmonic. At $I_{dc}=0$, only odd harmonics appear in the power spectrum. In Figs. 6 and 7, the points denote $S_V(m\omega)$, and the lines are merely to guide the eye and have no other

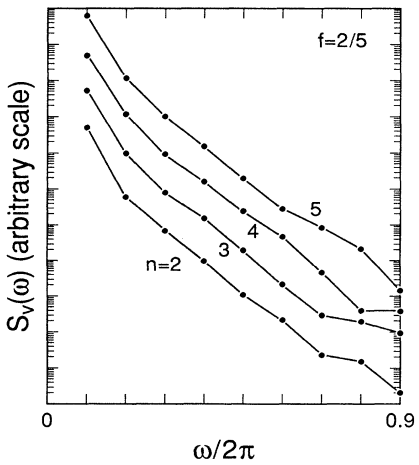

FIG. 7. Same as in Fig. 6, but for $f=\frac{2}{5}$ and currents corresponding to $n=2$ through 5 steps.

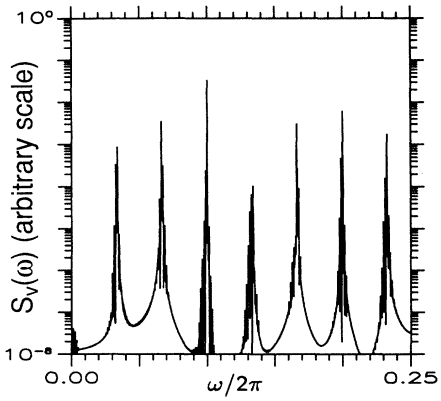

FIG. 8. Power spectrum at $f=0$ (logarithmic scale), as in Fig. 6, but for $I_{dc}=1.40I_c$ (between $n=1$ and 2 step). Since this power spectrum shows subharmonic structure, we give a continuous plot of $S_V(\omega)$.

significance. Figure 6 shows that, whereas the harmonics fall off monotonically with m for the $n=1$ step, they show an oscillatory dependence on m on the higher steps. For example, on the $n=2$ and 3 steps at $f=0$, $S_V(m\omega)$ has a secondary maximum at the second and third harmonic, respectively, with further oscillations at higher harmonics. This oscillatory behavior seems to be in qualitative agreement with the preliminary experimental results of Hebboul and Garland[21] for a 300×300 array. In the limit $I_{ac}/I_c \gg 1$, an analytic expression can be derived for the oscillatory dependence of the stepwidth of a *single junction* on the parameter I_{ac}/ν. This expression, given in the Appendix, gives an excellent fit to our calculations for *arrays* at $f=0$.

At dc currents off the steps, $V(t)$ is aperiodic and $S_V(\omega)$ is a quasicontinuous function that does not have period $1/\nu$. A representative power spectrum is shown in Fig. 8. The period tripling shown resembles behavior often seen in the power spectra of nonlinear dynamical variables. At other values of the voltage, we have found other multiples of the period (e.g., period octupling) as well as spectra that suggest the coexistence of sharp lines with a continuum.

IV. DISCUSSION

We have presented a numerical study of Shapiro steps in an array of resistively shunted Josephson junctions. We have verified that the time-dependent voltage is periodic on the steps, with many higher harmonics. The presence or absence of steps is sensitively affected by the transverse boundary conditions. The step widths are found to be oscillating functions of the amplitude and frequency of the ac driving currents.

At finite magnetic field of strength p/q flux quanta per plaquette, we find generally that *all* steps of the form $Nh\nu/(2eq)$ appear in the I-V characteristics. At all fields investigated, the vortex lattice is translated an integer number of plaquette lattice constants per cycle of the ac field. This seems consistent with the model of Benz *et al.*, as well as with a recent model of Shapiro steps due to Kvale and Hebboul,[22] which describes these steps in terms of only two degrees of freedom: a "vortex" and a "phase." At each magnetic field, the steps melt and disappear at temperatures no higher than the corresponding zero-current superconducting transition temperatures of the array.

Several groups[23] have suggested the possibility of anomalous half-integer steps of the form $\langle V\rangle = Nh\nu/(4e)$ at extremely low magnetic fields. We have no explanation for these anomalies, except to note that we see similar half-integer steps at $f=\frac{1}{5}$ with free boundary conditions. We may speculate that such anomalies may occur whenever the lattice is such that the vortex-pinning potential is significantly aperiodic. This will occur with free boundary conditions or with disorder in plaquette areas or coupling strengths. Another possibility is that even a periodic vortex-pinning potential will not be purely sinusoidal but will have higher harmonics. Such harmonics [analogous to terms of the form $\cos(2\phi)$ in the coupling energy of a single Josephson junction] produce subharmonic steps in single junctions. These speculative possibilities could be further investigated, if the anomalous half-integer steps are confirmed by further experiment.

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APPENDIX

The equation of motion for a single resistively shunted Josephson junction subjected to a current $I=I_{dc}+I_{ac}\sin(\omega t)$ can be expressed in the form

$$\frac{d\phi}{d\tau}=I_1+I_2\sin(z\tau)-\sin\phi\ , \tag{A1}$$

where $\tau=2eRI_c t/\hbar$ is a reduced time variable, and $z=\omega/(2eRI_c/\hbar)$ is a reduced frequency. I_1 and I_2 are the amplitudes of the applied dc and ac currents normalized to I_c, the critical current of the junction.

We obtain an approximate solution to Eq. (A1) in the limit of large I_2 ($I_2 \gg 1$), following the method of Kvale, Hebboul, and Garland.[24] Details of our derivation will be given elsewhere.[25] When $I_1=0$, the power spectrum is found to reside entirely in the odd harmonics and to take the form

$$S_V(m\omega)=\tfrac{1}{4}I_2^2\delta_{m,1}+J_m^2\left[\frac{I_2}{z}\right], \tag{A2}$$

where $J_m(I_2/z)$ is a Bessel function of order m, and $m=1, 3, 5, \ldots$. We have tested this form in our arrays at $f=0$ for the mth harmonics and $I_{ac}=200I_c$, and find that the power spectrum of the *array* is very well fitted by this prediction for a *single junction* for at least the first 50 odd harmonics.

In the limit $I_2 \gg 1$, Shapiro steps in a single junction occur at values of the dc current $I_1=nz$, with $n=1,2,3, \ldots$. As shown in Ref. 25, the power spectrum on the first step in this limit is

$$S_V^{(1)}(m\omega)=\begin{cases}\tfrac{1}{4}[I_2^2+(J_2-J_0)^2] & \text{for } m=1 \quad \text{(A3)}\\ \tfrac{1}{4}(J_{m+1}+J_{m-1})^2 & \text{for } m \text{ even and } \geq 2 \quad \text{(A4)}\\ \tfrac{1}{4}(J_{m+1}-J_{m-1})^2 & \text{for } m \text{ odd and } \geq 3\,. \quad \text{(A5)}\end{cases}$$

Likewise, for the second Shapiro step ($n=2$), we find that

$$S_V^{(2)}(m\omega)=\begin{cases}\tfrac{1}{4}[I_2^2+(J_1-J_3)^2] & \text{for } m=1 \quad \text{(A6)}\\ \tfrac{1}{4}(J_{m-2}-J_{m+2})^2 & \text{for } m \text{ even and } \geq 2 \quad \text{(A7)}\\ \tfrac{1}{4}(J_{m-2}+J_{m+2})^2 & \text{for } m \text{ odd and } \geq 3\,. \quad \text{(A8)}\end{cases}$$

In all cases, the argument of the Bessel functions is I_2/z.

We have compared these predictions for a *single* junction with the calculated power spectrum on the first and second step of a 10×10 array in the limit of large ac current. Agreement with the predictions is once again excellent over a range of at least 100 harmonics.

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