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Exact solution of the London equation in two dimensions

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We point out that for many simple geometries, corresponding to coordinate systems in which the Laplacian and boundary conditions are separable, exact solutions of the London equation can be obtained. We then present the solution for a circular inclusion embedded in a homogeneous infinite medium. We illustrate how the magnetic field associated with a vortex is distorted by the boundary of the inclusion. We also calculate the self-induced Lorentz force the vortex feels because of the inhomogeneity and show that the vortex is attracted or repelled to the inclusion boundary depending on whether the penetration depth is larger or smaller than that of the host material.

I. INTRODUCTION

It is now a well-established fact that the coherence length ξ of the high-T_c superconductors is considerably smaller than their London penetration depth λ. This means that, unless the external field approaches the upper critical field, the details of the vortex core structure can be disregarded, and hence that the magnetic field distribution in the material can be studied by solving the (linear) London equation, rather than the (nonlinear) Ginzburg-Landau equations. The linear character of the London equation reduces the problem of finding the magnetic field distribution in the superconductor to that of finding the magnetic field distribution for a single vortex. In a homogeneous superconductor, this distribution is well known.[1,2] However, the general case of a spatially varying penetration depth is more complex, and, to the best of our knowledge, little has been done toward obtaining analytical solutions to the London equation. Various attempts have been made using perturbation techniques[2] or direct numerical integration.[3,4] The position dependence of the penetration depth could correspond in practice to a variation of the superconducting electronic density near a grain boundary, a twinning plane, or various other defects present in the superconductor. In this paper we point out that for many relatively simple, but useful, geometries an exact solution of the London equation can be obtained. These solutions will occur in any of the well-documented coordinate systems in which the Laplacian operator and the boundary conditions are separable.[5] As an illustration, we present the exact solution to the London equation in two dimensions for an infinite homogeneous superconducting sheet with a circular inclusion having a different penetration depth than the host material. Using this exact solution we then determine the Lorentz force exerted on a single vortex caused by the spatial inhomogeneity of λ. This simple geometry may be viewed as a simple model of an isolated grain or precipitate region for which the interface region is small compared to all other lengths. We present several calculations that show the distortion of the magnetic field caused by the interface and show the attraction or repulsion of the vortex to this region depending on whether the penetration depth of the inclusion is greater than or less than that of the host. The plan of the paper is as follows. Section II outlines the determination of the solution and gives the exact expressions for the magnetic field distribution. In Sec. III we discuss the solution and calculate various quantities such as the current density distribution and the force acting on a vortex for various configurations. We state our conclusions in Sec. IV.

II. SOLUTION OF THE LONDON EQUATION

In an inhomogeneous London superconductor, the superconducting electronic density is position dependent. With this dependence, the same arguments that lead to the definition of the penetration depth λ and to the original London equation lead to the definition of a position dependent penetration depth $\lambda(\mathbf{r})$ and to a slightly more general form for the London equation. The so-called modified London equation one obtains have been used previously by other authors,[2–4] and is easily derived from the Ginzburg-Landau equations in the London limit:

$$\nabla\times[\lambda^2(\mathbf{r})\nabla\times B(\mathbf{r})]+\mathbf{B}(\mathbf{r})=\Phi_0\delta(\mathbf{r}-\mathbf{r}_0)\hat{\mathbf{z}}\ , \qquad (1)$$

where Φ_0 is the flux quantum and $\mathbf{r}_0$ is the position of the vortex. The penetration depth for a circular defect is given by

$$\lambda(\mathbf{r})=\begin{cases}\lambda_1, & \text{if } r<R\ ,\\ \lambda_2, & \text{if } r>R\ ,\end{cases} \qquad (2)$$

where λ_1 and λ_2 are two arbitrary (positive) constants.

The vortex is located either inside or outside the circular "defect." Let us consider the case $r_0<R$, where it is inside. The calculations for $r_0>R$ are similar and only the final result will be given. In each region, $r<R$ or $r>R$, the modified London equation, Eq. (1), becomes

$$-\lambda_i^2\nabla^2\mathbf{B}+\mathbf{B}=\Phi_0\delta(\mathbf{r}-\mathbf{r}_0)\hat{\mathbf{z}},\quad i=1,2\ . \qquad (3)$$

When $r>R$ the right-hand side of Eq. (1) vanishes, the resulting equation is homogeneous and can be solved without difficulty by separation of variables. For $r<R$ however, this is not the case. One can circumvent the difficulty by using the following ansatz for the solution:

$$B_z(r,\phi)=\frac{\Phi_0 k_1^2}{2\pi}K_0(k_1|\mathbf{r}-\mathbf{r}_0|)+B_z'(r,\phi)\ , \qquad (4)$$

where $k_i^2=\lambda_i^{-2}, i=1,2$. The first term on the right-hand side of Eq. (4) is the solution corresponding to a vortex in a homogeneous medium characterized by a penetration depth equal to λ_1. K_0 is the modified Bessel function of the second kind of 0th order. Upon substitution of Eq. (4) into Eq. (1), one gets, in cylindrical coordinates, a homogeneous equation for B_z':

$$r^2\frac{\partial^2 B_z'}{\partial r^2}+r\frac{\partial B_z'}{\partial r}-k_1^2 r^2 B_z'+\frac{\partial^2 B_z'}{\partial \phi^2}=0\ , \qquad (5)$$

which can be solved by separation of variables. Simple symmetry considerations together with the requirements that B_z be finite at $r=0$ and vanish when $r\to\infty$ lead to the following general form for the solution:

$$B_z(r,\phi)=\begin{cases}\dfrac{\Phi_0}{2\pi\lambda_1^2}K_0(k_1|\mathbf{r}-\mathbf{r}_0|)+\displaystyle\sum_{n=0}^{+\infty}\alpha_n I_n(k_1 r)\cos n\phi, & \text{if } r<R\ ,\\ \displaystyle\sum_{n=0}^{+\infty}\beta_n K_n(k_2 r)\cos n\phi, & \text{if } r>R\ ,\end{cases} \qquad (6)$$

where I_n and K_n are the modified Bessel functions of the first and second kinds, respectively. The coefficients α_n and β_n are determined by using boundary conditions for B_z.

The first boundary condition follows directly from Maxwell's equations, and reflects the continuity of the flux density B_z at the boundary $r=R$:

$$\lim_{r\to R-} B_z(r,\phi)=\lim_{r\to R^+} B_z(r,\phi)\ . \qquad (7)$$

This relation holds for all ϕ's. The second boundary condition is related to the discontinuity of the tangential component of the current density at the interface:

$$\lim_{r\to R-}\lambda^2(\nabla\times\mathbf{B})_t=\lim_{r\to R+}\lambda_2^2(\nabla\times\mathbf{B})_t\ . \qquad (8)$$

This condition can be derived easily by integrating Eq. (1) along a small circuit intersecting the boundary. Equation (8) follows by using Stokes' theorem and the continuity of the normal component of **B** (required by Maxwell's equations).

It is rather easy to obtain a set of linear equations for the coefficients α_n and β_n from the above boundary conditions, provided $K_0(k_1|\mathbf{r}-\mathbf{r}_0|)$ is expanded in a trigonometric series by means of Graf's addition theorem:[6]

$$K_0(k_1|\mathbf{r}-\mathbf{r}_0|)=\begin{cases}I_0(k_1 r)K_0(k_1 r_0)+2\displaystyle\sum_{n=1}^{+\infty}I_n(k_1 r)K_n(k_1 r_0)\cos n\phi, & \text{if } r<r_0\ ,\\ I_0(k_1 r_0)K_0(k_1 r)+2\displaystyle\sum_{n=1}^{+\infty}I_n(k_1 r_0)K_n(k_1 r)\cos n\phi, & \text{if } r>r_0\ .\end{cases} \qquad (9)$$

Upon equating the coefficients and $\cos n\phi$ term by term, a set of linear equations for α_n and β_n is obtained, and after some straightforward algebra, one gets the following expressions for these coefficients:

$$\alpha_0=\frac{\Phi_0}{2\pi}k_1^2 I_0(k_1 r_0)\frac{k_2 K_1(k_1 R)K_0(k_2 R)-k_1 K_0(k_1 R)K_1(k_2 R)}{k_1 I_0(k_1 R)K_1(k_2 R)+k_2 I_1(k_1 R)K_0(k_2 R)}\ , \qquad (10)$$

$$\beta_0=\frac{\Phi_0}{2\pi}k_1^2 I_0(k_1 r_0)\frac{k_2[I_0(k_1 R)K_1(k_1 R)+I_1(k_1 R)K_0(k_1 R)]}{k_1 I_0(k_1 R)K_1(k_2 R)+k_2 I_1(k_1 R)K_0(k_2 R)}\ , \qquad (11)$$

and for $n\neq 0$,

$$\alpha_n=\frac{\Phi_0}{\pi}k_1^2 I_n(k_1 r_0)\frac{n\dfrac{k_2^2-k_1^2}{k_1 k_2 R}K_n(k_1 R)K_n(k_2 R)+k_2 K_n(k_2 R)K_{n-1}(k_1 R)-k_1 K_n(k_1 R)K_{n-1}(k_2 R)}{n\dfrac{k_1^2-k_2^2}{k_1 k_2 R}I_n(k_1 R)K_n(k_2 R)+k_1 I_n(k_1 R)K_{n-1}(k_2 R)+k_2 K_n(k_2 R)I_{n-1}(k_1 R)}\ , \qquad (12)$$

$$\beta_n=\frac{\Phi_0}{\pi}k_1^2 I_n(k_1 r_0)\frac{k_2[I_n(k_1 R)K_{n-1}(k_1 R)+K_n(k_1 R)I_{n-1}(k_1 R)]}{n\dfrac{k_1^2-k_2^2}{k_1 k_2 R}I_n(k_1 R)K_n(k_2 R)+k_1 I_n(k_1 R)K_{n-1}(k_2 R)+k_2 K_n(k_2 R)I_{n-1}(k_1 R)}\ . \qquad (13)$$

The calculation is similar when the vortex is located outside the circular region $r < R$. The general solution in this case is given by

$$B_z(r,\phi)=\begin{cases}\sum_{n=0}^{+\infty}\gamma_n I_n(k_1 r)\cos n\phi, & \text{if } r<R\ ,\\ \dfrac{\Phi_0}{2\pi\lambda_1^2}K_0(k_2|\mathbf{r}-\mathbf{r}_0|)+\sum_{n=0}^{+\infty}\delta_n K_n(k_2 r)\cos n\phi, & \text{if } r>R\ ,\end{cases} \tag{14}$$

The boundary conditions are the same as before and they determine the coefficients γ_n and δ_n appearing in Eq. (14):

$$\gamma_0=\frac{\Phi_0}{2\pi}k_2^2K_0(k_2r_0)\frac{k_1[I_0(k_2R)K_1(k_2R)+I_1(k_2R)K_0(k_2R)]}{k_1I_0(k_1R)K_1(k_2R)+k_2I_1(k_1R)K_0(k_2R)}\ , \tag{15}$$

$$\delta_0=\frac{\Phi_0}{2\pi}k_2^2K_0(k_2r_0)\frac{k_1I_0(k_1R)I_1(k_2R)-k_2I_1(k_1R)I_0(k_2R)}{k_1I_0(k_1R)K_1(k_2R)+k_2I_1(k_1R)K_0(k_2R)}\ , \tag{16}$$

and for $n\neq 0$,

$$\gamma_n=\frac{\Phi_0}{\pi}k_2^2K_n(k_2r_0)\frac{k_1[I_n(k_2R)K_{n-1}(k_2R)+K_n(k_2R)I_{n-1}(k_2R)]}{n\dfrac{k_1^2-k_2^2}{k_1k_2R}I_n(k_1R)K_n(k_2R)+k_1I_n(k_1R)K_{n-1}(k_2R)+k_2K_n(k_2R)I_{n-1}(k_1R)}\ , \tag{17}$$

$$\delta_n=\frac{\Phi_0}{\pi}k_2^2K_n(k_2r_0)\frac{n\dfrac{k_2^2-k_1^2}{k_1k_2R}I_n(k_1R)I_n(k_2R)+k_1I_n(k_1R)I_{n-1}(k_2R)-k_2I_n(k_2R)I_{n-1}(k_1R)}{n\dfrac{k_1^2-k_2^2}{k_1k_2R}I_n(k_1R)K_n(k_2R)+k_1I_n(k_1R)K_{n-1}(k_2R)+k_2K_n(k_2R)I_{n-1}(k_1R)}\ . \tag{18}$$

III. RESULTS

It is relatively easy to evaluate numerically the exact expressions for B_z, Eqs. (6) and (14). Figures 1 and 2 show various configurations of the field for several positions of the vortex when either $\lambda_1 > \lambda_2$ or $\lambda_2 > \lambda_1$. Comparing Figs. 1(a) and 2(a) one sees that the distortion of the field caused by the boundary is more pronounced when the vortex position, defect size, and penetration depth are comparably sized. In Figs. 1(b) and 2(b), one sees the significant distortion caused by proximity to the interface.

From B_z, various other quantities, such as the current density, can be calculated directly. Indeed, j is given by

$$\mathbf{j}=\frac{1}{\mu_0}\nabla\times\mathbf{B}\ , \tag{19}$$

and has both a radial and a tangential component, the former being nonzero because of the inhomogeneity of the system. The current density is the superposition of a "homogeneous" component originating from the K_0 term in Eqs. (6) and (14), and an "inhomogeneous" component directly related to the existence of the inhomogeneity in λ. Although strictly speaking the homogeneous component diverges at the vortex core in the London approximation, the actual current density vanishes at $r=r_0$. The inhomogeneous component, however, is finite in general, and gives rise to a Lorentz force acting on the vortex:

$$\mathbf{f}=\mathbf{j}_{\rm inh}(0)\times\boldsymbol{\phi}_0\ . \tag{20}$$

Regardless of whether the system is homogeneous or not, the radial part of j vanishes at the vortex site, and hence does not contribute to f. (Of course, the radial component vanishes in our simple geometry merely because of the cylindrical symmetry. In general, this is not true and the radial component of the current could contribute to the net force exerted on the vortex.) This magnetic pinning force is caused by the supercurrent flow of the vortex itself, because the inhomogeneity in λ has "disturbed" the supercurrent flow (with respect to what it would have been in a homogeneous system) with the consequence that there is now a nonzero current flowing at the vortex site.

For the case considered here, the current density at $r=r_0$ (i.e., at the vortex site) is azimuthal, so that the force is purely radial:

$$f_r=\begin{cases}\dfrac{\phi_0}{2\mu_0}k_1\sum_{n=0}^{+\infty}\alpha_n[I_{n-1}(k_1r_0)+I_{n+1}(k_1r_0)], & \text{if } r_0<R\ ,\\ -\dfrac{\phi_0}{2\mu_0}k_2\sum_{n=0}^{+\infty}\gamma_n[K_{n-1}(k_2r_0)+K_{n+1}(k_2r_0)], & \text{if } r_0>R\ ,\end{cases} \tag{21}$$

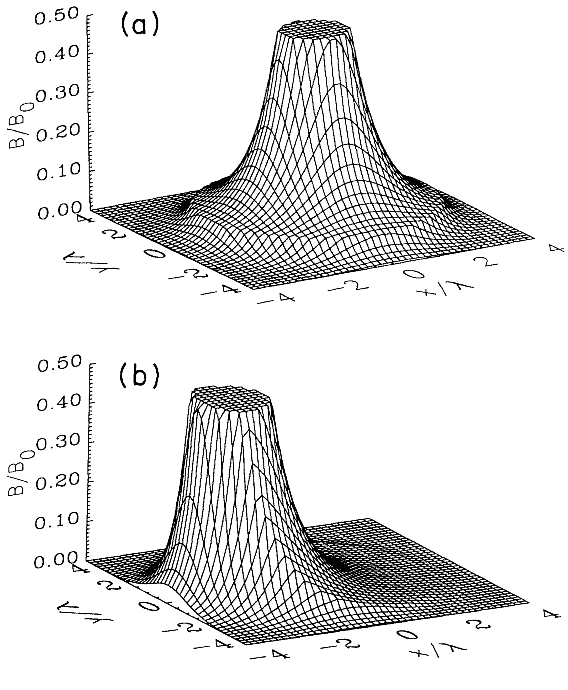

FIG. 1. Field distribution for various positions of the vortex, and $\lambda_1 > \lambda_2$, $(\lambda_1/\lambda_2 = 3)$. (a) $r_0 = 0$, (b) $r_0/R = 0.8$. In each case, the field is in units of $\Phi_0/(2\pi\lambda_1^2)$. The radius of the circular defect is equal to $3\lambda_1$.

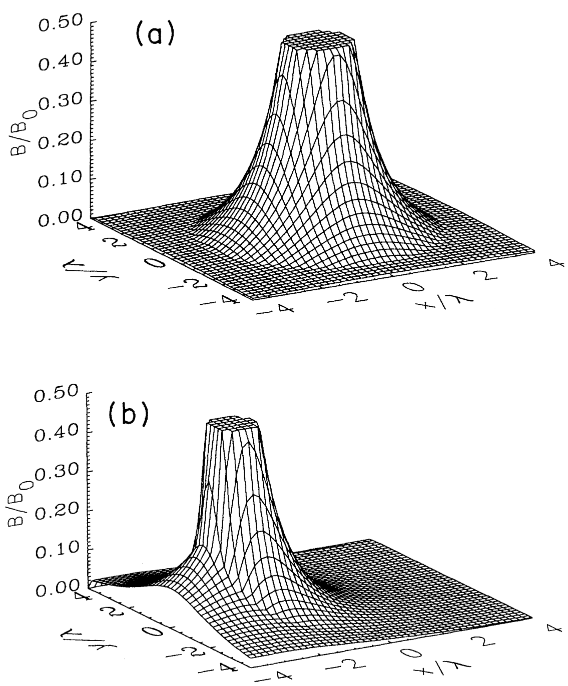

FIG. 2. Field distribution for various positions of the vortex, and $\lambda_1 < \lambda_2$, $(\lambda_1/\lambda_2 = \frac{1}{3})$. (a) $r_0 = 0$, (b) $r_0/R = 0.8$. In each case, the field is in units of $\Phi_0/(2\pi\lambda_1^2)$. The radius of the circular defect is equal to $3\lambda_1$.

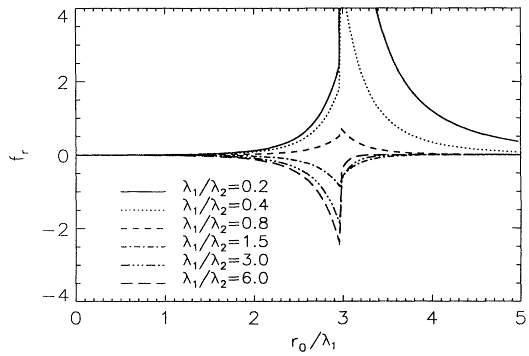

FIG. 3. Force acting on a single vortex as a function of its position with respect to the circular defect axis for different values of the ratio λ_1/λ_2. A positive force tends to move the vortex away from the axis. The force is units of $(1/\mu_0)(\Phi_0^2/2\pi\lambda_1^3)$. The radius of the circular defect is equal to $3\lambda_1$.

f_r can easily be evaluated numerically. The result is shown in Fig. 3 for various values of the ratio λ_1/λ_2. The sign of the force is such that the vortex is attracted toward the region corresponding to the largest penetration depth, i.e., to the more "normal" region of the sample. The force is large at the boundary, and decreases rapidly as one moves away from it. It is interesting to notice that when the penetration depth inside the inclusion is much larger than outside the defect, a vortex located outside the inclusion hardly "feels" its presence, unless the vortex is very close to the boundary. On the other hand, if the penetration depth outside the circular inclusion is much larger than inside the inclusion, there is a large repulsive force which tends to prevent a vortex located outside from penetrating in the more "superconducting" region. In this latter case, the vortex "feels" rather strongly the presence of the inclusion.

IV. CONCLUDING REMARKS

In this paper we pointed out that it is possible to solve exactly the London equation, which is an appropriate starting point for studying the phenomenological behavior of high-temperature superconductors, for the same geometries studied for classical field problems. With the vortices present, the solution is equivalent to finding the Green's function of the problem. We presented the solution for a particularly simple geometry, a circular inclusion embedded in an infinite, homogeneous host, and demonstrated that physically interesting and intuitively correct solutions are obtainable. In particular, we showed the importance of an interface region to the magnetic pinning of a vortex.

Other configurations having cylindrical symmetry could be investigated in a similar way. The simplest one besides a cylinder would be an annulus. The above expressions for the solution of the London equation remain identical except in the annular region where the I_n's and the K_n's appear in the expansion simultaneously. The boundary conditions are the same as before. This simple geometry could, for instance, model in a very simple way a grain of superconducting material embedded in another material, the annular region being the grain boundary.

Clearly, for other geometries and inhomogeneity models, exact solutions are obtainable in separable coordinate systems, and regions whose surfaces contour the coordinates of these systems. In the more general case, numeric and simulation methods are required.

Notice that because of the linearity of the London equation, the fields and pinning of several vortices can be found by superimposing the single vortex solution. The vortex configuration with the lowest energy is not determined by the London equation but would be determined by minimizing the configuration energy. Currently, we are studying such a problem, and plan to report our results elsewhere.

ACKNOWLEDGMENTS

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