High-field transport in semiconductors. I. Absence of the intra-collisional-field effect

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(Received 29 August 1990)

We show that the Levinson (Zh. Eksp. Teor. Fiz. **57**, 660 (1969) [Sov. Phys.—JETP **30**, 362 (1970)]) quantum transport equation reduces to the semiclassical Boltzmann equation in the limit of a strong homogeneous electrostatic field. To arrive at this conclusion we use a saddle-point approximation which has not been used in this field before. This approximation is exact in the high-field limit. We show that even for a relatively moderate field strength of 3×10^5 V/m in GaAs (even before the onset of the Gunn effect, when 90% of the electrons are still in the central valley) the saddle-point approximation can be used for a majority of the scattering events. The results in this paper are supported by numerical calculations based on realistic scattering mechanisms in GaAs.

I. INTRODUCTION

The size of modern electronic devices is shrinking continually while applied voltages remain the same. This translates into very strong electric fields with rapid spatial variations; however, the problem of high-field quantum transport in strongly inhomogeneous conditions is far from being solved at present. The problem of high-field transport in uniform electric fields, on the other hand, is more tractable and some headway has been made recently. However, important questions remain unanswered: At what field strength do quantum effects become important and the validity of the semiclassical Boltzmann equation break down?

In this paper we present an answer to the above question for nondegenerate electrons in a single parabolic band of infinite bandwidth interacting weakly with phonons. Our results can easily be extended to nonparabolic bands and other forms of weak scattering mechanisms. A quantum transport equation (QTE) for uniform but strong electric fields and weak electron-phonon scattering already exists in the literature and is commonly known as the Levinson[1] or Barker and Ferry[2,3] equation. In this paper our aim is not to question the validity of the QTE but instead to use it to study quantum transport in the limit of extremely high fields.

The QTE differs from the semiclassical Boltzmann equation (BE) mainly because the scattering rates in this equation depend on the electric-field strength, an effect known as the intra-collisional-field effect. Because of the complicated nature of the QTE, it is difficult to make an estimate of the field strength above which this field effect becomes observable. Previous treatments of the QTE depend on an approximation known as the "completed-collision" approximation[4,5] and show that the intra-collisional-field effect is not important up to field strengths of the order of 10^6 V/m for electrons in a nondegenerate single band suffering optic-phonon scattering with parameters appropriate for GaAs. Note that this result guarantees the validity of the semiclassical BE up to this field strength but does not exclude its validity above this value. Recent results obtained from Monte Carlo simulations indicate that the results of the QTE are identical to the semiclassical BE even for extremely high fields.[6]

In this paper we provide a theoretical justification for the use of the BE at high field strengths. Specifically, we concentrate on the following two aspects of the derivation. (i) We start with the integral form of the QTE, but instead of making the *ad hoc* completed-collision approximation we study the asymptotic high-field limit of this equation by using a more controlled "saddle-point" approximation. This approximation is valid in the high-field limit and its validity becomes better as the field strength increases. (ii) We find that in the high-field limit the QTE reduces to the semiclassical BE within the same accuracy as the completed-collision approximation. Thus the high-field transport equation derived within the completed-collision approximation and the BE are equivalent. The main conclusion of this paper is that for nondegenerate electrons in a single band of infinite bandwidth interacting weakly with phonons under the action of a homogeneous stationary field the semiclassical BE is valid for any field strength.

The results of this paper are supported by numerical calculations. We show that even for a moderate field of strength 3×10^5 V/m (before the onset of the Gunn effect), transport in GaAs is in the high-field regime and standard approximations that lead to the semiclassical Fermi golden rule break down. Nevertheless, as mentioned above, application of the appropriate high-field approximations leads to results which are identical to the semiclassical results that would have been obtained had the Fermi golden rule been used.

Section II presents the numerical model used to support the theoretical results of this paper. In Sec. III we present a detailed analysis of field-dependent scattering rates. This section is meant to introduce the two distinct regimes of scattering, semiclassical and high field. It will be shown that the two regimes are not analytically connected and require completely different approximations. The saddle-point approximation is introduced in this section. Section IV briefly reviews the semiclassical BE in integral forms which are not commonly used in the literature. Some important results of Monte Carlo simulations of the BE are also presented to verify the assumptions used in this paper. Section V contains the derivation of the main results of this paper where we apply the saddle-point approximation to the QTE. In Sec. VI we show that the QTE reduces to the BE in the high-field limit. We also derive the transport equation of Ref. 4 where it was derived by a less controlled completed-collision approximation. Finally, Sec. VII contains the conclusions of this paper.

II. NUMERICAL MODEL

The theory in this paper will be supported by numerical calculations. We use a time-independent, spatially uniform "high" field of 3×10^5 V/m, multiple bands of GaAs (central and satellite), and realistic intraband and interband scattering events caused by polar-optic and acoustic phonons at room temperature. Figure 1 shows the isotropic total scattering-out rate $1/\tau(\mathbf{k})$ as a function of energy [defined by (2) below] used in the numerical calculations. The discontinuity at lower energies is due to the onset of optical-phonon emission and at higher energies because of the onset of interband scatterings.

Monte Carlo simulations of the semiclassical BE show that at this field strength, 90% of the electrons are in the central valley and the satellite valleys are almost empty; i.e., we are operating before the onset of the Gunn effect. Thus we are justified in using our theoretical results, which are derived for a single central band for this field strength in GaAs.

We shall show in this paper that for a field strength of 3×10^5 V/m we are already in the high-field regime of the QTE and not in the semiclassical regime. Thus one would suspect the validity of the semiclassical BE for field strengths even before the start of the Gunn-effect regime. Since the onset of the Gunn effect provides a possible experimental test of the validity of the BE, the field strength of 3×10^5 V/m is of special interest.

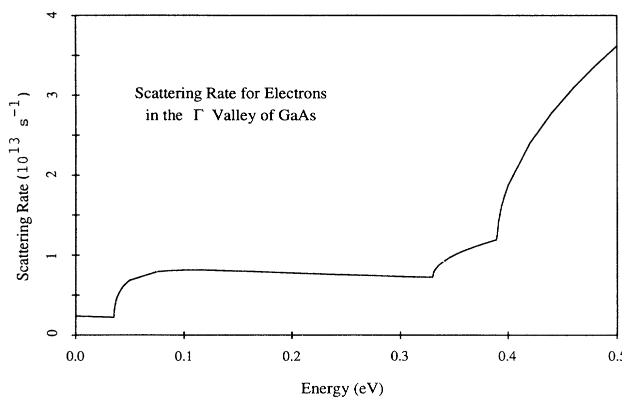

FIG. 1. Scattering rate for electrons in the Γ valley of GaAs including acoustic, polar-optic phonons, and interband scattering at room temperature. The discontinuity at lower energies is due to the onset of optical-phonon emission and at higher energies due to the onset of interband scattering.

III. EFFECT OF THE ELECTRIC FIELD ON SCATTERING, MOTIVATION FOR THE SADDLE-POINT APPROXIMATION

As an introduction to the intra-collisional-field effect and in order to get a physical feel for the associated time scales, we give a simple analysis of field-dependent scattering rates in this section. This section will illustrate the two distinct asymptotic regimes of the field-dependent scattering integral: the semiclassical regime and the extremely-high-field regime. We shall treat these limiting cases separately, recovering the Fermi golden rule in the semiclassical regime and motivating the introduction of the saddle-point approximation in the high-field regime. We want to emphasize that these field-dependent scattering rates are introduced for illustrative purposes only and are not derived from first principles. We do not use them in the QTE, which is derived independently from first principles. This section should serve as a convenient introduction to the methods of analysis used later to deal with the QTE whose time integrals will be only slightly different from the integrals treated here.

A. Equilibrium scattering rates

We introduce the probability $P^{\mathbf{F}}(\mathbf{k}-\mathbf{q}\rightarrow\mathbf{k})$ than an electron with momentum $\mathbf{k}-\mathbf{q}$ is scattered to momentum $\mathbf{k}$ (either by emitting a phonon of momentum $-\mathbf{q}$ or absorbing a phonon with momentum $\mathbf{q}$ and frequency $\omega_{\mathbf{q}}$) during a unit time interval in the presence of of an electric field $\mathbf{F}$. In the absence of the electric field, this probability evaluated within a self-consistent second-order perturbation approximation reads

$$P^{\mathbf{F}=0}(\mathbf{k}-\mathbf{q}\to\mathbf{k})=\frac{|M_{\mathbf{q}}|^2}{\hbar^2}\sum_{\eta=\pm}[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]2\,\mathrm{Re}\int_{-\infty}^{0}dt\,\exp\left[\frac{t}{\tau(\mathbf{k})}\right]\exp\left[\frac{i}{\hbar}[\varepsilon(\mathbf{k}-\mathbf{q})-\varepsilon(\mathbf{k})-\eta\hbar\omega_{\mathbf{q}}]t\right], \tag{1}$$

where $1/\tau(\mathbf{k})$ is the total scattering-out rate [i.e., $\tau(\mathbf{k})$ is the mean free time or the time an electron spends between two collisions],

$$\frac{1}{\tau(\mathbf{k})}=\frac{2\pi}{\hbar}\sum_{\mathbf{q},\eta}|M_{\mathbf{q}}|^2[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]\delta(\varepsilon(\mathbf{k}+\mathbf{q})-\varepsilon(\mathbf{k})+\eta\hbar\omega_{\mathbf{q}}), \tag{2}$$

and $N_{\mathbf{q}}$ is the Bose-Einstein statistics. Figure 1 contains a plot of $1/\tau(\mathbf{k})$ versus energy for GaAs including acoustic, polar-optic phonons, and interband scattering. The summation index $\eta=+1$ and -1 gives phonon emission and absorption, respectively.

B. Scattering rate in the presence of the electric field

According to the QTE, the momentum of an electron in the presence of an electrostatic field changes with time as $\mathbf{k}(t)=\mathbf{k}+e\mathbf{F}t$, where $\mathbf{k}$ is the initial momentum at time $t=0$. This statement can be precisely formulated within a vector potential description of the field $\mathbf{A}(t)=-\mathbf{F}t$. The energy $\varepsilon(\mathbf{k})$ becomes a time-dependent function due to the time dependence of the momentum which makes the scattering probaility $P^{\mathbf{F}}(\mathbf{k}-\mathbf{q}\to\mathbf{k})$ field dependent:[2]

$$P^{\mathbf{F}}(\mathbf{k}-\mathbf{q}\to\mathbf{k})=\frac{|M_{\mathbf{q}}|^2}{\hbar^2}\sum_{\eta=\pm}[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]2\,\mathrm{Re}\int_{-\infty}^{0}dt\,\exp\left[\frac{t}{\tau(\mathbf{k})}\right]\exp\left[\frac{i}{\hbar}\int_t^0 dt'[\varepsilon(\mathbf{k}(t')-\mathbf{q})-\varepsilon(\mathbf{k}(t'))-\eta\hbar\omega_{\mathbf{q}}]\right]. \tag{3}$$

For parabolic bands, one finds

$$\exp\left[\frac{i}{\hbar}\int_t^0 dt'[\varepsilon(\mathbf{k}(t')-\mathbf{q})-\varepsilon(\mathbf{k}(t'))-\eta\hbar\omega_{\mathbf{q}}]\right]=\exp\left[-\frac{i}{\hbar}[\varepsilon(\mathbf{k}-\mathbf{q})-\varepsilon(\mathbf{k})-\eta\hbar\omega_{\mathbf{q}}]t+i\frac{e\mathbf{F}\cdot\mathbf{q}}{2m\hbar}t^2\right]. \tag{4}$$

The effect of the electric field on the scattering probability (intra-collisional-field effect) is contained in the quadratic term [last term of (4)]. The importance of this field effect can be conveniently discussed in terms of a characteristic time scale τ_F associated with this term,

$$\tau_F=\left[\frac{m\hbar}{e|\mathbf{F}\cdot\mathbf{q}|}\right]^{1/2}. \tag{5}$$

We note that previous treatments of the field effect introduced an energy broadening ΔE. Here we find it more convenient to work with the time scale τ_F, which is related to the energy broadening via the relationship $\hbar/\tau_F=\Delta E$. In terms of τ_F, the field-dependent scattering rate (3) can be written as

$$P^{\mathbf{F}}(\mathbf{k}-\mathbf{q}\to\mathbf{k})=\frac{|M_{\mathbf{q}}|^2}{\hbar^2}\sum_{\eta=\pm}[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]2\,\mathrm{Re}\int_{-\infty}^{0}dt\,\exp\left[\frac{t}{\tau(\mathbf{k})}\right]\exp\left[-\frac{i}{\hbar}\Delta t\right]\exp\left[i\,\mathrm{sgn}(\mathbf{q}\cdot\mathbf{F})\frac{t^2}{2\tau_F^2}\right], \tag{6}$$

where

$$\Delta=\varepsilon(\mathbf{k}-\mathbf{q})-\varepsilon(\mathbf{k})-\eta\hbar\omega_{\mathbf{q}}. \tag{7}$$

The time integral in (6) contains three functions of time with characteristic time scales $\tau(\mathbf{k})$, τ_F, and $\hbar/\Delta$. We shall now show that the ratio τ_F/τ determines the crossover from one asymptotic region of (6) to another. On the one hand, when $\tau\ll\tau_F$, then the effect of the electric field on an electron during the time τ between two collisions is small and the scattering probability can be evaluated by assuming that the external field is a small perturbation on the zero-field result. On the other hand, when $\tau_F\ll\tau$, then the effect of the electric field on an electron between scattering events plays a dominant role. These two limiting cases are not analytically connected, and completely different approximations are valid in these two regimes as shown below. We now treat these two limits separately.

1. The semiclassical regime

We shall classify a scattering event to be in the semiclassical regime if the values of the momenta $\mathbf{k},\mathbf{q}$ and the electric field $\mathbf{F}$ are such that $\tau(\mathbf{k})\ll\tau_F$. Within the limits of time integration in (6) the function $\exp[t/\tau(\mathbf{k})]$ has an appreciable value only in the time interval of order $[-\tau(\mathbf{k}),0]$. Outside this region it decays to zero; in this interval the quadratic function which varies on a scale τ_F changes very little [since $\tau(\mathbf{k})\ll\tau_F$] and remains at a value near to unity, i.e., its value at $t=0$. Thus in this regime the magnitude of the lifetime sets the important scale, and the quadratic term due to the electric field plays a minor role. The problem can first be solved by treating the quadratic term in the electric field as a perturbation on the zero-field result. By setting the electric field equal to zero, the time integral in (6) becomes

$$2\,\mathrm{Re}\int_{-\infty}^{0} dt \exp\left[\frac{t}{\tau(\mathbf{k})}\right]\exp\left[-\frac{i}{\hbar}\Delta t\right]=\frac{2\hbar^2/\tau}{\Delta^2+\hbar^2/\tau^2}$$
$$\approx 2\pi\hbar\delta(\Delta)\ . \qquad (8)$$

Although the above equation leads to a well-known result, the Fermi golden rule, here we give a simplified discussion of the semiclassical regime to point out the difference between it and the high-field regime. Specifically we want to stress that one important characteristic of the low-field regime is that the **k** and **q** space is divided into two regions, one region (satisfying energy conservation) which contributes to the scattering integral while the other region contributes nothing. Depending on the values of the momenta **k** and **q** there are two cases. *Case (i),* $\hbar/\Delta<\tau(\mathbf{k})$. The time interval $[-\tau(\mathbf{k}),0]$ contains many periods of the oscillating term in (6) and the value of the integral is negligible. *Case (ii),* $\hbar/\Delta>\tau(\mathbf{k})$. The oscillating term varies little in the interval $[-\tau(\mathbf{k}),0]$ and the value of the time integral is finite. Thus the effect of the time integral is such that only those momenta **k** and **q** give finite scattering for which the energy difference Δ satisfies the relationship $\hbar/\Delta>\tau(\mathbf{k})$; i.e., in this regime the integral in (6) behaves like a broadened δ function in energy Δ with a broadening of $\hbar/\tau(\mathbf{k})$. We shall see below that in the high-field regime the momentum space cannot be divided in this way.

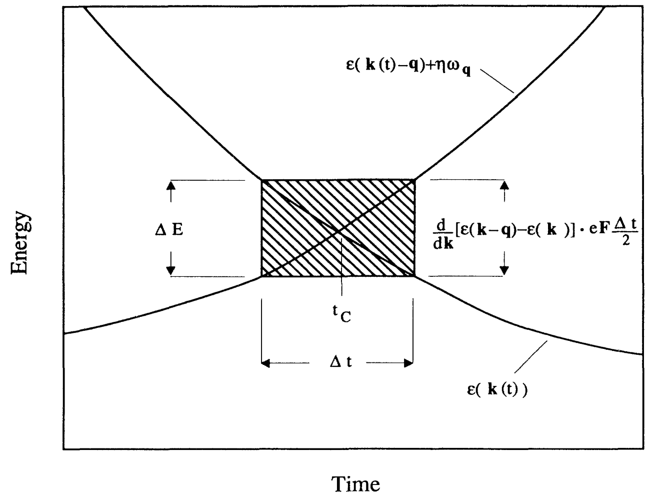

FIG. 2. A schematic representation of the time dependence of the initial energy state $\varepsilon(\mathbf{k}+e\mathbf{F}t)$ and the final energy state $\epsilon(\mathbf{k}-\mathbf{q}+e\mathbf{F}t)+\eta\omega_{\mathbf{q}}$. Both the energies coincide at time t_C and a transition is possible at this time in a semiclassical sense. The cross-hatched area represents the minimal area required by the uncertainty principle $\Delta E\,\Delta t=\hbar$. Since $\Delta E=\{\partial[\varepsilon(\mathbf{k}-\mathbf{q})-\varepsilon(\mathbf{k})]/\partial\mathbf{k}\}\cdot e\mathbf{F}\,\Delta t/2$, an application of this minimal principle directly gives $\Delta t=\tau_F$ defined in (5).

2. *The high-field regime*

In the high-field regime the values of the momenta **k**,**q** and the electric field **F** are such that $\tau_F \ll \tau(\mathbf{k})$. This condition says that for a scattering event to be in the high-field limit the transferred velocity in the direction of the field $v_T=\mathbf{q}\cdot\mathbf{F}/|\mathbf{F}|m$ has to satisfy the following condition:

$$v_T \gg \frac{\hbar}{e|\mathbf{F}|\tau^2(\mathbf{k})}\ . \qquad (9)$$

The functions which set the important time scales in this regime become more transparent if we rewrite the time integral in (6) as

$$\exp\left[-i\,\mathrm{sgn}(\mathbf{q}\cdot\mathbf{F})\frac{\Delta^2\tau_F^2}{2\hbar^2}\right]\int_{-\infty}^{0} dt \exp\left[\frac{t}{\tau(\mathbf{k})}\right]\exp\left[i\,\mathrm{sgn}(\mathbf{q}\cdot\mathbf{F})\frac{(t-t_C)^2}{2\tau_F^2}\right], \qquad (10)$$

where we have introduced an important time parameter t_C. Physically t_C is the time when the energy levels $\varepsilon(\mathbf{k}(t)-\mathbf{q})-\eta\hbar\omega_{\mathbf{q}}$ and $\varepsilon(\mathbf{k}(t))$ cross and we call it the "level-crossing time" defined by

$$\varepsilon(\mathbf{k}(t_C)-\mathbf{q})-\varepsilon(\mathbf{k}(t_C))-\eta\hbar\omega_{\mathbf{q}}=0\ . \qquad (11)$$

For parabolic bands we get

$$t_C=\mathrm{sgn}(\mathbf{q}\cdot\mathbf{F})\frac{\Delta}{\hbar}\tau_F^2\ . \qquad (12)$$

The physical relevance of the times t_C and τ_F is illustrated in Fig. 2.

The time integral (10) contains a positive definite exponentially decaying function for negative times and a strongly oscillating function. The exponentially decaying function has an appreciable value only in the time interval of order $[-\tau(\mathbf{k}),0]$; outside this region it decays to zero. The oscillatory function has a maximum period of oscillation of order τ_F for times t near t_C, and for times far from t_C the period decreases below this value. In the high-field regime, where $\tau_F \ll \tau(\mathbf{k})$, this function oscillates many times in the interval $[-\tau(\mathbf{k}),0]$ for any **k**. Thus we cannot separate the **k** and **q** space into [case (i)] and [case (ii)] as was done while discussing the semiclassical regime. Since this separation in the semiclassical limit gave us energy conservation, one might naively (and incorrectly) conclude that energy conservation is lost in the high-field regime. In the semiclassical regime this separation is possible because the energies of the final and initial states change very little over the lifetime τ and one can associate the time $t=0$ with the instant of scattering. In the high-field limit the time dependence of the initial and final states is very strong and one has to be careful in interpreting this time of scattering. There is nothing physically special about $t=0$ in the high-field limit; instead the time t_C is the only physically significant time associated with a given $\mathbf{k}$ and $\mathbf{q}$ via (11). If one interprets the time of transition as the time t_C, then energy conservation is satisfied, as can be seen from Eq. (11).

C. The saddle-point approximation

Now we are ready to simplify (10) by making the saddle-point approximation. The maximum contribution to the integral comes from the region where the integrand oscillates the least number of times, i.e., when t lies near the "saddle-point" t_C; for values of t away from t_C the integrand oscillates and the contribution to the integral is small. This suggests the saddle-point approximation, where one can replace the time t in the slowly varying exponentially decaying function by t_C, provided that the saddle point lies in the range of integration, i.e.,

$$\int_{-\infty}^{0} dt \exp\left[\frac{t}{\tau(\mathbf{k})}\right]\exp\left[i\,\mathrm{sgn}(\mathbf{q}\cdot\mathbf{F})\frac{(t-t_C)^2}{2\tau_F^2}\right]=\min\left\{\exp\left[\frac{t_C}{\tau(\mathbf{k})}\right],1\right\}\int_{-\infty}^{0} dt \exp\left[i\,\mathrm{sgn}(\mathbf{q}\cdot\mathbf{F})\frac{(t-t_C)^2}{2\tau_F^2}\right], \quad (13)$$

where the function $\min\{a,b\}$ is defined as the minimum of a or b. This function is introduced because for $t_C>0$ the saddle point t_C is outside the range of integration; thus t never achieves the value t_C and the integral on the left-hand side of (13) is negligible. The min{ } function on the right-hand side of (13) ensures that the integral on the right-hand side also becomes small when $t_C>0$. Without the presence of this term the right-hand side would diverge for positive t_C. Perhaps a more elegant way of writing the min{ } function in the above equation would be to use the following identity (which we shall use in the following sections):

$$\min\left\{\exp\left[\frac{t_C}{\tau(\mathbf{k})}\right],1\right\}=\exp\left[\theta(-t_C)\frac{t_C}{\tau(\mathbf{k})}\right]. \quad (14)$$

D. Error estimate of the saddle-point approximation

To give a more rigorous justification for the saddle-point approximation we define $R_\pm$ as the ratio of the exact integral [left-hand side of (13)] and its saddle-point approximation:

$$R_\pm=\left[\int_{-\infty}^{0} dt \exp\left[\frac{t}{\tau}\right]\exp\left[\pm i\frac{(t-t_C)^2}{2\tau_F^2}\right]\right]\Bigg/\left[\exp\left[\frac{t_C}{\tau}\right]\int_{-\infty}^{0} dt \exp\left[\pm i\frac{(t-t_C)^2}{2\tau_F^2}\right]\right]. \quad (15)$$

The integrals in this expression can be written in terms of the error function erf [ ]:

$$R_\pm=\frac{\exp(\pm i\tau_F^2/2\tau^2)\left\{\mathrm{erf}\left[\sqrt{\mp i/2}\left[\frac{t_C}{\tau_F}\pm i\frac{\tau_F}{\tau}\right]\right]-1\right\}}{\mathrm{erf}\left[\sqrt{\pm i/2}\frac{t_C}{\tau_F}\right]-1}, \quad (16)$$

where $\sqrt{\pm i}=(1\pm i)/\sqrt{2}$. This expression formally depends on the ratio τ_F/τ and t_C/τ_F; however, an analysis of this expression shows that in the high-field limit ($\tau_F/\tau<1$) the relevant parameter is in fact t_C/τ and not t_C/τ_F. To the lowest order in τ_F/τ we get for $t_C>0$

$$R_\pm\sim\exp(-t_C/\tau), \quad (17)$$

and for $t_C<0$ we get

$$R_\pm\sim 1+\frac{\exp(\pm it_C^2/2\tau_F^2)}{\sqrt{\mp 2\pi i}}\left[\frac{\tau_F}{t_C}\right][\exp(-t_C/\tau)-1]. \quad (18)$$

From this analysis we conclude that in the high-field limit the saddle-point approximation is valid for $|t_C/\tau|<1$. For $|t_C/\tau|>1$ this approximation is not valid. This is because for large values of t_C the integral on the left-hand side of (13) goes to zero algebraically while the saddle-point approximation becomes proportional to $\exp(t_C/\tau)$. Thus the saddle-point approximation exponentially underestimates this integral for negative t_C, and exponentially overestimates this integral for positive values of t_C. The underestimation for negative t_C does not cause a problem since the left-hand side of (13) is small anyhow in this regime, but for positive and large values of t_C the exponential overestimation has to be killed "by hand" as was done in (13).

Figure 3 shows a numerical example of this approximation; the figure contains plots of the real and imaginary parts of the left-hand and the right-hand sides of (13) as functions of t_C for the high-field limiting case $\tau_F/\tau(\mathbf{k})=\frac{1}{3}$. Although this value of $\frac{1}{3}$ does not represent the limiting case $\tau_F/\tau\ll 1$, we chose this value as representative for GaAs below the Gunn-effect regime (see Fig. 4 and discussion below). Numerical studies show that already at the higher-field value of $\tau_F/\tau(\mathbf{k})=0.1$ the two curves are indistinguishable.

E. Summary of the section

We have shown that the parameter $\tau_F/\tau(\mathbf{k})$ determines whether we are in the semiclassical or high-field regime. In the semiclassical regime $\tau_F/\tau(\mathbf{k})\gg 1$, we use the

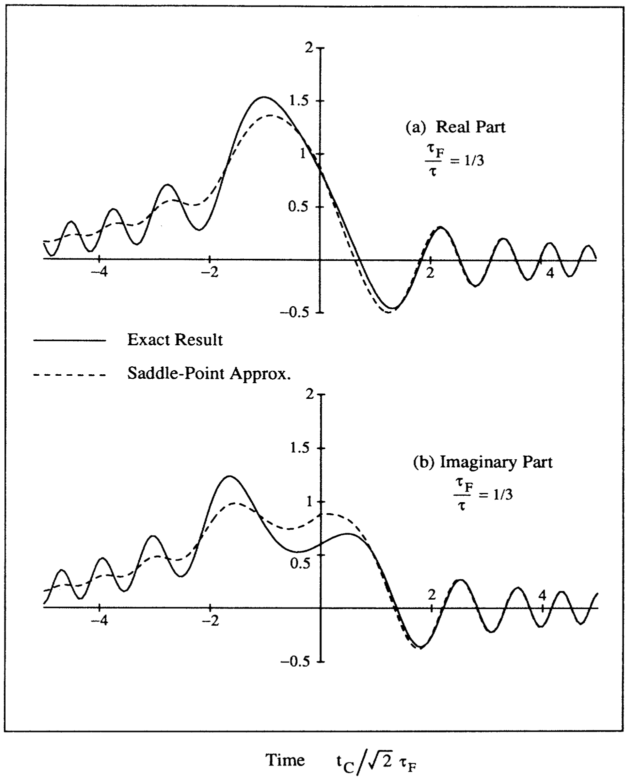

FIG. 3. Numerical demonstration of the accuracy of the saddle-point approximation. The figure contains plots of the real and imaginary parts of the left-hand (solid line) and right-hand (dashed line) side of (13) as a function of t_C. The right-hand side is the saddle-point approximation of the left-hand side. A value of $\tau_F/\tau=\frac{1}{3}$ appropriate for GaAs at a field strength of 3×10^5 V/m is used.

"standard" methods to deal with the transport equation; we shall not be concerned with this regime in this paper. In the high-field regime $[\tau_F/\tau(\mathbf{k})\ll 1]$ we can make use of the saddle-point approximation to simplify the integrals.

Notice that $\tau(\mathbf{k})$ depends on $\mathbf{k}$, the electron momentum before scattering, while τ_F [see Eq. (5)] depends on the electric-field strength as well as the magnitude of the transferred momentum $|\mathbf{F}\cdot\mathbf{q}|/|\mathbf{F}|$ in the direction of the field. Thus for a given electric-field strength there are some scattering processes for which the initial momentum and transferred momentum are such that the semiclassical condition $\tau_F/\tau(\mathbf{k})\gg 1$ is satisfied and some scattering processes for which the high-field condition $\tau_F/\tau(\mathbf{k})\ll 1$ is satisfied. Figure 4 shows how the region of scattering "phase space" is divided into semiclassical and high-field regions for GaAs at a field strength of 3×10^5 V/m, i.e., just before velocity saturation or the Gunn-effect regime. The figure contains contours of the ratio $\tau_F/\tau(\mathbf{k})$ where the x axis shows the electron initial velocity $\mathbf{k}/m$ and the y axis the magnitude of the transferred velocity $|\mathbf{F}\cdot\mathbf{q}|/|\mathbf{F}|m$ in the direction of the electric field; here m is the effective mass of an electron in the central valley of GaAs. The shaded area shows the semiclassical regime where the ratio is larger than one; scattering events in the rest of the figure is in the high-field limit. We see that for this field strength of 3×10^5 V/m most of the scattering phase space is in the high-field regime. For higher field strengths the high-field region would grow even more and the semiclassical region would shrink. Note that because of energy conservation some parts of the region shown in the figure are not accessible for scattering.

IV. THE SEMICLASSICAL BOLTZMANN EQUATION

The semiclassical BE plays two important roles in this paper. First, we shall show in this paper that the QTE can be expressed in various forms which are not strictly identical but are equivalent within the same order of τ_F/τ; one of these forms will turn out to be the semiclassical BE (22). Second, the BE provides us with estimates of the electronic distribution function which will be used to estimate the time scales within the QTE.

Since the integral form of the BE is not commonly used in the literature we first present this equation in its integral form. For stationary and homogeneous fields the BE can be written as

$$f(\mathbf{k})=\frac{2\pi}{\hbar}\sum_{\mathbf{q},\eta}|M_{\mathbf{q}}|^2[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]\int_{-\infty}^{0}dt\,f(\mathbf{k}-\mathbf{q}+e\mathbf{F}t)\times\exp\left[-\int_t^0 dt'\frac{1}{\tau(\mathbf{k}+e\mathbf{F}t')}\right]\delta(\varepsilon(\mathbf{k}-\mathbf{q}+e\mathbf{F}t)-\varepsilon(\mathbf{k}+e\mathbf{F}t)-\eta\hbar\omega_{\mathbf{q}})\ , \quad (19)$$

where $1/\tau(\mathbf{k})$ is the total scattering-out rate (2) and $f(\mathbf{k})$ is the distribution function. A derivation of this equation can be found (for example) in Ref. 7.

In Sec. III we introduce the level crossing time t_C, where we were able to replace t by t_C by making use of the saddle-point approximation in the high-field regime. This level-crossing time t_C can also be introduced in the semiclassical BE by using Eq. (11) to get

$$t_C-t=\frac{\tau_F^2}{\hbar}\mathrm{sgn}(\mathbf{q}\cdot\mathbf{F})[\varepsilon(\mathbf{k}-\mathbf{q}+e\mathbf{F}t)-\varepsilon(\mathbf{k}+e\mathbf{F}t)-\eta\hbar\omega_{\mathbf{q}}]\ . \quad (20)$$

This result can be used to convert the energy argument of the δ function into a time argument:

$$\delta(\varepsilon(\mathbf{k}-\mathbf{q}+e\mathbf{F}t)-\varepsilon(\mathbf{k}+e\mathbf{F}t)-\eta\hbar\omega_q)=\delta(t_C-t)\tau_F^2/\hbar\ . \quad (21)$$

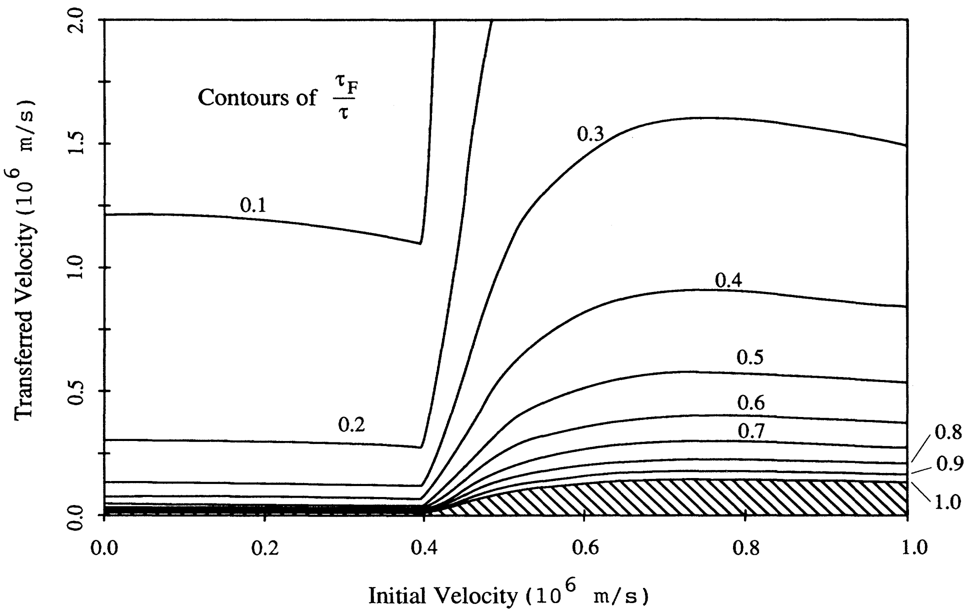

FIG. 4. Contour plots of the ratio τ_F/τ appropriate for GaAs at a field strength of 3×10^5 V/m. The x axis shows the electron initial velocity $\mathbf{k}/m$ before scattering and the y axis the magnitude of the transferred velocity (i.e., change in the velocity due to scattering) in the direction of the electric field. The cross-hatched area ($\tau_F/\tau>1$) shows scattering events which should be treated semiclassically. The rest of the area ($\tau_F\tau<1$) represents scattering events which should be treated in the high-field limit. Note that because of energy conservation some parts of the region shown in this figure are not accessible for scattering.

Substituting this result in (19) and explicitly integrating out the δ function turns the BE into

$$f(\mathbf{k})=2\pi\sum_{\mathbf{q},\eta}\frac{|M_{\mathbf{q}}|^2}{\hbar^2}[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]f(\mathbf{k}-\mathbf{q}+e\mathbf{F}t_C)\times\exp\left[-\int_{t_C}^{0}dt'\frac{1}{\tau(\mathbf{k}+e\mathbf{F}t')}\right]\times\theta(-t_C)\tau_F^2\ . \qquad (22)$$

In the next section we shall use the saddle-point approximation to simplify the QTE; there, the times t_C and τ_F will appear naturally as a consequence of the saddle-point approximation. The results obtained in later sections will be compared with this form of the BE.

A. Semiclassical estimate of the variation of the distribution function

The BE is usually derived under the assumption that the distribution function varies slowly along the free-particle trajectories. In our case this means the $f(\mathbf{k}+e\mathbf{F}t)$ varies slowly with time t. In particular we shall need a semiclassical estimate of the change in the distribution function $f(\mathbf{k})$ as the momentum changes from $\mathbf{k}$ to $\mathbf{k}+e\mathbf{F}\tau_F$ during a time interval equal to the characteristic time scale τ_F set by the electric field. For this estimate we use the differential form of the BE [obtained by taking the derivative of (19) with respect to $\mathbf{k}$], which gives

$$\left|e\mathbf{F}\tau_F\cdot\frac{\nabla f(\mathbf{k})}{f(\mathbf{k})}\right|=\left|\frac{\tau_F}{\tau(\mathbf{k})}-\frac{\tau_F(2\pi/\hbar)\sum_{\mathbf{q},\eta}|M_{\mathbf{q}}|^2[N_{\mathbf{q}}+\frac{1}{2}(1+\eta)]f(\mathbf{k}+\mathbf{q})\delta(\varepsilon(\mathbf{k}-\mathbf{q})-\varepsilon(\mathbf{k})-\eta\hbar\omega_{\mathbf{q}})}{f(\mathbf{k})}\right|\ . \qquad (23)$$

The first term on the right-hand side is due to scattering out and the second term is due to scattering in or backscattering. These two terms, which are typically of the same order of magnitude, contribute with opposite signs. Therefore, we assume that

$$\left|e\mathbf{F}\tau_F\cdot\frac{\nabla f(\mathbf{k})}{f(\mathbf{k})}\right|\le\frac{\tau_F}{\tau(\mathbf{k})} \qquad (24)$$

for values of $\mathbf{k}$ where $f(\mathbf{k})$ is appreciable.

We now support this assumption by numerical calculations. We have performed Monte Carlo simulations of the semiclassical BE (19) to obtain $f(\mathbf{k})$ at an electric-field strength of 3×10^5 V/m. Realistic scattering mechanisms (polar optic, acoustic, and interband scattering) as well as multiple bands (central and satellite) were used. The simulations were performed over 20 million real collisions to ensure convergence of the results. At this field strength, about 90% of the electrons turn out to be in the central valley and only about 10% in the satellite valley (i.e., regime just before the onset of the Gunn effect and velocity saturation). Thus we are justified in using these numerical results to support our theoretical claims which are based on a single parabolic band.

For a homogeneous effect field the distribution function depends on the momentum in the direction of the electric field $k_{\parallel}$, and perpendicular to the field $k_{\perp}$. Figure 5 contains 3D plots of the function $e\mathbf{F}\tau(\mathbf{k})\cdot\nabla f(\mathbf{k})/f(\mathbf{k})$ versus parallel velocity $v_{\parallel}=k_{\parallel}/m$ and perpendicular velocity $v_{\perp}=k_{\perp}/m$ [note that the mean free time $\tau(\mathbf{k})$ is isotropic, i.e., it depends only on the magnitude of the momentum $\mathbf{k}$]. The electric field is directed in the direction of decreasing $v_{\parallel}$. The figure contains a range of velocites for which the distribution function has an appreciable value [specifically $f(\mathbf{k})\geq f_{\max}/100$, where $f_{\max}$ is the maximum value of the distribution function for all momenta]. The figures show that the magnitude of the plotted function is of order unity or less, which implies that (24) is satisfied by the results of our numerical simulations.

V. THE QUANTUM TRANSPORT EQUATION

The integral form of the QTE (Refs. 1–4 and 8) is the starting point of our treatment of the high-field regime. Equation (13) can be rewritten as

$$f(\mathbf{k})=\sum_{\mathbf{q},\eta}\frac{|M_{\mathbf{q}}|^2}{\hbar^2}[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]\int_{-\infty}^{0}dt_1\int_{-\infty}^{0}dt_2\exp[g(t_1,t_2)]\exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2}dt'[\varepsilon(\mathbf{k}-\mathbf{q}+e\mathbf{F}t')-\varepsilon(\mathbf{k}+e\mathbf{F}t')-\eta\hbar\omega_{\mathbf{q}}]\right], \tag{25}$$

where

$$g(t_1,t_2)=\ln f(\mathbf{k}-\mathbf{q}+e\mathbf{F}t_<)-\int_{t_<}^{0}dt'\frac{1}{\tau(\mathbf{k}+e\mathbf{F}t')}+\tfrac{1}{2}\int_{t_<}^{t_>}dt'\left[\frac{1}{\tau(\mathbf{k}+e\mathbf{F}t')}-\frac{1}{\tau(\mathbf{k}-\mathbf{q}+e\mathbf{F}t')}\right], \tag{26}$$

and $t_>$ and $t_<$ are the larger and smaller of t_1 and t_2. We have written this equation in a form so that comparison with (3) is more transparent. This equation is valid for nondegenerate electrons in a single, unbounded band interacting weakly with phonons in the presence of a static, homogeneous electric field. The reader should refer to Refs. 4 and 8 for details about the techniques and approximations involved in the derivation of this equation. In this paper we shall take the validity of this equation for granted and base the rest of the analysis on it.

The approach we follow in the rest of this section can be motivated by noticing that the QTE (25) has two integrations over time compared with none in the BE (22). These time integrations will be removed by making the saddle-point approximation.

A. Saddle-point approximation

To handle (25) in the high-field regime, $\tau_F \ll \tau(\mathbf{k})$, we follow the steps introduced in Sec. III and write this equation as

$$f(\mathbf{k})=\sum_{\mathbf{q},\eta}|M_{\mathbf{q}}|^2[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]\times\int_{-\infty}^{0}dt_1\int_{-\infty}^{0}dt_2\exp[g(t_1,t_2)]\exp\left[i\,\mathrm{sgn}(\mathbf{q}\cdot\mathbf{F})\frac{(t_1-t_C)^2}{2\tau_F^2}\right]\exp\left[-i\,\mathrm{sgn}(\mathbf{q}\cdot\mathbf{F})\frac{(t_2-t_C)^2}{2\tau_F^2}\right], \tag{27}$$

where τ_F is the time scale (5) introduced by the electric field and t_C is the level-crossing time (11). Note that no approximation has been made in deriving this equation from the original QTE (25).

Let us now consider one of the time integrals in (27), say, integration over t_1. This integral contains an oscillating function $\exp[i\,\mathrm{sgn}(\mathbf{q}\cdot\mathbf{F})(t_1-t_C)^2/2\tau_F^2]$ with a time scale τ_F and the function $\exp[g(t_1,t_2)]$ which, as we shall soon show, varies on a scale τ. This situation is exactly analogous to the time integration in (10) encountered in Sec. III. We will concentrate on momenta $\mathbf{k},\mathbf{q}$ which correspond to the high-field region and follow the same steps and arguments that we used in Sec. III to motivate the saddle-point approximation.

Let us now examine the scale of variation of $\exp[g(t_1,t_2)]$ as a function of t_1. Specifically we need to know the fractional change in this function as the time t_1 changes by an amount τ_F, i.e., an order-of-magnitude estimate of the quantity

$$\tau_F\frac{d}{dt_1}\exp[g(t_1,t_2)]/\exp[g(t_1,t_2)]=\tau_F\frac{d}{dt_1}g(t_1,t_2)\,. \tag{28}$$

For $t_1<t_2$ this quantity equals

$$\tau_F \frac{d}{dt_1} g(t_1,t_2) = \frac{\tau_F}{2\tau(\mathbf{k}-\mathbf{q}+e\mathbf{F}t_1)} + \frac{\tau_F}{2\tau(\mathbf{k}+e\mathbf{F}t_1)} + e\mathbf{F}\tau_F \cdot \frac{\nabla f(\mathbf{k}-\mathbf{q}+e\mathbf{F}t_1)}{f(\mathbf{k}-\mathbf{q}+e\mathbf{F}t_1)} , \quad (29)$$

and for $t_1 > t_2$ we have

$$\tau_F \frac{d}{dt_1} g(t_1,t_2) = \frac{\tau_F}{2\tau(\mathbf{k}+e\mathbf{F}t_1)} - \frac{\tau_F}{2\tau(\mathbf{k}-\mathbf{q}+e\mathbf{F}t_1)} . \quad (30)$$

The first two terms on the right-hand sides of both of the above equations are small because $\tau_F/\tau(\mathbf{k}) \ll 1$ by definition in the high-field regime. By making use of the semiclassical result (24) of the last section we see that the remaining term [third term on the right-hand side of (29)] is also very small in the high-field limit. We are justified in using these results based on the semiclassical BE because the main conclusion of this paper is that the QTE gives results which are identical to the semiclassical BE even in the high-field limit. Moreover, we use these results only as an order-of-magnitude estimate. Thus we conclude that $\exp[g(t_1,t_2)]$ as a function of t_1 varies on a scale τ. These results suggest that we can make the saddle-point approximation in the t_1 integral, i.e., replace t_1 by t_C in the function $g(t_1,t_2)$.

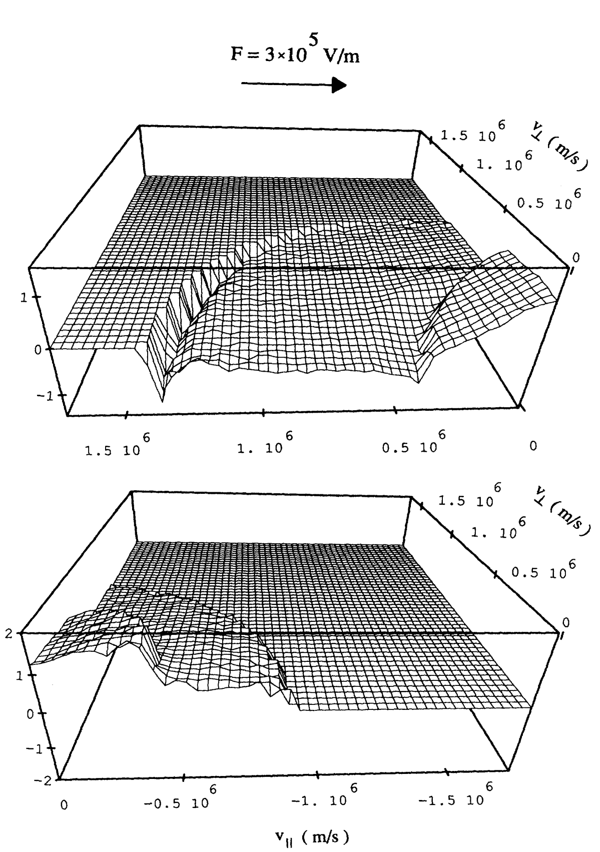

FIG. 5. Plot of the function $e\mathbf{F}\tau(\mathbf{k})\cdot\nabla f(\mathbf{k})/f(\mathbf{k})$ vs parallel velocity $v_\parallel = k_\parallel/m$ and perpendicular velocity $v_\perp = k_\perp/m$. The data for this plot are obtained from Monte Carlo simulations of the semiclassical Boltzmann equation at a field strength of 3×10^5 V/m for electrons in GaAs. The electric field is pointing in the direction of decreasing $v_\parallel$. The plotted function is of order 1 or less; this validates the assumption (24) made in the main body of this paper. In other words, this figure confirms that the fractional change in the distribution function $f(\mathbf{k})$ as the momentum changes from $\mathbf{k}$ to $\mathbf{k}+e\mathbf{F}\tau_F$ during a time interval equal to the characteristic time scale τ_F set by the electric field is small.

B. Analysis of the discontinuity in the integrand

One minor point is worth noting: Although the function $g(t_1,t_2)$ is continuous at the point $t_1=t_2$, its derivative is not, as can be seen from (29) and (30). However, this discontinuity is of order $1/\tau(\mathbf{k})$ and its effect is very small in the high-field limit when integrated with the strongly oscillating function; therefore, the saddle-point approximation can be used even in the presence of the discontinuity in the derivative of $g(t_1,t_2)$.

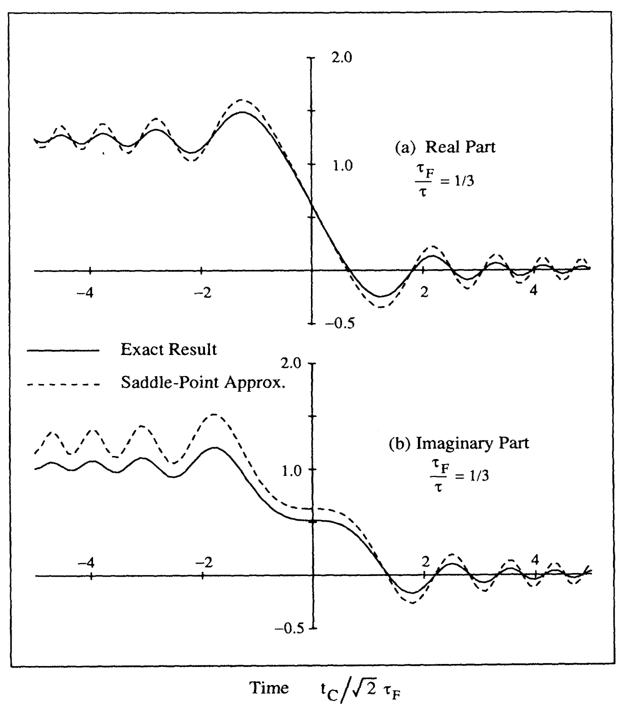

FIG. 6. Numerical demonstration of the accuracy of the saddle-point approximation with a discontinuity, as discussed in Sec. V. The figure contains plots of the real and imaginary parts of the numerator (solid line) and denominator (dashed line) of Eq. (31) as a function of t_C. The denominator is the saddle-point approximation of the numerator. A value of $\tau_F/\tau = \frac{1}{3}$ appropriate for GaAs at a field strength of 3×10^5 V/m is used.

Since the main contribution of the integral over t_2 comes from the region $t_2 \sim t_C$, we discuss the discontinuity at the time $t_2 = t_C$. We now justify this point more rigorously by using the function $\exp(-|t-t_C|/2\tau)$ as the slowly varying test function which varies on a scale τ and has a discontinuity in its derivative of order $1/\tau$. Specifically we shall test the accuracy of the saddle-point approximation by evaluating the ratio $R_\pm$ defined as

$$R_\pm = \left[\int_{-\infty}^{0} dt \exp \frac{-|t-t_C|}{2\tau} \exp \left[\pm i \frac{(t-t_C)^2}{2\tau_F^2} \right] \right] \Bigg/ \left[\int_{-\infty}^{0} dt \exp \left[\pm i \frac{(t-t_C)^2}{2\tau_F^2} \right] \right] . \tag{31}$$

Here the denominator is the saddle-point approximation to the integral in the numerator. We have placed the discontinuity in the derivative at time t_C, i.e., the saddle point, because the effect of this discontinuity is expected to be most pronounced at this point. Also, later in this section we shall make the saddle-point approximation for the t_2 integration in (27) and replace t_2 by t_C. Following the steps introduced in Sec. III we express (31) in terms of the error function erf[ ]:

$$R_\pm = \frac{\exp(\pm i\tau_F^2/8\tau^2) \left\{ 1 - \frac{t_C}{|t_C|} \operatorname{erf} \left[\sqrt{\mp i/2} \left[\frac{|t_C|}{\tau_F} \pm i \frac{\tau_F}{2\tau} \right] \right] + 2\theta(-t_C) \operatorname{erf} \left[\frac{\mp i \sqrt{\mp i}\, \tau_F}{2\sqrt{2}\tau} \right] \right\}}{1 - \operatorname{erf} \left[\sqrt{\pm i/2} \frac{t_C}{\tau_F} \right]} , \tag{32}$$

where $\sqrt{\pm i} = (1 \pm i)/\sqrt{2}$. To lowest order in τ_F/τ we get for $t_C > 0$

$$R_\pm \sim \exp(-t_C/2\tau) , \tag{33}$$

and for $t_C < 0$ we get

$$R_\pm \sim 1 + \frac{\exp(\pm i t_C^2/2\tau_F^2)}{\sqrt{\pm 2\pi i}} \left[\frac{\tau_F}{t_C} \right] [\exp(t_C/2\tau) - 1] . \tag{34}$$

Thus we conclude that in the high-field limit the saddle-point approximation is valid for all values of t_C except $t_C > \tau$ where this approximation overestimates the correct integral. This is because for large values of positive t_C the numerator of (31) becomes proportional to $\exp-(t_C/2\tau)$ while the denominator, i.e., the saddle-point approximation, goes to zero algebraically. This does not cause a problem for the application of the saddle-point approximation to the QTE since the actual integral and its saddle-point approximation are both small in this regime and contribute very little to the QTE.

Figure 6 demonstrates the accuracy of this approximation via a numerical calculation; the figure shows plots of the real and imaginary parts of the numerator and denominator of (31) as functions of t_C for the high-field limiting case $\tau_F/\tau(\mathbf{k}) = \frac{1}{3}$. This figure shows that the two functions are close to each other for all values of t_C. As mentioned before, although this value of $\frac{1}{3}$ does not represent the limiting case $\tau_F/\tau \ll 1$, we chose this value as representative for GaAs. Numerical studies show that for a value of $\tau_F/\tau(\mathbf{k}) = 0.1$ the two curves are already indistinguishable.

C. Transport equation within the saddle-point approximation

An identical argument holds for the t_2 integration in (27). Thus we can apply the saddle-point approximation to both the t_1 and t_2 integrations in (27), i.e., repalce $t_1 \to t_C$ and $t_2 \to t_C$ in the slowly varying function $\exp[g(t_1, t_2)]$ to get

$$f(\mathbf{k}) = \sum_{\mathbf{q},\eta} \frac{|M_\mathbf{q}|^2}{\hbar^2} [N_\mathbf{q} + \tfrac{1}{2}(1+\eta)] f(\mathbf{k} - \mathbf{q} + e\mathbf{F}t_C) \times \exp \left[-\int_{t_C}^{0} dt \frac{\theta(-t_C)}{\tau(\mathbf{k} + e\mathbf{F}t)} \right] \left| \int_{-\infty}^{0} dt \exp \left[i \operatorname{sgn}(\mathbf{q}\cdot\mathbf{F}) \frac{(t-t_C)^2}{2\tau_F^2} \right] \right|^2 . \tag{35}$$

The presence of the theta function $\theta(-t_C)$ in the exponential is to avoid the unphysical divergence in the region $t_C > 0$ introduced by the saddle-point approximation, as was demonstrated in the last section in Eqs. (13) and (14).

In order to make this equation more suitable for later discussion we introduce a dimensionless function

$$S(t) = \frac{1}{2\pi} \left| \int_{-\infty}^{t/\tau_F} dx \exp \left[i \frac{x^2}{2} \right] \right|^2 , \tag{36}$$

in terms of which (35) reads

$$f(\mathbf{k})=2\pi\sum_{\mathbf{q},\eta}\frac{|M_{\mathbf{q}}|^2}{\hbar^2}[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]f(\mathbf{k}-\mathbf{q}+e\mathbf{F}t_C)\times\exp\left[-\int_{t_C}^{0}dt\frac{\theta(-t_C)}{\tau(\mathbf{k}+e\mathbf{F}t)}\right]S(-t_C)\tau_F^2 \ . \tag{37}$$

This equation is the final form of the QTE after making the saddle-point approximation. At this point this equation does not seem to resemble any previously derived transport equation, but in the following section we shall show that to the lowest order in the parameter τ_F/τ this equation is equivalent to the high-field transport equation derived in Ref. 4 and also to the BE.

VI. DERIVATION OF THE BE FROM THE QTE

Before we proceed with the derivation of the BE we shall first outline the main step of the derivation. The necessary algebra will be filled in in the rest of this section.

First we notice that the QTE (37) differs from the BE (22) by only one term; i.e., if $S(-t_C)$ in the QTE could be replaced by $\theta(-t_C)$ then one would recover the semiclassical BE. Using Fresnel integrals it can be shown that $S(t)$ has the limits $S(-\infty)=0$ and $S(\infty)=1$. Between these limits this function varies on a scale τ_F; thus if the other functions in (37) vary slowly on the scale τ_F, then $S(t)$ can be replaced by $\theta(t)$ and one would recover the BE. However, we need to be careful in following this argument since the only integration in (37) is over the variable $\mathbf{q}$ while t_C is a function of $\mathbf{q}$, as can be seen from (11).

In order to follow the argument and be able to use the limit $\tau_F/\tau<1$ we must decouple the explicit $\mathbf{q}$ dependence of the integrand of (37) from t_C dependence. We accomplish this by introducing an additional time integration by defining $A(t)$:

$$S(-t_C)=\int_{-\infty}^{0}dt\ A(t-t_C) \tag{38}$$

and rewrite (37) as

$$f(\mathbf{k})=2\pi\int_{-\infty}^{0}dt\sum_{\mathbf{q},\eta}\frac{|M_{\mathbf{q}}|^2}{\hbar^2}[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]\times f(\mathbf{k}-\mathbf{q}+e\mathbf{F}t_C)\times\exp\left[-\int_{t_C}^{0}dt'\frac{\theta(-t_C)}{\tau(\mathbf{k}+e\mathbf{F}t')}\right]\times\tau_F^2A(t-t_C) \ . \tag{39}$$

The definition of the function $A(t)$ has an additional important property: When $A(t)$ is integrated with a function varying on a time scale larger than τ_F then $A(t)$ behaves like a delta function $\delta(t)$. Now we use this property of $A(t)$ in (39). As a function of t_C the fractional variation of $f(\mathbf{k}-\mathbf{q}+e\mathbf{F}t_C)$ on a time scale τ_F is of order τ_F/τ or larger, as was demonstrated in Sec. IV, Eq. (24). Here again we are justified in using a semiclassical result because we shall show that the QTE reduces to the semiclassical BE in the high-field limit. The fractional variation of the exponential term in (39) within this time is also of order τ_F/τ as a function of t_C. Since in the high-field limit $\tau_F/\tau<1$, both of these functions vary on a scale larger than τ_F and therefore we can (a) replace t_C in these equations by t and (b) replace $A(t-t_C)$ by a delta function $\delta(t-t_C)$ without changing the value of the integral in (39). We perform these replacements in two separate steps to connect this work to previous results in the literature.[4]

Replacement (a) gives

$$f(\mathbf{k})=2\pi\int_{-\infty}^{0}\sum_{\mathbf{q},\eta}|M_{\mathbf{q}}|^2[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]f(\mathbf{k}-\mathbf{q}+e\mathbf{F}t)\times\exp\left[-\int_{t}^{0}dt'\frac{1}{\tau(\mathbf{k}+e\mathbf{F}t')}\right]\times\tau_F^2A(t-t_C) \ , \tag{40}$$

which is the QTE derived in Ref. 4 [Eq. (4.12) of that reference] by using the *ad hoc* completed-collision approximation. Here this equation has been derived in a more controlled manner. Note that the theta function $\theta(-t)$ in the exponential decay term can now be replaced by one since t is always negative.

Performing step (b) turns this equation into

$$f(\mathbf{k})=2\pi\int_{-\infty}^{0}\sum_{\mathbf{q},\eta}|M_{\mathbf{q}}|^2[N_{\mathbf{q}}+\tfrac{1}{2}(1+\eta)]f(\mathbf{k}-\mathbf{q}+e\mathbf{F}t)\times\exp\left[-\int_{t}^{0}dt'\frac{1}{\tau(\mathbf{k}+e\mathbf{F}t')}\right]\times\tau_F^2\delta(t-t_C) \ . \tag{41}$$

It can be seen by using (21) that the above equation is in fact the semiclassical BE (19).

Therefore we conclude that in the high-field limit ($\tau_F\ll\tau$) the QTE reduces to the semiclassical BE. This result was demonstrated via Monte Carlo simulations in Ref. 6 for very high electric fields; here we have supplied an analytic proof. Thus in both the low-field limit (a case which we have not treated in this paper) and the high-field limit the QTE gives results identical to the BE. Therefore a reasonable assumption is that this should also be true for intermediate values of the electric field.

VII. CONCLUSIONS

In this paper we have studied the effect of the electric field on scattering in the Γ valley of GaAs below the onset of the Gunn effect. We have shown that scattering events with large momentum transfer (i.e., large-angle scattering which gives the most important contribution to resistivity) are already in the high-field regime even at a moderately low field strength of 3×10^5 V/m. By using a saddle-point approximation which is exact in the high-field limit we recover the scattering rates obtained earlier in Ref. 4 by using the *ad hoc* completed-collision approximation. We show that within the same accuracy (τ_F/τ) these rates are equal to those obtained from the Fermi golden rule in the absence of the electric field.

Although we have specifically treated GaAs before the onset of the Gunn effect, a more general theoretical conclusion of this paper is that the BE is valid for large homogeneous electrostatic fields and nondegenerate electrons in a single unbounded band interacting weakly with phonons.

ACKNOWLEDGMENTS

This work was supported in part by the Office of Naval Research.

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