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Conductivity of random resistor networks: An investigation of the accuracy of the effective-medium approximation

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We investigate the electrical conductivity of disordered random resistor networks. A systematic perturbative weak-disorder expansion of the conductivity is derived, in terms of the moments of the probability distribution of the random-bond conductances. This diagrammatic technique applies to any distribution of (possibly complex) conductances, and any regular lattice of arbitrary dimensionality d. Explicit quantitative results are given, up to the sixth order of perturbation theory, for (hyper)cubic lattices in all dimensionalities of physical interest, and compared with the predictions of the effective-medium approximation (EMA). The conductivity generically departs from the EMA formula at the fourth order of perturbation theory. On the square lattice, due to the duality symmetry, the EMA starts to be incorrect only at the fifth order. In all the situations considered, the discrepancy between the EMA prediction and the exact conductivity is affected by a very small numerical factor. The limit of a large dimensionality d is also investigated. The conductivity is shown to have a systematic $1/d$ expansion, three terms of which are given explicitly. The discrepancy with the EMA formula is again very weak, and starts with the $1/d^3$ terms. This study yields, therefore, a quantitative understanding of the currently observed fact that the EMA prediction is very accurate for a large class of conductance distributions.

I. INTRODUCTION

Electrical transport properties of disordered media are often modeled by random resistor networks. Neglecting the geometrical randomness of the material, one maps it onto a regular lattice, where the disorder only enters the distribution of the conductances living on the bonds of the lattice, considered as independent random variables. Of special interest is the case of binary composite media, such as the conductor-dielectric mixture, which exhibits a percolation transition. General reviews on these topics can be found in Refs. 1–6. More recently, random-network models have also been proposed to study the frequency-dependent (ac) conductivity and dielectric constant of binary systems, and other random mixtures. In this latter case, the bond conductances assume, in general, complex values. Reference 7 presents a recent review of ac electrical properties of disordered media.

One of the key issues is the determination of the macroscopic conductivity of the random network, as a function of the probability distribution of the bond conductances. Although very few analytical results are available, several efficient and accurate numerical approaches have been proposed, which will not be reviewed here. The present analysis concerns mostly the effective-medium approximation (EMA). This approach, by now over 50 years old,[8,9] is a self-consistent approximate scheme, which owes its success to both advantages of being easily applicable to many various situations, and its ability to yield quantitatively good predictions (see Ref. 10 for a review), except for conductance distributions that are too singular, such as, e.g., the binary mixture close to the percolation threshold.

In spite of the many successful applications of the EMA, the analytical study of its validity does not seem to have motivated much work so far. Kirkpatrick[1] argued in a qualitative way that the EMA formula is expected to be a very good approximation, since it only ceases to be exact, in the case of weakly disordered networks, at the fourth order of perturbation theory. This argument has then been confirmed on a quantitative basis by Bergman and Kantor,[11] in the case of a binary distribution of the bond conductances, on the three-dimensional cubic lattice, by means of a weak-disorder expansion of the conductivity, up to the seventh order of perturbation theory. A somewhat similar perturbative analysis has also been performed in a slightly different context, namely hopping models on random chains.[12,13] These models describe the diffusion of classical particles hopping in a disordered medium, characterized by static random transition rates between neighboring sites. In one dimension, the static diffusion coefficient D_0 is known exactly, and the weak-disorder expansion of the frequency-dependent diffusion coefficient $D(z)$ can be recast into a systematic low-frequency expansion in powers of $z^{1/2}$, of the form $D(z)=D_0(1-a_1 z^{1/2}+a_2 z+\cdots)$. Here z denotes a dimensionless variable, proportional to the squared frequency. It turns out that the second correction coefficient a_2 is, in general, not correctly predicted by the EMA formula.

The goal of the present work is to investigate, from a very general viewpoint, the validity of the EMA prediction for the conductivity of random resistor networks, built from (hyper)cubic lattices in any dimension d, and

with arbitrary (possibly complex) conductance distributions. We aim, in particular, at a systematic comparison between the predictions of the EMA formula and of the perturbative weak-disorder expansion, thus extending the results of Ref. 11, which concerned only the binary distribution for $d=3$.

The content of this article is as follows. Section II presents some general formalism about circuit equations in random resistor networks, in a form adapted to derive a weak-disorder expansion. The diagrammatic expansion of the conductivity is described in Sec. III, whereas Sec. IV is devoted to a general comparison between the EMA prediction and a perturbative weak-disorder results. We show, in particular, how the EMA formula amounts to performing a resummation of one-impurity effects, often referred to as a T-matrix resumation. In Sec. V we discuss, from a more quantitative viewpoint, several specific situations, namely the linear chain, where the conductivity is known exactly; the square lattice, where the consequences of the duality symmetry are discussed; the cubic lattice; and finally the limit of a large dimensionality. A short summary is presented in Sec. VI, where the rather peculiar case of weakly diluted lattices is also discussed.

II. GENERAL FORMALISM

In this section we present some general formalism about circuit equations, which will be used later to derive the perturbative weak-disorder expansion of the conductivity of random resistor networks.

We denote by x the sites of the hypercubic lattice in any dimension d, and by e_i $(1\le i\le d)$ a basis of unit vectors. The lattice spacing is taken equal to unity. The bond joining the sites x and $x+e_i$ carries a random conductance $\sigma_i(x)$. These bond conductances are independent random (possible complex) quantities, drawn from some probability distribution $\rho(\sigma)d\sigma$. Let $V(x)$ be the electric potential (voltage) of the site x. The current intensity $I_i(x)$, flowing out of site x in the positive ith direction, is related to the voltage drop by Ohm's law,

$$I_i(x)=\sigma_i(x)[V(x+e_i)-V(x)] , \tag{2.1}$$

whereas the conservation of electric charge reads

$$\sum_{i=1}^{d}[I_i(x)-I_i(x-e_i)]=0 . \tag{2.2}$$

Thus we obtain the circuit equations for the potentials in the form

$$\sum_{i=1}^{d}\{\sigma_i(x)[V(x+e_i)-V(x)] +\sigma_i(x-e_i)[V(x-e_i)-V(x)]\}=0 . \tag{2.3}$$

In order for the problem to be well behaved, and in analogy with previous studies of diffusion problems on random lattices,[14,15] we restrict the problem for a while to a finite sample with length L lattice spacings in each direction, and periodic boundary conditions. This sample then has the topology of a torus and consists of L^d sites. We impose an arbitrary electric field E_i, and look for a solution to the circuit equations (2.3) of the form

$$V(x)=E\cdot x+W(x) \quad \text{with } E\cdot x=\sum_{i=1}^{d}E_i x_i , \tag{2.4}$$

where the disorder-induced potential fluctuations $W(x)$ have the same periodicity L in each direction as the random bond conductances. Since the electric potential is only determined up to an additive constant, we can impose the condition that the average of the fluctuations $W(x)$ over the sample vanishes: $\sum_x W(x)=0$.

The conductivity tensor Σ_{ij} of the finite sample is defined as the set of coefficients of the linear relation between the mean current intensities $\mathcal{J}_i$ and the applied electric field, namely

$$\mathcal{J}_i=\frac{1}{L^d}\sum_x I_i(x)=\sum_{j=1}^{d}\Sigma_{ij}E_j . \tag{2.5}$$

This conductivity tensor can be shown to be a self-averaging quantity in the thermodynamic limit. Namely, although Σ_{ij} varies from sample to sample as long as the size L is finite, it converges to a well-defined limit, with probability 1 as L goes to infinity. In the present situation, since the distribution of the microscopic bond conductances is isotropic, it can be argued that the conductivity tensor of the infinite random network is an isotropic quantity, of the form $\Sigma_{ij}=\Sigma\delta_{ij}$, where δ_{ij} is Kronecker's symbol. Throughout the following, Σ will denote the conductivity of the model in the thermodynamic limit.

In the simple (nonrandom) case of a uniform network, where each bond conductance takes the value σ, it can be easily checked that the function $W(x)$ vanishes identically in such a way that $I_i(x)=\sigma E_i$, implying $\Sigma=\sigma$. The conductivity of the network is just the common value of the bond conductances, with our choice of units where the lattice spacing is 1.

We now set more precisely the formalism which will be used in the perturbative weak-disorder analysis of the problem. We assume that the bond conductances take the form

$$\sigma_i(x)=\sigma_0[1+\epsilon_i(x)] , \tag{2.6}$$

where the dimensionless fluctuations $\epsilon_i(x)$ are considered as small and have zero average. In other terms, σ_0 is, by definition, the average of $\sigma_i(x)$ over the distribution $\rho(\sigma)d\sigma$ of the conductances.

We introduce the moments of the conductance fluctuations

$$\mu_k=\langle[\epsilon_i(x)]^k\rangle \quad (k\ge 1) , \tag{2.7}$$

where the average is taken over the distribution $\rho(\sigma)d\sigma$. We have thus, in particular, $\mu_1=0$. Our aim is to evaluate the conductivity Σ of the random network as a systematic series expansion in the moments μ_k $(k\ge 2)$.

With the above definitions, the key equation (2.3) can be recast as

$$-\Delta W(x)=\sum_{i=1}^{d} E_i[\epsilon_i(x)-\epsilon_i(x-e_i)] + \sum_{i=1}^{d} \{\epsilon_i(x)[W(x+e_i)-W(x)] + \epsilon_i(x-e_i)[W(x-e_i)-W(x)]\} , \tag{2.8}$$

with the notation

$$\Delta W(x)=\sum_{i=1}^{d} [W(x+e_i)+W(x-e_i)-2W(x)] \tag{2.9}$$

for the discrete Laplace operator on the lattice. The local current intensities take the form

$$I_i(x)=\sigma_0[1+\epsilon_i(x)][E_i+W(x+e_i)-W(x)] . \tag{2.10}$$

It turns out to be advantageous to rewrite the above equations in reciprocal space. We define the Fourier transform $\tilde{W}(q)$ of the function $W(x)$ through

$$\tilde{W}(q)=\sum_{x} e^{-iq\cdot x}W(x) . \tag{2.11}$$

The reciprocal formula reads

$$W(x)=\frac{1}{L^d}\sum_{q} e^{iq\cdot x}\tilde{W}(q) , \tag{2.12}$$

where the sum runs over the L^d discrete values $q_i=2\pi m_i/L$ of the wave vector q, with $1\le m_i\le L$. In the limit of a large sample, this normalized sum becomes an integral over the first Brillouin zone, namely

$$\frac{1}{L^d}\sum_{q}\rightarrow\int d\tilde{q} \quad \text{with } d\tilde{q}=\frac{dq_1\cdots dq_d}{(2\pi)^d} , \tag{2.13}$$

where each component q_i of the wave vector varies between 0 and 2π.

We define in a similar way the Fourier transforms $\tilde{I}_i(q)$ and $\tilde{\epsilon}_i(q)$. Equations (2.8)–(2.10) are then fully equivalent, in the limit of a large system size L, to

$$K(q)\tilde{W}(q)=\sum_{i=1}^{d}(1-e^{-iq_i})\left[E_i\tilde{\epsilon}_i(q)+\int d\tilde{p}\,\tilde{\epsilon}_i(p)(e^{i(q-p)_i}-1)\tilde{W}(q-p)\right] \tag{2.14}$$

and

$$\mathcal{J}_i=\Sigma E_i=\sigma_0\left[E_i+\frac{1}{L^d}\int d\tilde{q}\,\tilde{\epsilon}_i(-q)(e^{iq_i}-1)\tilde{W}(q)\right] \tag{2.15}$$

with the definitions

$$K_{ij}(q)=(1-e^{iq_i})(1-e^{-iq_j}) =4e^{i(q_i-q_j)/2}\sin\left[\frac{q_i}{2}\right]\sin\left[\frac{q_j}{2}\right] , \tag{2.16}$$

$$K(q)=\sum_{i=1}^{d}K_{ii}(q)=4\sum_{i=1}^{d}\sin^2\frac{q_i}{2}=2\sum_{i=1}^{d}(1-\cos q_i) .$$

III. PERTURBATIVE WEAK-DISORDER EXPANSION

This section is devoted to the derivation of the weak-disorder expansion of the conductivity Σ of random resistor networks, in terms of the moments μ_k of the distribution of the bond-conductance fluctuations, up to the sixth order of perturbative theory, using the general formalism exposed in the preceding section.

The mechanism underlying this expansion is the structure of Eq. (2.14), which reads symbolically $\tilde{W}=\tilde{\epsilon}+\tilde{\epsilon}_*\tilde{W}$, hiding integrals and prefactors in a convolution-like operation, denoted symbolically by an $*$. This implicit equation can be solved iteratively, yielding $\tilde{W}=\tilde{\epsilon}+\tilde{\epsilon}_*\tilde{\epsilon}+\tilde{\epsilon}_*\tilde{\epsilon}_*\tilde{\epsilon}+\cdots$. We obtain thus the solution $\tilde{W}(q)$ as an infinite power series in the function $\tilde{\epsilon}(p)$, which involves more and more complex multiple integrals.

For the sake of clarity, we present first a very detailed calculation of the leading nontrivial term in the perturbative analysis, which turns out to bring a second-order contribution to the conductivity, proportional to μ_2.

This leading contribution originates in the term linear in $\tilde{\epsilon}_i(q)$, which can be directly read from the first term on the right-hand side of Eq. (2.14), namely

$$\tilde{W}(q)=\sum_{i=1}^{d}E_i\frac{1-e^{-iq_i}}{K(q)}\tilde{\epsilon}_i(q)+\cdots . \tag{3.1}$$

By inserting this expression into Eq. (2.15), we are left with

$$\mathcal{J}_i=\sigma_0\left[E_i-\frac{1}{L^d}\sum_{j=1}^{d}E_j\int d\tilde{q}\frac{K_{ij}(q)}{K(q)}\tilde{\epsilon}_i(-q)\tilde{\epsilon}_j(q)+\cdots\right] . \tag{3.2}$$

Let us consider for a while the quantity

$$S_{ij}(q)=\frac{1}{L^d}\tilde{\epsilon}_i(-q)\tilde{\epsilon}_j(q) =\frac{1}{L^d}\sum_{x,y}e^{iq\cdot(y-x)}\epsilon_i(x)\epsilon_j(y) . \tag{3.3}$$

Since the conductances of any two different bonds are statistically independent, the only terms which yield a nonzero contribution in the limit of a large sample correspond to both ϵ's being on the same bond, namely $x=y$

and $i=j$. We thus have the following limit behavior, with probability 1,

$$S_{ij}(q)\to\mu_2\delta_{ij} \quad \text{as } L\to\infty \tag{3.4}$$

for any value of the wave vector q, where δ_{ij} is again the Kronecker symbol. We are thus left with

$$\mathcal{J}_i=\sigma_0 E_i\left[1-\mu_2\int d\tilde{q}\frac{K_{ij}(q)}{K(q)}\right]. \tag{3.5}$$

For reasons of symmetry, the value of the integral is independent of the index i. This value is therefore $1/d$, in such a way that we obtain the following expression for the conductivity to the lowest nontrivial order:

$$\Sigma=\sigma_0\left[1-\frac{\mu_2}{d}+\cdots\right]. \tag{3.6}$$

This first correction term induced by the randomness is thus proportional to the variance $\Delta\sigma^2$ of the distribution of the conductances, since we have $\Delta\sigma^2=\langle\sigma^2\rangle-\sigma_0^2=\mu_2\sigma_0^2$.

The next terms in the weak-disorder expansion of the conductivity Σ can be obtained in a systematic way as follows. One derives from Eq. (2.14) an expression of $\tilde{W}(q)$ as a power series in $\tilde{\epsilon}_i(p)$, as explained above. Let us write explicitly the solution up to terms quadratic in $\tilde{\epsilon}_i(p)$

$$\tilde{W}(q)=\sum_{i=1}^{d}E_i\frac{1-e^{-iq_i}}{K(q)}\tilde{\epsilon}_i(q) - \sum_{i=1}^{d}\frac{1-e^{-iq_i}}{K(q)}\int d\tilde{p}\,\tilde{\epsilon}_i(p)\sum_{j=1}^{d}E_j\frac{K_{ij}(q-p)}{K(q-p)} \times\tilde{\epsilon}_j(q-p)+\cdots. \tag{3.7}$$

The conductivity is the obtained by inserting this expansion into Eq. (2.15). Let us notice that it is sufficient to expand the function $\tilde{W}(q)$ up to terms of order ϵ^{k-1} included, in order to obtain the conductivity up to terms of order ϵ^k.

The next step consists of evaluating the multiple sums of the random conductance fluctuations $\epsilon_i(x)$, which generalize the sum considered in Eq. (3.3). Since these random variables are independent and have zero mean, it is easy to realize for instance that the sum involving three ϵ's is proportional to μ_3, since the ϵ's are bound to be on the same bond, whereas the sum involving four ϵ's splits into a term in μ_4 (corresponding to the four ϵ's on the same bond) and a term in μ_2^2 (corresponding to two pairs of ϵ's on two different bonds).

It turns out that each contribution obtained through this procedure is advantageously represented by a *diagram,* analogous to those used, e.g., in Refs. 11 and 12. Figure 1 shows all the diagrams which contribute up to the fourth order of perturbation theory. Each dot along the solid line represents one function $\tilde{\epsilon}_i(p)$, which can be thought of as one elementary scattering of the electric potential with the disordered bond conductances. The dashed lines joining the dots show how the averaging over the random conductance fluctuations is performed. Any connected set of k dots joined by dashed lines corresponds to a factor of c_k, the kth cumulant of the distribution of $\epsilon_i(x)$. Let us recall here for further reference the relationship between the cumulants c_k and the moments μ_k of a probability distribution, up to $k=6$, in the particular case $\mu_1=0$:

$$c_2=\mu_2,\quad c_3=\mu_3,\quad c_4=\mu_4-3\mu_2^2, \qquad c_5=\mu_5-10\mu_2\mu_3,\quad c_6=\mu_6-15\mu_2\mu_4-10\mu_3^2+30\mu_2^3. \tag{3.8}$$

The present expansion shares all the common features of field-theoretical diagrammatic expansions. The contribution of each diagram takes the form of a multiple integral over several internal wave vectors, each of them running over the first Brillouin zone. The integrands in-

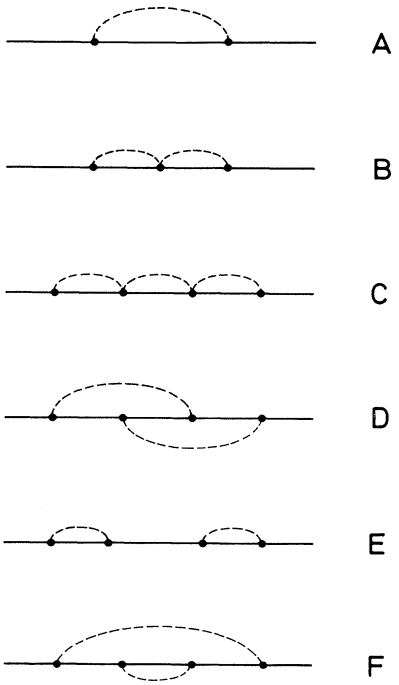

FIG. 1. The six diagrams which contribute to the weak-disorder expansion of the conductivity, up to the fourth order of perturbation theory. Their contributions are listed in Table I.

TABLE I. Contribution of the six diagrams shown in Fig. 1 to the weak-disorder expansion of the conductivity up to the fourth order of perturbation theory.

Term	Diagram label	Contribution
$-c_2/d$	A	1
c_3/d^2	B	1
$-c_4/d^3$	C	1
$-c_2^2/d^3$	D	I_1
	E	0
	F	d

volve products of the functions $K_{ij}(p)$ and $K(p)$, defined in Eq. (2.16), respectively, in their numerators and denominators. We prefer not to bother the reader with a detailed exposition of these technicalities. The contributions to the conductivity of the diagrams shown on Fig. 1 are listed in Table I. The actual contribution of a diagram is the product of two factors, namely (1) the product of cumulants and powers of the spatial dimension d given for each diagram in the "Term" column of Table I, and (2) the result of the integration over internal wave vectors, given in the "Contribution" column. The expression of the quantity I_1, which enters the contribution of the diagram labeled D, will be given in Eq. (3.9). It is worth noting that the diagram labeled E brings a vanishing contribution to the conductivity. More generally, it can be shown that only the "one-particle-irreducible" diagrams have a nonzero contribution. Those diagrams are defined by the property that they cannot be separated into two disconnected parts by cutting the solid line once.

In pursuing this diagrammatic expansion up to higher orders, one realizes soon that several different diagrams may have identical contributions. These *a priori* casual coincidences become evident when using another graphical representation of the various terms. We are thus led to introduce *graphs,* analogous to the Feynman graphs of quantum field theory. A graph is obtained from a diagram by curling it up, in such a way that all the dots which are connected by a given dashed line are brought to coincidence. The dashed lines vanish in this procedure, so that a graph is made of a single curved solid line, with two external legs. Any given graph corresponds to several diagrams, which contribute to the conductivity for equal amounts. In the following, the number of such contributions is called the *multiplicity* of the graph.

Figure 2 shows the 24 different one-particle-irreducible graphs that contribute a nonzero amount to the conductivity, up to the sixth order of perturbation theory. The contribution of each graph to the conductivity is given in Table II as the product of three factors, namely (1) the product of cumulants and powers of d given in the "Term" column, (2) the result of the integration over internal wave vectors, given in the "Contribution" column, and (3) the number of identical terms represented by the graph, given in the "multiplicity" column.

The contributions of these graphs involve four nontrivial numerical quantities, denoted I_1, I_2, J_1, and J_2, and called "lattice integrals" in the following. These numbers are actually the only ingredient of the weak-disorder expansion that depends explicitly on the geometry of the underlying lattice. They are defined as the following integrals over several wave vectors:

$$
\begin{aligned}
I_1 &= d^3 \int d\tilde{p} \int d\tilde{q} \int d\tilde{r}\, \tilde{\delta}(p+q+r) \frac{K_{ii}(p)}{K(p)} \frac{K_{ii}(q)}{K(q)} \frac{K_{ii}(r)}{K(r)} , \\
I_2 &= d^5 \int d\tilde{p} \int d\tilde{q} \int d\tilde{r} \int d\tilde{s} \int d\tilde{t}\, \tilde{\delta}(p+q+r+s+t) \frac{K_{ii}(p)}{K(p)} \frac{K_{ii}(q)}{K(q)} \frac{K_{ii}(r)}{K(r)} \frac{K_{ii}(s)}{K(s)} \frac{K_{ii}(t)}{K(t)} , \\
J_1 &= d^4 \sum_{j=1}^{d} \int d\tilde{p} \int d\tilde{q} \int d\tilde{r} \int d\tilde{s}\, \tilde{\delta}(p+q+r+s) \frac{K_{ij}(p)}{K(p)} \frac{K_{ij}(q)}{K(q)} \frac{K_{ij}(r)}{K(r)} \frac{K_{ij}(s)}{K(s)} , \\
J_2 &= d^5 \sum_{j=1}^{d} \int d\tilde{p} \int d\tilde{q} \int d\tilde{r} \int d\tilde{s} \int d\tilde{t}\, \tilde{\delta}(p+q+r) \tilde{\delta}(p-s-t) \frac{K_{ii}(p)}{K(p)} \frac{K_{ij}(q)}{K(q)} \frac{K_{ij}(r)}{K(r)} \frac{K_{ij}(s)}{K(s)} \frac{K_{ij}(t)}{K(t)} .
\end{aligned}
\tag{3.9}
$$

In these formulas, i is any fixed direction index ($1 \le i \le d$); the results do not depend on i, for obvious symmetry reasons. The functions $K(p)$ and $K_{ij}(p)$ have been defined in Eq. (2.16), and $\tilde{\delta}$ is a notation for the normalized Dirac δ function on the Brillouin zone:

$$
\tilde{\delta}(p) = (2\pi)^d \delta(p_1) \cdots \delta(p_d) = \sum_x e^{ip \cdot x} , \tag{3.10}
$$

where the summation runs over all the points x of the infinite d-dimensional hypercubic lattice.

In the following we will need, both for analytical and numerical purposes, an alternative representation in real space for the four lattice integrals I_1, I_2, J_1, and J_2.

Let us introduce first the lattice Green's function $G(x)$, defined by

$$
G(x) = \begin{cases} \int d\tilde{p} \dfrac{e^{ip \cdot x}}{K(p)} & \text{for } d > 2, \\ \int d\tilde{p} \dfrac{e^{ip \cdot x} - 1}{K(p)} & \text{for } d \le 2 . \end{cases} \tag{3.11}
$$

This function obeys the difference equation $-\Delta G(x) = \delta_{x,0}$ where Δ is the lattice Laplace operator defined in Eq. (2.9). We can thus interpret $G(x-y)$ as the kernel of the inverse of the operator $(-\Delta)$, hence the name "Green's function." Since the basic equation (2.8) involves the difference operator Δ, the Green's function $G(x)$ can be expected to play some part in a real-space reformulation of our results.

In any dimension $d > 2$, $G(x)$ is maximal for $x = 0$, and falls off for large x as the Green's function in the continu-

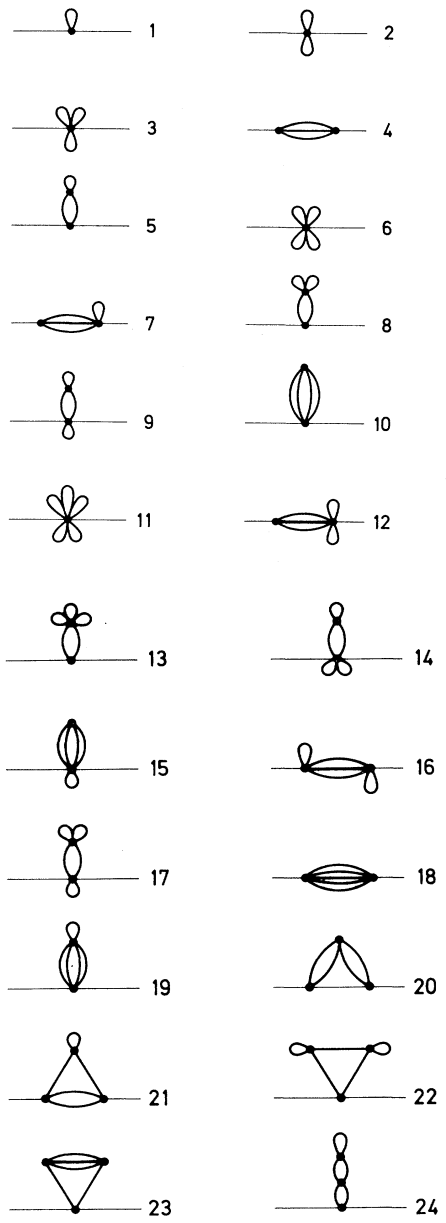

FIG. 2. The 24 one-particle-irreducible graphs which contribute to the weak-disorder expansion of the conductivity, up to the sixth order of perturbation theory. Their contributions are listed in Table II.

um, namely $G(x)\approx|x|^{2-d}/[(d-2)S_d]$, where S_d denotes the area of the unit sphere in d dimensions. For $d=1$ and 2, one subtraction has to be performed, as indicated in Eq. (3.11), in order to avoid a long-distance, or "infrared," divergence. One thus has $G(0)=0$. In one dimension, the Green's function can be evaluated exactly, namely $G(x)=-|x|/2$. In two dimensions, $G(x)$ grows logarithmically at large separations, as $G(x)\approx-\ln|x|/(2\pi)$.

We define then the following second-order differences:

$$G_{ij}(x)=G(x)-G(x+e_i)-G(x-e_j)+G(x+e_i-e_j)$$
$$=\int d\tilde{p}\, e^{ip\cdot x}\frac{K_{ij}(p)}{K(p)}\ . \tag{3.12}$$

The integral representation of the term on the second line of this equation holds in any dimension $d\geq 1$, and the $G_{ij}(x)$ always fall off as $|x|^{-d}$. By inserting into the definitions (3.9) of the lattice integrals the plane-wave representation (3.10) for the Dirac δ functions, we are left, after some elementary manipulations, with the following alternative definitions of I_1, I_2, J_1, and J_2 in real space:

$$\begin{aligned}
I_1&=d^3\sum_x[G_{ii}(x)]^3\ ,\\
I_2&=d^5\sum_x[G_{ii}(x)]^5\ ,\\
J_1&=d^4\sum_x\sum_{j=1}^{d}[G_{ij}(x)]^4\ ,\\
J_2&=d^5\sum_{x,y}\sum_{j=1}^{d}G_{ii}(x-y)[G_{ij}(x)]^2[G_{ij}(y)]^2\ .
\end{aligned} \tag{3.13}$$

These expressions will prove to be very useful in the following, both for analytical and numerical purposes.

We close this section with our main general result, namely the weak-disorder expansion of the conductivity Σ of an arbitrary random resistor network, up to the sixth order of perturbation theory. The following expression has been obtained by adding up the contributions, given in Table II, of the 24 graphs shown in Fig. 2. In terms of the moments μ_k, we have

$$\Sigma=\sigma_0(1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5+\Sigma_6+\cdots)\ \text{with}\ \Sigma_2=-\frac{\mu_2}{d},\ \Sigma_3=\frac{\mu_3}{d^2}\ ,$$
$$\Sigma_4=-\frac{1}{d^3}[\mu_4+(d-3+I_1)\mu_2^2]\ ,$$
$$\Sigma_5=\frac{1}{d^4}[\mu_5+(3d-10+4I_1+J_1)\mu_2\mu_3]\ , \tag{3.14}$$
$$\Sigma_6=-\frac{1}{d^5}\{\mu_6+(4d-15+6I_1+3J_1)\mu_2\mu_4+(2d-10+4I_1+2J_1+I_2)\mu_3^2$$
$$+[2d^2-12d+30+3(d-6)I_1+(d-9)J_1+4J_2]\mu_2^3\}\ .$$

IV. THE EFFECTIVE-MEDIUM APPROXIMATION

This section is devoted to a detailed comparison of the systematic weak-disorder expansion derived above and the effective-medium approximation (EMA). As stated in the Intoduction, the EMA is an approximate self-consistent scheme that was proposed long ago[1,8–10] to describe the conductivity of inhomogeneous materials.

Our first goal is to show how the EMA formula can be viewed as a self-consistent one-impurity resummation of the weak-disorder expansion exposed above. To do so, we go back to the general formalism of Sec. II. We split the random bond conductances according to

$$\sigma_i(x)=\Sigma^{\mathrm{EMA}}[1+\eta_i(x)] , \tag{4.1}$$

just as in the definition (2.6), but *without* the hypothesis that Σ^{EMA} is the average of $\sigma_i(x)$ with respect to its distribution $\rho(\sigma)d\sigma$. Under this circumstance, we can recast the central equations (2.3) and (2.5) into the following form, which is more appropriate for a real-space analysis. We denote by $F_i(x)$ the total voltage drops across the bonds,

$$F_i(x)=V(x+e_i)-V(x)=E_i+W(x+e_i)-W(x) . \tag{4.2}$$

TABLE II. Contribution of the 24 graphs shown in Fig. 2 to the weak-disorder expansion of the conductivity up to the sixth order of perturbation theory.

Term	Graph number	Contribution	Multiplicity
$-c_2/d$	1	1	1
c_3/d^2	2	1	1
$-c_4/d^3$	3	1	1
$-c_2^2/d^3$	4	I_1	1
	5	d	1
c_5/d^4	6	1	1
c_2c_3/d^4	7	I_1	4
	8	d	1
	9	d	2
	10	J_1	1
$-c_6/d^5$	11	1	1
$-c_2c_4/d^5$	12	I_1	6
	13	d	1
	14	d	3
	15	J_1	3
$-c_3^2/d^5$	16	I_1	4
	17	d	2
	18	I_2	1
	19	J_1	2
$-c_2^3/d^5$	20	J_2	4
	21	dI_1	3
	22	d^2	1
	23	dJ_1	1
	24	d^2	1

In terms of these quantities, the circuit equations (2.3) or (2.8) assume the form

$$F_i(x)=E_i-\sum_{y,j} G_{ij}(x-y)\eta_j(y)F_j(y) , \tag{4.3}$$

where $G_{ij}(x)$ denotes the combination of lattice Green's functions defined in Eq. (3.12). The mean current intensities read

$$\mathcal{J}_i=\frac{\Sigma^{\mathrm{EMA}}}{L^d}\sum_x [1+\eta_i(x)]F_i(x) . \tag{4.4}$$

In order to recover the weak-disorder expansion of the conductivity within the present formalism, one would just have to solve Eq. (4.3) iteratively, thus obtaining $F_i(x)$ as a power-series expansion in the $\eta_j(y)$, to insert this expansion into Eq. (4.4), and finally to average over the distribution of the $\eta_j(y)$. This approach, which is fully equivalent to that exposed in Sec. II, although less elegant, has the advantage of showing more explicitly the part played by each individual random bond conductance.

The EMA approach consists of taking into account—in the successive terms of the weak-disorder expansion—only the random part $\eta_i(x)$ of one single bond conductance. One thus approximates the quantity $F_i(x)$ that shows up in Eq. (4.4) by the solution $F_i^{\mathrm{EMA}}(x)$ of Eq. (4.3), obtained by neglecting all terms on the right-hand side of that equation, except for the term with $(y,j)=(x,i)$.

Since $G_{ii}(0)=1/d$—this quantity is nothing but the integral on the right-hand side of Eq. (3.5)—we obtain

$$F_i^{\mathrm{EMA}}(x)=\frac{d}{d+\eta_i(x)}E_i , \tag{4.5}$$

so that the conductivity of the network is given by

$$\Sigma=\Sigma^{\mathrm{EMA}}d\left\langle\frac{\sigma}{\sigma+(d-1)\Sigma^{\mathrm{EMA}}}\right\rangle \tag{4.6}$$

where the angular brackets denote an averaging with respect to the probability distribution $\rho(\sigma)d\sigma$. This approximate result for the conductivity Σ still depends on the choice of Σ^{EMA}. Recall that we have not assumed $\Sigma^{\mathrm{EMA}}=\langle\sigma_i(x)\rangle$. The usual assumption which leads to the EMA formula is the rather natural self-consistency requirement that $\Sigma=\Sigma^{\mathrm{EMA}}$. The standard EMA formula is thus recovered, in the form of the implicit equation

$$\left\langle\frac{\sigma-\Sigma^{\mathrm{EMA}}}{\sigma+(d-1)\Sigma^{\mathrm{EMA}}}\right\rangle=0 , \tag{4.7}$$

where the angular brackets again denote an averaging over the probability distribution $\rho(\sigma)d\sigma$.

In order to compare the EMA prediction for the conductivity with our perturbative results of Sec. III, we have to derive the weak-disorder expansion of the EMA prediction. To do so, we go back to the notation (2.6) for the bond conductances, with $\sigma_0=\langle\sigma_i(x)\rangle$, and we assume that both $\epsilon_i(x)$ and $\Sigma^{\mathrm{EMA}}-\sigma_0$ are small. We can thus determine recursively from the implicit formula (4.7) an expansion of the EMA conductivity, in terms of the moments μ_k. The result reads, up to the sixth order,

$$\Sigma^{\mathrm{EMA}}=\sigma_0(1+\Sigma_2^{\mathrm{EMA}}+\Sigma_3^{\mathrm{EMA}}+\Sigma_4^{\mathrm{EMA}}+\Sigma_5^{\mathrm{EMA}}+\Sigma_6^{\mathrm{EMA}}+\cdots) \text{ with } \Sigma_2^{\mathrm{EMA}}=-\frac{\mu_2}{d}\ ,\ \Sigma_3^{\mathrm{EMA}}=\frac{\mu_3}{d^2}\ ,$$

$$\Sigma_4^{\mathrm{EMA}}=-\frac{1}{d^3}[\mu_4+(d-2)\mu_2^2],\quad \Sigma_5^{\mathrm{EMA}}=\frac{1}{d^4}[\mu_5+(3d-5)\mu_2\mu_3]\ , \tag{4.8}$$

$$\Sigma_6^{\mathrm{EMA}}=-\frac{1}{d^5}[\mu_6+2(2d-3)\mu_2\mu_4+(2d-3)\mu_3^2+(2d^2-8d+7)\mu_2^3]\ .$$

Since we have derived the EMA formula along the lines of the general formalism introduced in Sec. II, we are able to point out, in a quantitative way, which approximation the EMA procedure corresponds to, on the contribution of each diagram in the perturbative expansion. Since the EMA theory only takes into account one-impurity effects, it is easily realized that it amounts to approximating the values of the lattice integrals defined in Eqs. (3.9) and (3.13), by the contribution of the single term in Eq. (3.13) where all bonds are identified, namely $x=0$ for I_1 and I_2, $(x=0,\ j=i)$ for J_1, and $(x=y=0,\ j=i)$ for J_2. Since we have $G_{ii}(0)=1/d$, as mentioned earlier in this section, the EMA approximation just corresponds to the values

$$I_1=I_2=J_1=J_2=1\ (\mathrm{EMA}) \tag{4.9}$$

of the four lattice integrals involved in the weak-disorder expansion of the conductivity. Indeed, it is easy to check that the expansion (4.8) of the EMA formula is recovered, for the hypercubic lattice in any dimension d, by inserting the EMA values (4.9) into the exact expansion (3.14).

A more quantitative comparison of the exact perturbative expansion and the EMA prediction will be presented in Sec. V, where the dimensionalities 1,2,3, and the $d\to\infty$ limit, will be considered successively. Let us just mention that the EMA prediction ceases to be exact, in general, at the fourth order of perturbation theory. Indeed, the term in μ_2^2 involves the lattice integral I_1, which is, in general, different from its EMA value, given in Eq. (4.9). One has therefore, in general, the following estimate,

$$\Sigma-\Sigma^{\mathrm{EMA}}\approx\frac{1}{d^3}(1-I_1)\sigma_0\mu_2^2\ , \tag{4.10}$$

for the leading correction to the EMA formula for the conductivity of an arbitrary, weakly disordered resistor network.

V. SPECIFIC RESULTS

In this section we discuss the general results presented in Secs. III and IV in several specific cases, namely the linear chain ($d=1$), where the conductivity is known exactly, the square lattice ($d=2$), for which the duality symmetry yields some constraints on the conductivity, the cubic lattice ($d=3$), and finally the limit of a large dimensionality ($d\to\infty$).

A. One dimension

The one-dimensional situation is that of a linear chain, a trivial geometry as far as random resistor networks are concerned. The general formalism of Sec. II is not needed in this case, since we know from elementary circuit theory that the resistance of several elements put in series is the sum of their individual resistances. The conductivity of the infinite random chain is therefore given by the exact formula

$$\Sigma=\frac{1}{\langle 1/\sigma\rangle}\ . \tag{5.1}$$

For weakly disordered random conductances of the form (2.6), the general result can be expanded as a power series in the moments μ_k. We thus obtain

$$\Sigma=\sigma_0[1-\mu_2+\mu_3+(-\mu_4+\mu_2^2)+(\mu_5-2\mu_2\mu_3)+(-\mu_6+2\mu_2\mu_4+\mu_3^2-\mu_2^3)+\cdots]\ . \tag{5.2}$$

The linear chain thus provides an explicit example, on which our general perturbative result (3.14) can be tested. To do so, we just have to evaluate the four lattice integrals involved in the diagrammatic expansion and defined in Eq. (3.9). Since the indices i,j can assume only one value, the integrands in Eq. (3.9) are identically equal to unity, and the measure $d\tilde{p}$ is normalized, so that we have

$$I_1=I_2=J_1=J_2=1\ (d=1)\ . \tag{5.3}$$

It is easily checked, once these values are inserted into Eq. (3.14), with $d=1$, that the expansion of the conductivity coincides with Eq. (5.2), as it should.

The exact value (5.1) for the conductivity of the one-dimensional chain is also correctly predicted by the EMA formula (4.7), for any distribution of the bond conductances. In particular, going back to the weak-disorder analysis, the values (5.3) of the lattice integrals for $d=1$ coincide with their EMA values (4.9).

B. Two dimensions

The two-dimensional case is that of the planar square lattice. This situation is also peculiar, because of the existence of the duality symmetry. Duality is a geometrical transform acting on two-dimensional graphs, defined as follows. To each planar graph G, one associates its dual graph $\tilde{G}$, obtained by drawing one dual bond across each bond of G, and matching these dual bonds in a consistent way, taking special care with boundaries (see, e.g., Ref. 16). The infinite square lattice is self-dual, i.e., dual to itself. This symmetry has many consequences concerning the physics of various lattice models. It was used long ago by Kramers and Wannier[17] to determine the critical temperature of the Ising model. Reference 18 contains a

complete review of the applications of duality to statistical mechanics.

Now consider graph G as an electrical circuit, with arbitrary conductances σ_l on its bonds labeled by l, and build a dual circuit $\tilde{G}$ by putting conductances $\tilde{\sigma}_{\tilde{l}}=1/\sigma_l$ on the bond $\tilde{l}$ dual to l. It can be shown that the conductances Σ and $\tilde{\Sigma}$ of both networks, with appropriate positions for the electrodes, are also reciprocal to each other:

$$\Sigma\tilde{\Sigma}=1 \ . \tag{5.4}$$

Straley[19] noticed that this identity still holds when Σ denotes the conductivity of an infinite square random resistor network, the bond conductances having any probability distribution $\rho(\sigma)$, and $\tilde{\Sigma}$ denotes the conductivity of the random network with the "dual" distribution of bond conductances, obtained from the initial one by changing σ to $1/\sigma$. This result has, in particular, remarkable consequences for binary distributions of the bond conductances, in connection with percolation theory (see Ref. 7 for a recent review).

In the present context, the duality symmetry imposes some constraints on the weak-disorder expansion derived in Sec. II, in the case of the square lattice. Indeed, the identity (5.4) has to be obeyed by the power-series expansion (3.14) of both conductivities Σ and $\tilde{\Sigma}$. Since the bond conductances σ of the original network are given by Eq. (2.6), we are led to set

$$\tilde{\sigma}=\frac{1}{\sigma}=\frac{1}{\sigma_0(1+\epsilon)}=\tilde{\sigma}_0(1+\tilde{\epsilon}) \ . \tag{5.5}$$

The condition $\langle\tilde{\epsilon}\rangle=0$ implies

$$\tilde{\sigma}_0=1/\sigma_0\left\langle\frac{1}{1+\epsilon}\right\rangle, \quad \tilde{\epsilon}=\frac{1}{\left\langle\frac{1}{1+\epsilon}\right\rangle(1+\epsilon)}-1 \ . \tag{5.6}$$

Equation (5.6) allows one to express the moments $\tilde{\mu}_k$ of the dual variable $\tilde{\epsilon}$ in terms of the μ_k, and to obtain thus the power series of $\tilde{\Sigma}$. If we now assume that the conductivity Σ admits the following power-series expansion,

$$\begin{aligned}\Sigma=\sigma_0[1&+a_2\mu_2+a_3\mu_3+(a_{41}\mu_4+a_{42}\mu_2^2)\\&+(a_{51}\mu_5+a_{52}\mu_2\mu_3)\\&+(a_{61}\mu_6+a_{62}\mu_2\mu_4+a_{63}\mu_3^2+a_{64}\mu_2^3)+\cdots] \ ,\end{aligned} \tag{5.7}$$

with unknown coefficients a_2, etc., a systematic expansion of the identity (5.4) yields expressions of the coefficients of the even orders in terms of those of the previous odd orders, namely

$$\begin{aligned}&a_2=-\tfrac{1}{2}, \quad a_{41}=\tfrac{1}{4}-\tfrac{3}{2}a_3 \ ,\\&a_{42}=-\tfrac{3}{8}+\tfrac{3}{2}a_3 \ ,\\&a_{61}=-\tfrac{1}{2}+\tfrac{5}{2}a_3-\tfrac{5}{2}a_{51} \ ,\\&a_{62}=\tfrac{11}{8}-6a_3+\tfrac{5}{2}a_{51}-\tfrac{3}{2}a_{52} \ ,\\&a_{63}=\tfrac{1}{2}-2a_3+\tfrac{1}{2}a_3^2-a_{52} \ ,\\&a_{64}=-\tfrac{15}{16}+\tfrac{7}{2}a_3+\tfrac{3}{2}a_{52} \ .\end{aligned} \tag{5.8}$$

Roughly speaking, one-half of the terms of the weak-disorder expansion of the conductivity of the square lattice are determined by the duality symmetry.

It can be checked that all the conditions (5.8) are obeyed by the general expression (3.14), provided the four lattice integrals obey the following identities:

$$I_1=I_2=1, \quad J_1=J_2 \ (d=2) \ . \tag{5.9}$$

As a matter of fact, it is rather easy to prove that the integrals defined in Eq. (3.9) indeed obey Eq. (5.9). Let us give an explicit proof for the first lattice integral I_1. Consider the quantity

$$A_{ijk}=\int d\vec{p}\int d\vec{q}\int d\vec{r}\,\tilde{\delta}(p+q+r)\times\frac{K_{ii}(p)}{K(p)}\frac{K_{jj}(q)}{K(q)}\frac{K_{kk}(r)}{K(r)} \ , \tag{5.10}$$

where the direction labels i,j,k take the values 1 or 2. For reasons of symmetry, among these eight numbers, there are only two different quantities, namely $A_{ijk}=X$ for $i=j=k$, and $A_{ijk}=Y$ in all other cases. On the other hand, by summing over any one of the three indices, we obtain a factorized integral over two wave vectors, which is very easily evaluated. This remark yields the sum rule $A_{1jk}+A_{2jk}=\frac{1}{4}$, for any values of the indices j and k, and two other analogous relations. The first sum rule, taken for $(j,k)=(1,1)$ and $(j,k)=(1,2)$, yields $X+Y=2Y=\frac{1}{4}$, implying $X=Y=\frac{1}{8}$, and finally $I_2=8X=1$. Q.E.D.

Thus the weak-disorder expansion of the conductivity up to sixth order on the square lattice,

$$\begin{aligned}\Sigma=\sigma_0\{1&-\tfrac{1}{2}\mu_2+\tfrac{1}{4}\mu_3-\tfrac{1}{8}\mu_4+\tfrac{1}{16}(\mu_5+J_1\mu_2\mu_3)\\&-\tfrac{1}{32}[\mu_6+(3J_1-1)\mu_2\mu_4+(2J_1-1)\mu_3^2\\&+(-3J_1+2)\mu_2^3]\} \ ,\end{aligned} \tag{5.11}$$

involves only one quantity that has not been calculated explicitly, namely the integral J_1.

We have evaluated the numerical value of J_1, by means of its representation (3.13) in terms of the lattice Green's function $G(x)$. It is known (see Ref. 20 for more details) that the exact value of the Green's function of the Laplace operator on the square lattice can be determined via a recursive scheme, using only two inputs, namely the difference equation $-\Delta G(x)=\delta_{x,0}$, and the values on the main diagonals, which are known explicitly: $G(\pm ne_1\pm ne_2)=-(1/\pi)[1+\frac{1}{3}+\frac{1}{5}+\cdots+1/(2n-1)]$. We have used this scheme to generate very accurate numerical values of $G(x)$, and performed the sum given in Eq. (3.13). The result $J_1\approx 1.092\,958\,179$ is reproduced in Table III.

Let us now turn to a quantitative comparison with the EMA prediction. The conductivity of the square lattice ceases to be correctly described by the EMA approach only at the fifth order of perturbation theory, since the value of the lattice integral I_1, given in Eq. (5.9), coincides with its EMA value, given in Eq. (4.9), whereas this is not the case for J_1. The leading discrepancy for weakly disordered networks therefore reads

$$\Sigma-\Sigma^{\mathrm{EMA}}\approx\tfrac{1}{16}(J_1-1)\sigma_0\mu_2\mu_3\ , \tag{5.12a}$$

or, numerically,

$$\Sigma-\Sigma^{\mathrm{EMA}}\approx 5.809\,886\times10^{-3}\sigma_0\mu_2\mu_3\ . \tag{5.12b}$$

The leading fifth-order contributions to the conductivity that are not correctly taken into account by the EMA approach therefore come into play with a very small numerical coefficient, of the order of 0.5%.

C. Three dimensions

The case of the three-dimensional cubic lattice is fully generic, from a purely technical viewpoint. Indeed, the four nontrivial lattice integrals, involved in the result of the weak-disorder expansion up to the sixth order of perturbation theory, and defined in Eq. (3.9), do not possess any remarkable property for $d=3$. We have evaluated these quantities numerically, using their expressions (3.13) in terms of the real-space Green's function $G(x)$. Unlike the two-dimensional case, the difference equation obeyed by the function $G(x)$ is not sufficient to determine it entirely. We have therefore made use of the integral formula (3.12) to evaluate the required differences of lattice Green's functions. The numerical values of the four lattice integrals are listed in Table III. Although this procedure is less accurate than the "exact" one used in the case of the square lattice, a precision better than the seven significant digits quoted in Table III is easily achieved, without having to use more sophisticated techniques, such as those developed in Refs. 21 and 22.

As far as the comparison with the EMA prediction goes, since the three-dimensional case is generic, the EMA formula begins to depart from the exact conductivity at the fourth order of perturbation theory, with the terms in μ_2^2, according to Eq. (4.10), which yields, numerically,

$$\Sigma-\Sigma^{\mathrm{EMA}}\approx 2.822\,87\times10^{-3}\sigma_0\mu_2^2\ . \tag{5.13}$$

This leading correction to the EMA formula again involves a very small numerical prefactor.

D. Large dimensionality

Let us finally examine the behavior of the weak-disorder expansion of the conductivity, in the limit of a large dimensionality d. The key point of the following analysis is that the lattice Green's function $G(x)$, defined in Eq. (3.11), possesses a regular expansion in powers of $1/d$, the inverse of the dimensionality, e.g., for $x=0$, and more generally for any fixed value of the position x, with a finite number of nonzero coordinates, such as $x=e_1+e_2$.

The simplest way of deriving the $1/d$ expansion of the Green's function is to introduce a Schwinger-like variable t, conjugate to the denominator $K(p)$ in the definition (3.11). Let us exemplify this derivation in the simplest case of the origin $x=0$:

$$\begin{aligned} G(0)&=\int d\tilde{p}\,\frac{1}{K(p)}=\int d\tilde{p}\int_0^\infty dt\,e^{-tK(p)} \\ &=\int_0^\infty dt\,[e^{-2t}I_0(2t)]^d\ , \end{aligned} \tag{5.14}$$

where I_0 is the modified Bessel function, namely

$$I_0(z)=\int_0^{2\pi}\frac{dp}{2\pi}e^{z\cos p}=\sum_{k\geq0}\frac{(z/2)^{2k}}{(k!)^2}\ . \tag{5.15}$$

When the dimensionality d of the lattice becomes large, the t integral on the second line of Eq. (5.14) is dominated by the vicinity of the origin $t=0$. To leading order, the Bessel function I_0 can be replaced by 1, so that we have $G(0)\approx\int_0^\infty dt\,e^{-2dt}=1/(2d)$. This procedure can be pursued, by keeping higher orders of the series expansion (5.15). We obtain thus a systematic $1/d$ expansion of the Green's function at the origin, namely

$$G(0)=\frac{1}{2d}+\frac{1}{(2d)^2}+\frac{3}{(2d)^3}+\cdots\ . \tag{5.16}$$

For a generic lattice point x, with a fixed number of nonzero coordinates, $G(x)$ can be shown to fall off proportionally to an integer power of the lattice dimensionality d, namely $G(x)\sim d^{-|x|-1}$, where $|x|=|x_1|+|x_2|+\cdots$ denotes the sum of the absolute values of the coordinates of the vector x. In other terms, only the value $G(0)$ of the Green's function at the origin decays as $1/d$, whereas its values $G(\pm e_i)$ at the first neighbors of the origin decay as $1/d^2$, and the values $G(\pm2e_i)$, or $G(\pm e_i\pm e_j)$ for $i\neq j$, decay as $1/d^3$, etc.

Therefore, in order to evaluate the lattice integrals to a given order of the $1/d$ expansion, starting from their representation (3.13), we have to consider only a finite num-

TABLE III. Values of the four lattice integrals, defined in Eq. (3.9), that enter the weak-disorder expansion of the conductivity, in the case of (hyper)cubic lattices of various dimensionalities. For $d=1$, 2, or 3, the values 1 are exact, whereas the other numbers are very accurate numerical estimates. The last column gives a few terms of the large-d expansion of the integrals.

Integral	$d=1$	$d=2$	$d=3$	$d\to\infty$
I_1	1	1	0.923 782 5	$\frac{3}{4}(1+1/2d+1/2d^2+\cdots)$
I_2	1	1	0.986 912 8	$\frac{15}{16}(1+1/6d+\cdots)$
J_1	1	1.092 958 178	1.263 574 7	$\frac{d}{4}(1+3/2d+3/2d^2+\cdots)$
J_2	1	1.092 958 179	1.056 113 3	$\frac{d}{8}(1+7/d+\cdots)$

ber of types of neighbors of the origin. We do not give the full derivations here. The last column of Table III gives the first two or three terms of the $1/d$ expansion of the four lattice integrals involved in the perturbative expression of the conductivity.

We now have the more ambitious aim of recasting the result (3.14) of the general weak-disorder expansion into a systematic $1/d$ expansion of the conductivity of a random resistor network in a large dimensionality, without reference to perturbation theory. This reinterpretation of our result is made possibly by the following observation. The apparent structure of the terms of Eq. (3.14) is fully general. Namely, the sum Σ_k of the terms of order k has an explicit prefactor of $(-1/d)^{k-1}$, whereas the rest of the expression only takes up one power of d for every even order in perturbation theory. Therefore, it makes sense to recast our result (3.14) in the following form of a systematic $1/d$ expansion, using the results of Table III:

$$\Sigma=\sigma_0\left[1-\frac{\mu_2}{d}+(\mu_3-\mu_2^2)\frac{1}{d^2}+(-4\mu_4+9\mu_2^2+13\mu_2\mu_3-9\mu_2^3)\frac{1}{4d^3}+\cdots\right]. \tag{5.17}$$

More generally, the term in d^{-n} is a linear combination of terms of various orders k ranging from $k=n+1$ to $k=2n$ in the perturbative weak-disorder expansion.

We finally notice that the large-d behavior of the lattice integrals, as given in the last column of Table III, is not correctly reproduced by the EMA values (4.9). As a consequence, there is a discrepancy between the exact large-d expansion (5.17) of the conductivity, and the expansion of the EMA prediction, starting with the terms of order $1/d^3$, namely

$$\Sigma-\Sigma^{\mathrm{EMA}}=\sigma_0\left[(\mu_2^2+\mu_2\mu_3-\mu_2^3)\frac{1}{4d^3}+\cdots\right]. \tag{5.18}$$

VI. DISCUSSION

In this paper we have shown how a systematic perturbative weak-disorder expansion could be derived for the conductivity of random resistor networks, built from (hyper)cubic lattices in arbitrary dimension d. The outcome of this diagrammatic expansion only involves the moments of the probability distribution of the (possibly complex) microscopic bond conductances, whereas the structure of the underlying lattice enters through some integrals involving lattice propagators. The explicit results given up to the sixth order of perturbation theory involve four such nontrivial integrals, denoted I_1, I_2, J_1, and J_2. Either exact or very accurate numerical values of these quantities are given in Table III for all dimensions of physical interest ($d=1$, 2, or 3) and in the limit of a large dimensionality. These results generalize therefore those presented in Ref. 11, which were derived using a somewhat similar diagrammatic approach, but concerned only the binary distribution on the three-dimensional cubic lattice.

We have also checked explicitly that the effective-medium approximation (EMA) corresponds to a self-consistent resummation of the one-impurity terms (T-matrix resummation) of the perturbative expansion. From a quantitative viewpoint, this approximate scheme amounts to replacing all the lattice integrals by 1, namely the value obtained by taking into account only the contribution of coinciding points, using a real-space formalism. Under general circumstances, in agreement with Refs. 1 and 11, we have shown that the EMA formula ceases to be exact at the fourth order of perturbation theory, and this discrepancy is affected by a very small numerical prefactor. This analysis explains, therefore, on a quantitative basis, the current observation that the EMA prediction is very accurate for quite a broad variety of distributions of the microscopic conductances (see, e.g., Refs. 2, 7, and 10).

These general results are then discussed in more detail in several more specific situations. In the case of the linear chain ($d=1$), the conductivity is known exactly for any distribution of the conductances. This exact result is correctly reproduced by the EMA. This example provides therefore a test case for the formalism used in this work. The case of the square lattice ($d=2$) is also particular, since the duality symmetry imposes rather severe constraints on the expression of the conductivity. One consequence of this symmetry is that the EMA prediction departs from the exact value of the conductivity only at the fifth order of perturbation theory. In both two and three dimensions, the leading terms discarded by the EMA formula show up with a very small numerical prefactor, of a fraction of a percent (also see Ref. 11).

In the limit of a large dimensionality ($d\to\infty$), it turns out that our perturbative results can be recast in a systematic expansion of the conductivity in negative powers of the dimension. The terms calculated so far yield the explicit values of the first three terms of this $1/d$ expansion. The EMA prediction is also shown to be very accurate in the limit of a large dimensionality, since it ceases to be exact only at the level of the coefficient of $1/d^3$.

We want to discuss finally in some detail the specific case of a binary distribution of conductances, where each bond conductance $\sigma_i(x)$ equals either σ_1, with probability p, or σ_2, with the complementary probability $q=1-p$. This binary model has been studied extensively, especially in connection with percolation theory (see Ref. 7 for a recent review). In particular, the limiting cases $\sigma_2=0$ and $\sigma_1=\infty$ correspond to the conductor-insulator and the conductor-superconductor mixtures, respectively, which undergo a percolation transition for $p=p_c$, where p_c denotes the geometrical (connectivity) threshold of the bond-percolation problem.

Denoting by h the (possibly complex) dimensionless conductance ratio σ_2/σ_1, along the lines of Ref. 7, we expect the EMA prediction to be reasonably good for all

values of p and h, except in rather limited critical regions around both isolated percolation critical points ($p=p_c$, $h=0$) and ($p=1-p_c$, $h=\infty$). Going back to a weak-disorder analysis, one realizes that there are two limiting cases where the binary model becomes weakly disordered, namely (1) $h\to 1$, and (2) $p\to 0$ or $p\to 1$. In the first situation, our weak-disorder analysis applies fully, and generalizes that of Ref. 11 to an arbitrary dimensionality. We do not aim at being more quantitative about the general binary model in the present article.

The second situation is more peculiar. Let $p\to 0$ for definiteness. It can be shown that all the moments μ_k, defined in Sec. II, vanish linearly with p. As a consequence, the successive terms of our perturbative expansion do not become smaller and smaller in a nicely controlled way. Hence the present version of perturbation theory cannot be viewed as a systematic small-p approach. It is known that the EMA prediction correctly describes the term linear in p in the conductivity of the binary model, whereas the evaluation of the term proportional to p^2 is a difficult task, which has been achieved in a series of papers,[23-30] for both site and bond dilution, using another perturbative approach, and a formalism inspired from kinetic theory. It would be desirable to work out a more general perturbative scheme, which would embrace both our approach (which holds for general distributions, under the assumption that the moments μ_k of the fluctuating part of the microscopic conductances fall off for high values of the order k), and the kinetic-theoretical one (which is valid for a peculiar kind of weak randomness, namely weak dilution).

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