Conductance of a disordered narrow wire in a strong magnetic field

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We have studied numerically the two-probe conductance of a narrow wire in a strong magnetic field using a tight-binding Hamiltonian with random disorder. We found that for a square-well confining potential the interband scattering between edge states is significant and leads to sharp minima in conductance near the transitions from one quantized conductance plateau to another. Within these minima the conductance fluctuates quasiperiodically as a function of the magnetic field.

INTRODUCTION

In the past few years it has become possible to fabricate increasingly narrow two-dimensional electronic devices, which have shown physical properties quite different from those of a two-dimensional electron gas. These devices consist typically of a metal-oxide-semiconductor field-effect transistor structure with a very narrow (less than 100 nm) gate electrode that can be used to control the electron density in the inversion layer under the gate. The conducting channel can be several micrometers long, hence the aspect ratio of the device is typically of the order of 100. Much interest has been focused on the transport properties of these mesoscopic systems in strong magnetic fields, when the magnetic length is comparable to the width of the channel, and quantum effects play an important role.

In a strong magnetic field the two-dimensional electron gas is well known to exhibit quantized Hall effect with vanishing longitudinal conductance and transverse conductance equal to Ne^2/h [integer quantum Hall effect (IQHE)], where N is the number of occupied Landau levels.[1] It has been shown that the IQHE can be understood in terms of edge states propagating along the boundaries of the sample.[2-4] In this paper we study the conductance of a quasi-one-dimensional electron gas in a strong magnetic field using the edge state picture, and we expect to find finite-size corrections to the behavior of the two-dimensional electron gas.

CLEAN SYSTEM

The eigenstates of a two-dimensional electron gas in a perpendicular magnetic field in the $\hat{\mathbf{y}}$ direction can be written in the Landau gauge $\mathbf{A}=-Bx\hat{\mathbf{z}}$ as

$$\phi_{nk}(x,z)=e^{ikz}\chi_{nk}(x) = A_{nk}e^{ikz}H_n\left[\frac{x-kl_B^2}{l_B}\right]\exp\left[-\frac{(x-kl_B^2)^2}{2l_B^2}\right] \quad (1)$$

with energies $E_n=(n+\frac{1}{2})\hbar\omega_c$, where ω_c is the cyclotron frequency eB/mc. To ensure current conservation and unitarity of the scattering matrix it is convenient to choose the normalization coefficients A_{nk} so that each eigenstate carries unit flux. The states are extended plane waves in the longitudinal z direction, but in the transverse x direction they are localized within a few magnetic lengths $l_B=\sqrt{\hbar c/eB}$ from their guiding centers $x_0=kl_B^2$. In an infinite system the energy of a state is independent of k and the guiding center. If a confining potential is included in the x direction, we can continue to label the states with n (subband index) and k, but the energy increases with $|k|$ and hence depends on the guiding center $x_0=\langle x\rangle$, as can be easily derived for a parabolic confining potential. At a given energy a clean system supports N_p propagating modes, where N_p is defined by $E>E_{N_p-1}(k=0)$, $E<E_{N_p}(k=0)$. For a square-well confining potential the energy bands are displayed in Fig. 1.

The conductance of a device can be related to the probability of an incoming electron being transmitted through it.[2-7] It has been shown that in linear response the (two-probe) conductance is given by

$$\Gamma=\frac{e^2}{h}\mathrm{Tr}(t^\dagger t) , \quad (2)$$

where t is the transmission matrix across the system. Analogous formulas have been derived for multiprobe measurements as well.[4] We can immediately see that the conductance of a clean, noninteracting electron gas with N_p propagating modes is

$$\Gamma=N_p\frac{e^2}{h} , \quad (3)$$

independent of the length of the sample. The conductance of a disordered system is always less than or equal to this limiting value.

MODEL FOR A DISORDERED SYSTEM

In a disordered system the energy levels broaden to bands the widths of which depend on the amount of disorder. Now the situation is very different depending on if the Fermi level lies near the bulk energy levels or well between them. In the former case states on the Fermi sur-

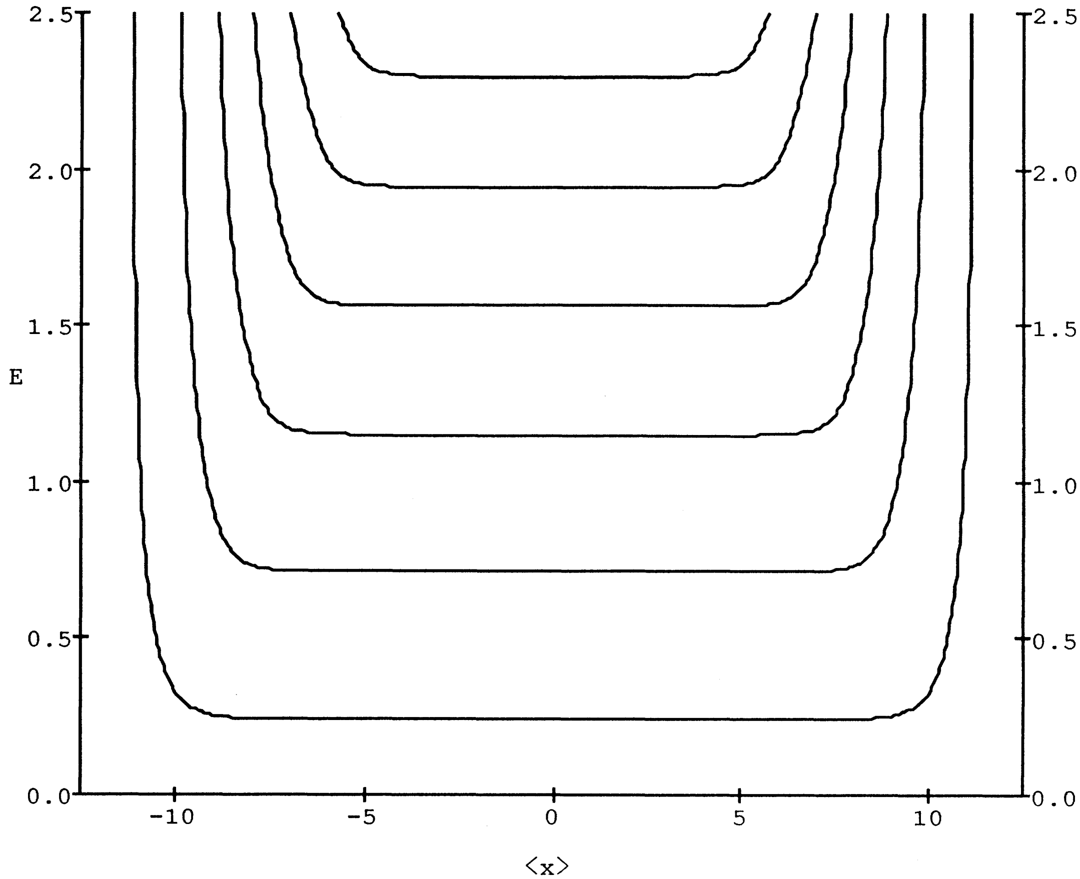


FIG. 1. Lowest energy bands in arbitrary units as a function of the guiding center $\langle x \rangle$ for electrons in a square-well confining potential in a strong magnetic field. The width of the wire is $W=25a$ and the magnetic field is $0.25\hbar c/ea^2$.

face can easily scatter elastically among themselves, but in the latter case they either have to scatter across the entire sample, or to a different subband. As can be seen from Fig. 1, the guiding centers of the latter states are located near the boundaries of the sample, and they can be classified as edge states. Classically these states correspond to electron orbits hopping along the boundaries of the sample (Fig. 2). The bulk states are more likely to be reflected back than the edge states, as the latter ones can only reverse their direction by scattering across the sample. Hence the conductance (2) of a disordered system in a magnetic field is primarily due to the propagation of edge states.

For weakly disordered systems that are much wider than the magnetic length the interedge scattering is likely to be negligible and the conductance is expected to be quantized to $\Gamma=N_e e^2/h$, where N_e is the number of edge states. However, if the magnetic length is long compared to the width of the system, we expect the magnetic-field effects to be unimportant, leading to a length-dependent conductance that goes to zero as the length of the systems increases. The crossover from the low- to the high-field regime as the magnetic field is increased is the main topic of the remainder of this paper.

We studied the effect of impurities on a discrete lattice using a tight-binding Anderson Hamiltonian

$$H=\sum_{j,k} U_{jk} a_{jk}^\dagger a_{jk} + \sum_{j,k,j',k'} V_{jk,j'k'} a_{jk}^\dagger a_{j'k'} , \quad (4)$$

where

$$U_{jk}=\begin{cases} 0, & k<0 \text{ or } k>L \text{ and } |j|<\frac{1}{2}N \\ u_{jk}, & 0<k<L \text{ and } |j|<\frac{1}{2}N \\ +\infty, & |j|>\frac{1}{2}N \end{cases} \quad (5)$$

and

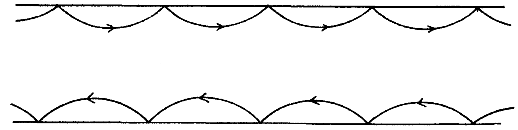


FIG. 2. Schematic illustration of the classical hopping orbits that propagate along the edges.

$$V=-\delta_{|j-j'|,1}\delta_{kk'}-e^{-iBj(k-k')}\delta_{jj'}\delta_{|k-k'|,1} \;. \tag{6}$$

This Hamiltonian describes a noninteracting spinless electron gas in an infinite wire of width $W=Na$ with clean leads attached to a disordered region of length La. We have chosen an infinite square-well potential for computational simplicity. The actual confining potential in real devices is considerably smoother, but we believe that a square-well potential is sufficient to describe the devices qualitatively. The entire wire is in a uniform magnetic field along the y axis. The magnetic field is represented by the complex hopping terms and is expressed in units $\hbar c/ea^2$. The Hamiltonian is periodic in B and can be regarded as an approximation of the continuum system if $B \ll \pi$, i.e., if the magnetic flux through a lattice cell is much less than half a flux quantum. The disorder parameter u is a uniformly distributed random variable with $|u_{ij}|<V_D/2$. In addition to this rough impurity potential we also considered a smoothened model, where random impurities were placed on a coarse lattice and u_{ij} was obtained by interpolating between the points of the coarse lattice.

We use the Lippmann-Schwinger equation to connect the states ψ of the disordered system to the states ϕ of the clean system with the same energy.[8] On a lattice the equation becomes

$$\psi_n(\mathbf{r})=\phi_n(\mathbf{r})+\sum_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\phi_n(\mathbf{r}') \;, \tag{7}$$

where G is the Green's function for the disordered system. In a continuum system we can use the equation of motion

$$\left\{E-\left[\frac{1}{2m}\left[-i\hbar\nabla-\frac{e}{c}Bx\hat{\mathbf{z}}\right]^2+V(x)\right]\right\} G(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}') \tag{8}$$

for the Green's function G to write Eq. (7) in a more convenient form. The boundary conditions for G are chosen so that to the right of the disordered region we only have states propagating to the right. Hence the appropriate function to use is G^+, the retarded Green's function. Applying (8) to eliminate the G^+V term from Eq. (7) we obtain for the continuum system

$$\psi_n(\mathbf{r})=[1-P_D(\mathbf{r})]\phi_n(\mathbf{r}) +\frac{\hbar^2}{2m}\oint_{\partial D} d\hat{\mathbf{n}}'\cdot\left[\phi_n\nabla' G^+(\mathbf{r},\mathbf{r}')-G^+(\mathbf{r},\mathbf{r}')\nabla'\phi_n +2i\frac{e}{\hbar}\mathbf{A}(\mathbf{r}')\phi_n(\mathbf{r}')G^+(\mathbf{r},\mathbf{r}')\right] , \tag{9}$$

where ∂D denotes the boundary of the disordered region D. The projection operator $P_D(\mathbf{r})$ is 1 for $\mathbf{r}\in D$ and 0 otherwise. The corresponding transformation can be performed on a lattice as well. The final result only contains sums over the ends of the disordered region and is more suitable for numerical work than expression (7).

The states ψ can be written as linear combinations of the states ϕ

$$\psi_n(\mathbf{r})=\sum_{n'} t_{nn'}\phi_{n'}(\mathbf{r}) \;, \tag{10}$$

where t is the transmission matrix in the basis formed by the eigenstates of the clean system. The conductance is given by the trace of the transmission matrix, which is independent of the basis, but the calculation is most conveniently done in this basis, since we can immediately distinguish propagating states from exponentially decaying evanescent modes, which are generated by scattering processes in the disordered region. In an infinitely long wire the evanescent modes die out in the leads before reaching the probes and hence do not contribute to the conductance. We solve Eq. (10) for a fixed z and consider only a finite number of transverse states χ_n. It should be noted that χ_n are not eigenstates of the same Hermitian operator and are in general not orthogonal to one another; however, they are linearly independent. By placing the entire system in a uniform magnetic field we can avoid any reflection due to impedance mismatch caused by the changing field and isolate the effect of impurities.

The problem of calculating the two-probe conductance has now been reduced to evaluating the various Green's functions that appear in (9). It is convenient to write them as complex $N\times N$ matrices (N is the number of transverse lattice points) that depend on two parameters z and z'. The Green's functions for the disordered system can be evaluated recursively starting from the Green's function for one of the semi-infinite clean leads by growing the disordered region slice by slice.[9,10] In a uniform magnetic field the Dyson equation for the left lead becomes, in this model, a second-order matrix equation

$$G_L G_0^{-1}-G_L V G_L V^{\dagger}=I \;, \tag{11}$$

where G_0 is the Green's function for an isolated slice, G_L is the propagator that describes an electron in the last slice of the semi-infinite clean lead, and V is the diagonal hopping matrix

$$V_{xx'}=\delta_{xx'}e^{-iBx} \;. \tag{12}$$

Equation (11) can be solved analytically for zero magnetic field, but for a nonzero field we must solve it numerically. Solving this system of $2N^2$ coupled nonlinear equations becomes very time consuming even for rather small values of N. The Green's functions that describe an electron propagating within the last slice of a semi-infinite disordered system that extends from $-\infty$ to $z+1$ can be constructed recursively from

$$G_L(z+1;z+1)=[G_0^{-1}(z+1;z+1)-VG_L(z;z)V^{\dagger}]^{-1} \;. \tag{13}$$

The other clean lead can be attached to the end of the disordered region in the same fashion to give the propagator for the rightmost disordered slice in the infinite system with both leads attached,

$$G(L;L)=[G_L^{-1}(L;L)-V^{\dagger}G_R(L+1;L+1)V]^{-1} \;. \tag{14}$$

$G(1;1)$ can be constructed analogously starting from the

Green's function for the right lead. Off-slice Green's functions can be evaluated using the formulas

$$G(L;J)=G(L;J+1)V^{\dagger}G_L(J;J)\ , \quad (15)$$

$$G(1;J)=G(1;J-1)VG_R(J;J)\ . \quad (16)$$

These formulas can be used to construct all propagators that appear in Eq. (9) and consequently to calculate the transmission matrix and the conductance.

RESULTS OF A NUMERICAL STUDY

We performed a series of numerical calculations for $N=13$ and $L=104$, 208, 325, 650, and 1300 varying the

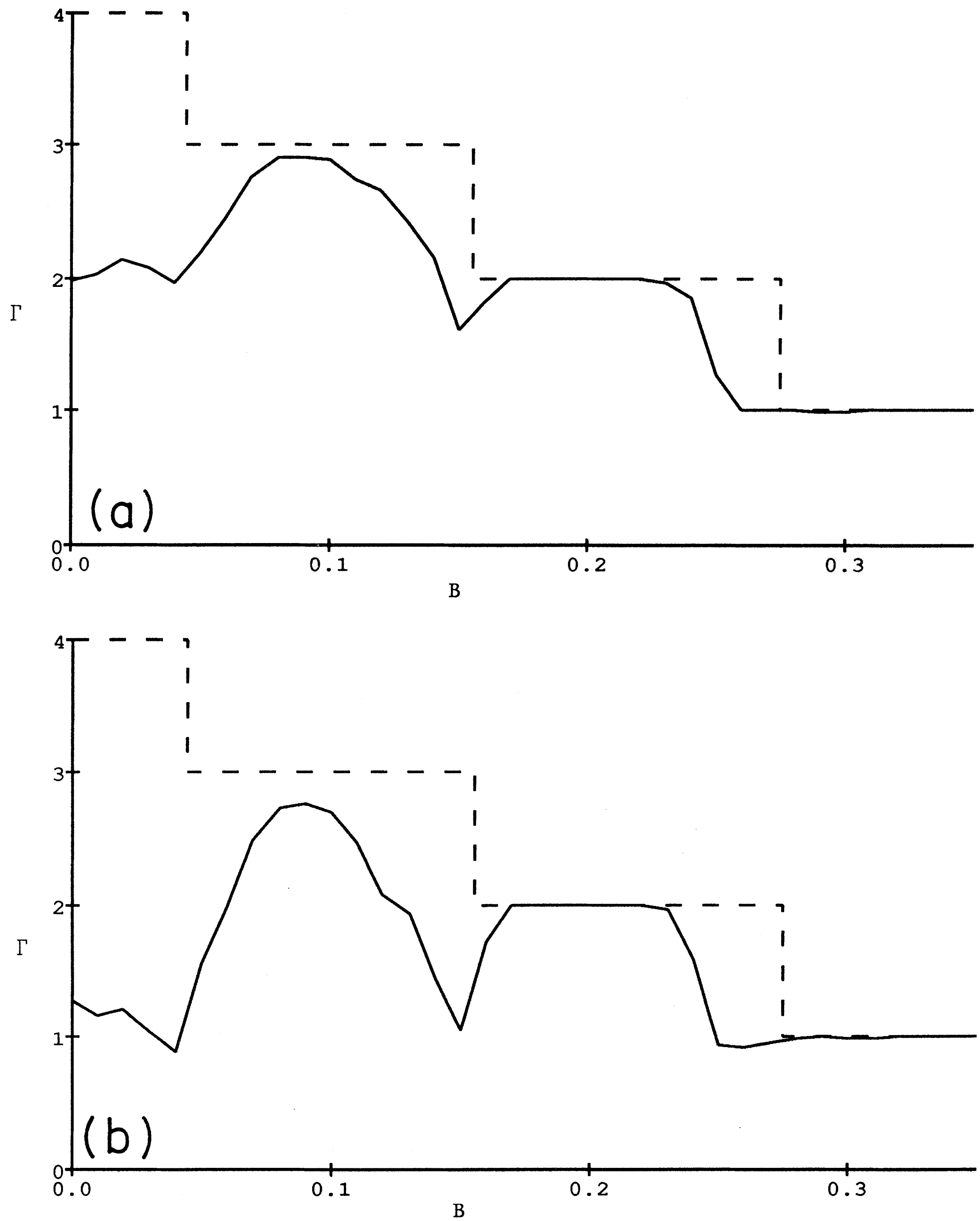


FIG. 3. Ensemble-averaged conductances for systems of variable lengths and rough disorder potentials of different strengths: (a) $L=208$ and $V_D=0.30$, (b) $L=650$ and $V_D=0.30$, and (c) $L=650$ and $V_D=0.15$. The dashed line indicates the conductance of a clean system.

magnetic field and keeping the chemical potential fixed. We also studied the effect of the width of the sample by repeating some of the calculations for $N=11$ and $L=325$ and 650. The calculations were done for rough and smooth impurity potentials and the two models were found to yield qualitatively similar results, although the smooth potential was found to have smaller effects for the same value of V_D, as could be expected. Ensemble-averaged quantities were calculated for ensembles of 10, 25, or 100 systems.

In the clean system the conductance is quantized to $N_p e^2/h$. When the magnetic field B is increased, the subband energies increase, and when the bottom of a subband passes above the Fermi level, the conductance is reduced by e^2/h. We found that the primary effect of impurities is to reduce the conductance at low magnetic fields and near conductance steps. At low magnetic fields the states are very wide, and states traveling in opposite directions can scatter to one another quite easily, hence even a small amount of impurities causes significant backscattering. Near a conductance step the situation is quite similar. The states on the highest occupied subband on the higher plateau are located near the center of the wire, and again the state with longitudinal wave vector **k** can scatter to state with $-\mathbf{k}$, since they are within the effective range of the scattering potential. As the impurity potential becomes stronger, the bulk energy levels acquire finite widths and states nearer the edges have other states with the same energy within the scattering range of the potential, and can therefore effectively take part in scattering processes. The electrons in these states start to scatter across the wire and the conductance decreases. The transition from one plateau to another becomes more gradual and the plateaus themselves narrower. The dependence of Γ of L and V_D is shown in Fig. 3. The dashed line in Figs. 3(a)–3(c) indicates the conductance of a clean system, which is given by the number of occupied subbands.

The intraband scattering processes described above cause the contribution of a particular channel to the conductance to decay exponentially with the length of the wire. We observed that also the total conductance of the wire decays according to a simple exponential law

$$\Gamma(B,L)=\Gamma_0(B)e^{-L/\lambda(B)}, \tag{17}$$

as shown in Fig. 4. This simple decay law was confirmed over a wide range of L and V_D for B in the transition regions between plateaus. We note that in the middle of a conductance plateau Γ differs by less then e^2/h between the shortest and longest wires studied, and it is difficult to determine the precise length dependence of Γ. If the edge states did not mix, the conductance would be given by a sum of exponentially decaying terms with a different decay constant for each edge state. The observed simple exponential decay indicates that the edge states thermalize and the interband scattering is important in square-well wires. More direct evidence for the interband scattering and the mixing of edge states is given by the conductance minima near transitions from one conductance plateau to another (Fig. 3). The conductance falls well below the expected value for the lower plateau, meaning that several edge states can reverse their directions of propagation by scattering across the wire. They can do so by scattering via a neighboring intermediate state on a different subband. It has been shown that at least for a parabolically confined wire the interband

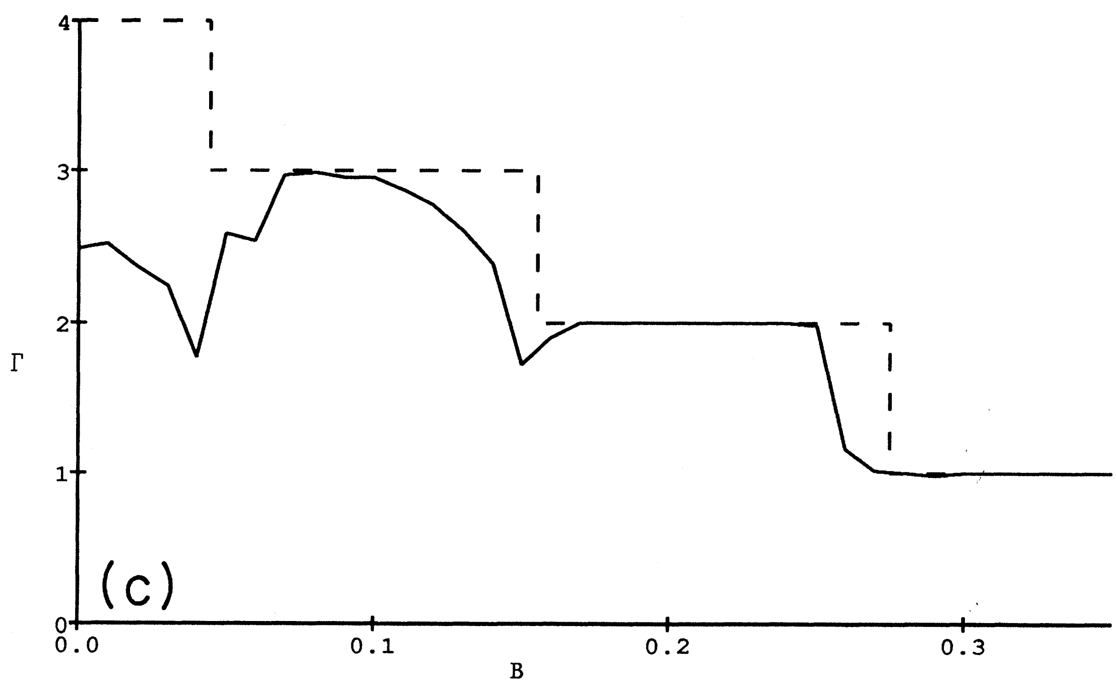


FIG. 3. (*Continued*).

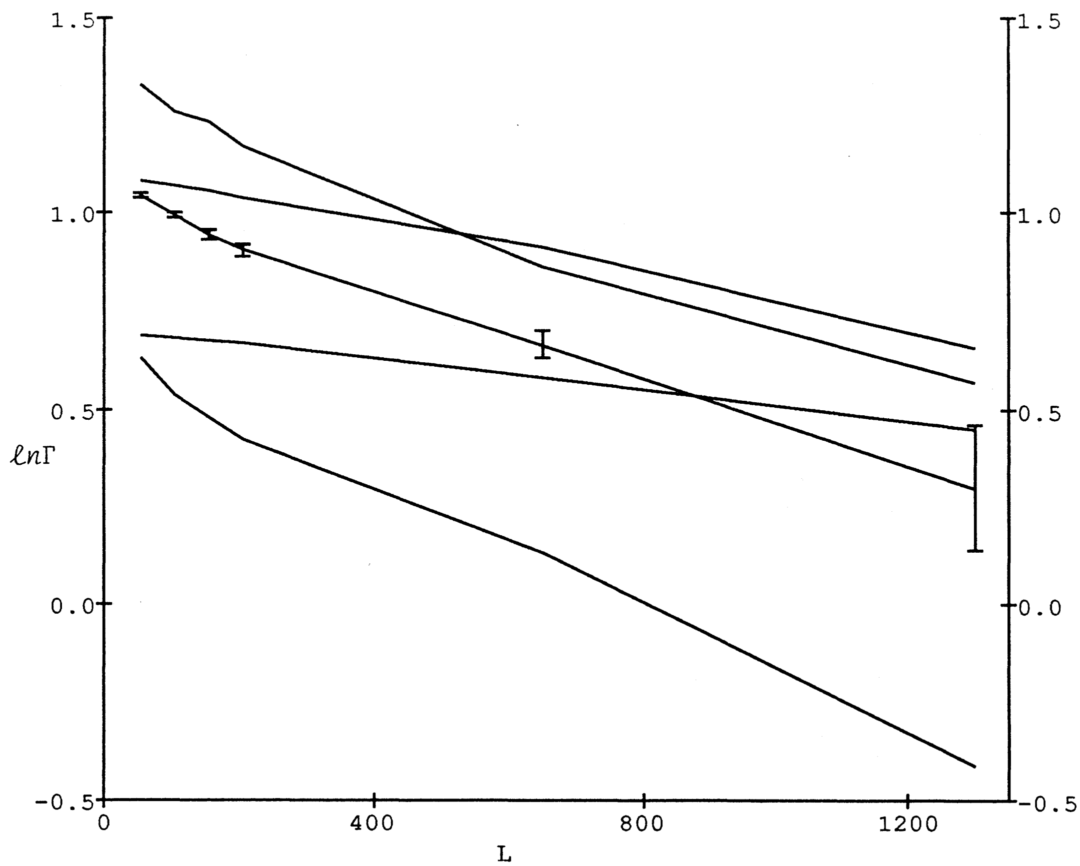


FIG. 4. Conductance as a function of the length of the wire. The widths of the wires are $W=13a$ and the disorder potential is smooth with $V_D=0.30$. Magnetic fields are $B=0.02$, 0.07, 0.15, 0.22, and 0.27 (from top to bottom on the left). Statistical error bars are shown for $B=0.15$.

scattering rate is a Gaussian function of the separation between edge states[11] and, for a slowly varying potential, the interband scattering rate is reduced by a Gaussian overlap. In a square-well potential the edge states are packed closely together and the scattering rate is enhanced. This enhanced scattering rate near the conductance steps is reflected by sharp minima of $\lambda(B)$. The conductance is only quantized if the backscattering of the edge states is small, so the quantization condition can be written as $\lambda(B) \gg L$. This shows that the quantization of the two-probe conductance is first lost at low magnetic fields and near the conductance steps, where the interband processes increase the backscattering rates of edge states. At strong magnetic fields the edge states are very narrow and consequently the interedge scattering is weak, resulting in quantized two-probe conductance even in long wires. In Fig. 5 we have plotted the inverse localization length $1/\lambda$ as a function of the magnetic field for a fixed strength of the impurity potential $V_D=0.3$. For this impurity potential the two-probe conductance remained quantized at $\Gamma=1$ even in the longest wires we studied for magnetic fields exceeding $0.3\hbar c/ea^2$.

Between the quantized plateaus the conductance was found to be a quasiperiodic function of magnetic field (Fig. 6). The oscillations are caused by interference effects between different paths through the sample. In the systems that we studied it is difficult to determine the characteristic period ΔB of these oscillations accurately because of the narrowness of the transition region where the oscillations occur. In wires that are short compared to the localization length λ the oscillations are similar to universal conductance oscillations seen in metals.[12,13] The period of the oscillations is given by $LW\Delta B=2\pi$, which yields $\Delta B=0.002$ for a wire of length $L=208$ and width $W=13$, in agreement with Fig. 6(a). The amplitude of these oscillations is about $0.3e^2/h$, which agrees well with theoretical estimates for universal conductance fluctuations for spinless electrons. It is interesting to note that the character of the oscillations changes qualitatively from smooth oscillations to sharp peaks as the length of the wire becomes comparable with λ. In long wires the oscillations are quite rapid and the accuracy in determining their period is limited by the magnetic-field resolution of the simulations. Qualitatively we can say that the period of oscillations decreases with increasing length of the wire and increases with increasing disorder. We do not have a quantitative understanding of these structures at the present.

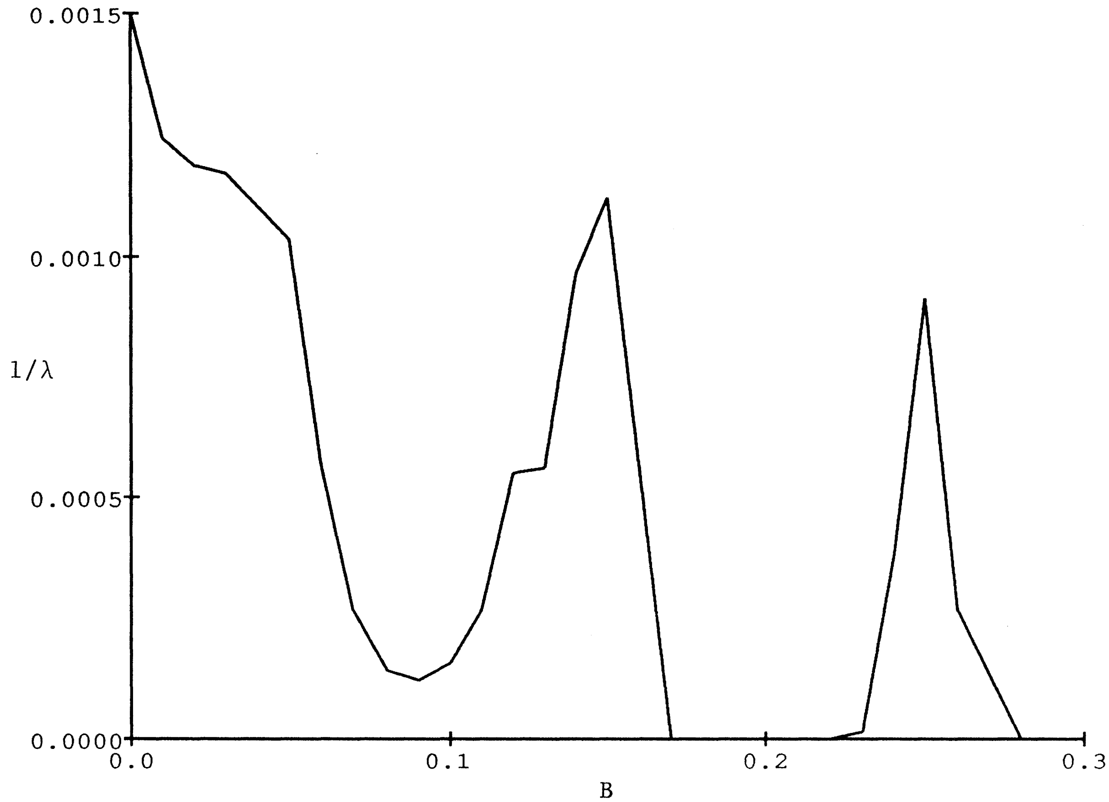


FIG. 5. Inverse localization length $1/\lambda$ as a function of B. The width of the wire is $W=13a$ and the disorder potential is smooth with $V_D=0.30$.

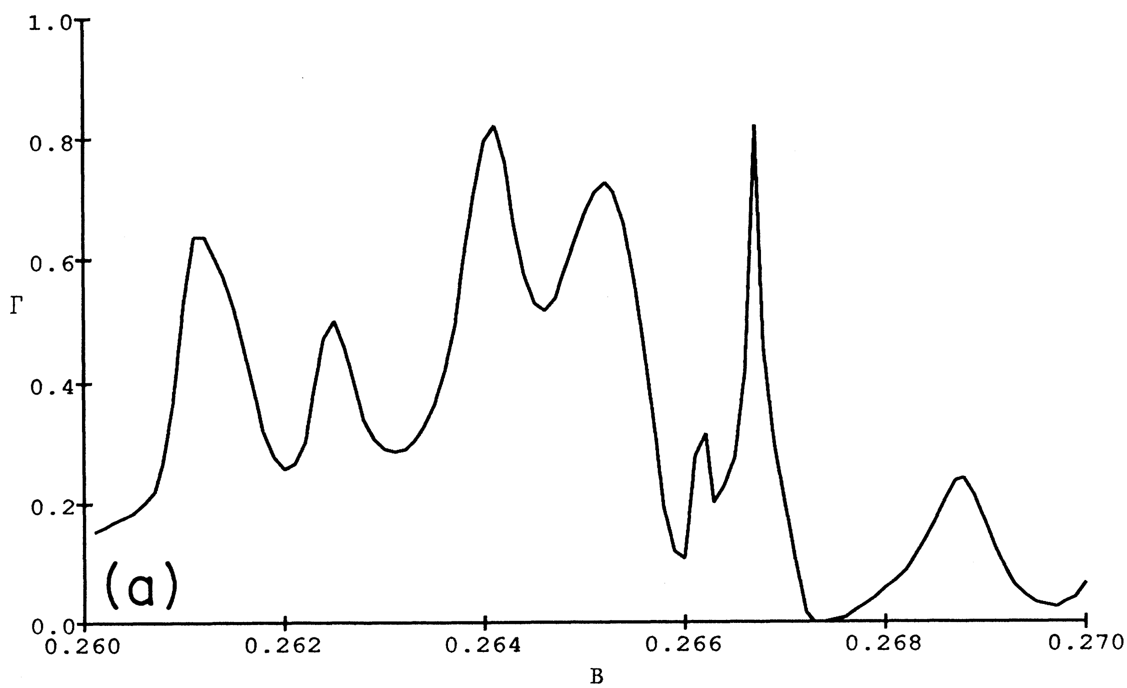
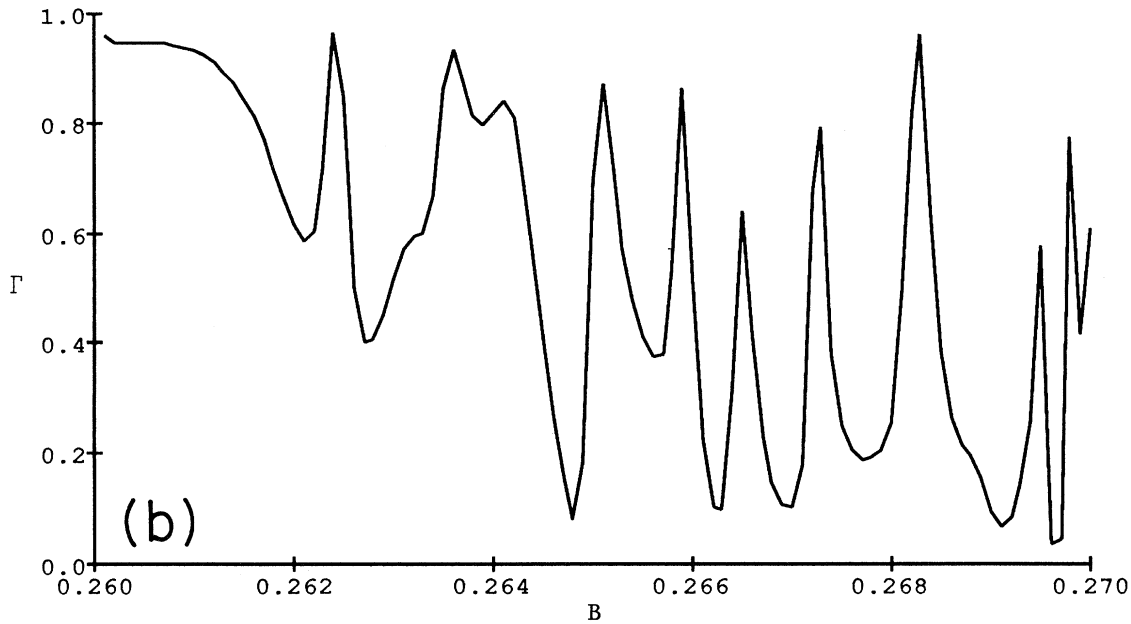


FIG. 6. Conductances of individual samples near the transition from one plateau to another for systems of variable lengths and disorders: (a) $L=325$ and $V_D=0.60$, (b) $L=325$ and $V_D=0.30$, and (c) $L=1300$ and $V_D=0.30$.



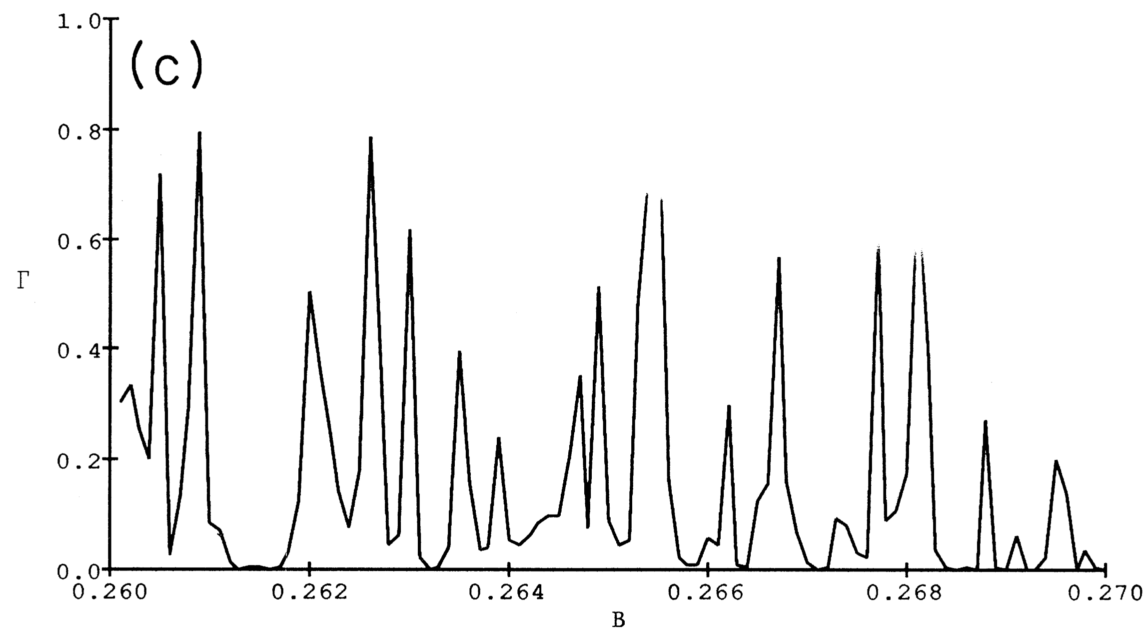


FIG. 6. (*Continued*).

COMPARISON WITH EXPERIMENTS

Experimentally the conductance is usually measured with a four-probe technique which uses different contacts as current and voltage probes, and few experimental results using two probe technique have been published.

The measurements by Kastner *et al.*[14] show that the conductance of a narrow Si wire in a magnetic field is quantized in units of $2e^2/h$. Our interpretation of the extra factor of 2 is that it is due to two degenerate energy valleys in silicon that are occupied equally. We have shown elsewhere that symmetric occupation of valleys is energetically favorable provided that the wire is narrow enough and the electron density is sufficiently high.[15] At low electron densities the exchange interactions between electrons make it favorable to fill the valleys asymmetrically and consequently the conductance should be quantized in units of e^2/h in narrow Si wires with low enough electron density.

In the experiments it was observed that different conductance plateaus disappear at different temperatures. This indicates that there are two different energy gaps in the system. We interpret the smaller gap to be due to Zeeman energy, which leads to conductance steps that disappear at quite low temperatures (1.5 K). The larger gap is due to transverse modes (subbands). Hence we conclude that the first conductance step is due to the formation of edge states with spin up in the lowest subband, the second step is caused by lowest subband edge states with spin down, and the third step is related to edge states in the second subband. This is in contrast with the original suggestion[14] that all observed conductance steps could be explained in terms of orbital effects. This point can be checked by introducing a magnetic field parallel to the interface. Parallel fields do not affect orbital states, but they do change the Zeeman splitting and hence move conductance steps due to edge states with opposite spins to opposite directions when measured as functions of gate voltage. If the density of electrons is sufficiently high and the wire is narrow enough, we expect that both spin states will be filled symmetrically and the conductance will be quantized in steps of $4e^2/h$. To our knowledge this has not yet been observed.

The downward cusps in the conductance were not observed by Kastner *et al.* This indicates that the interband scattering is less important than the simulation implies. We believe that this is because the confining potential in the experimental system is smoother than the infinite square well we used in the simulation, and the edge states are further apart. Moderate cusps have been observed in an earlier experiment by Fowler *et al.*[16] Because of the weakness of the observed conductance minima and the insufficient magnetic field resolution of the measurements neither group saw conductance oscillations between the quantized plateaus. However, oscillations similar to those we found in the simulations have been observed in four-probe measurements[17,18] near transitions from one plateau to another. Chang *et al.*[17] also observed deviations from the exact quantization R_{xy} in long narrow wires. Their results are in qualitative agreement with our simulations, though quantitative comparison is difficult as the experiments were conducted using a four-probe technique.

CONCLUSION

We have shown that the two-probe conductance of a narrow wire in a strong magnetic field can be explained in terms of edge states that are scattered across the wire by impurities. The conductance is quantized if each state is either fully transmitted or fully reflected. We found that in an infinite square-well potential the interband scattering is important and leads to partial transmission of edge states that would be fully transmitted in the absence of interband scattering. This destroys the quantization in transition regions between the plateaus. In these regions the conductance is lower than expected and decays exponentially with the length of the wire. The transition regions are broader for longer wires and stronger impurity potentials, destroying the quantization of the two-probe conductance at higher and higher magnetic fields. Within the minima the conductance oscillates quasiperiodically as a function of the magnetic field with a period that depends both on the area of the wire and on the impurity concentration.

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[1]K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
[2]B. I. Halperin, Phys. Rev. B **25**, 2185 (1982).
[3]P. Středa, J. Kucera, and A. H. MacDonald, Phys. Rev. Lett. **59**, 1973 (1987).
[4]M. Büttiker, Phys. Rev. Lett. **57**, 1761 (1986); M. Büttiker, IBM J. Res. Dev. **32**, 317 (1988).
[5]R. Landauer, IBM J. Res. Dev. **1**, 233 (1957); R. Landauer, Philos. Mag. **21**, 863 (1970).
[6]D. S. Fisher and P. A. Lee, Phys. Rev. B **23**, 6851 (1981).
[7]H. Baranger and A. D. Stone, Phys. Rev. B **40**, 8169 (1989).
[8]L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, Tokyo, 1968).
[9]P. A. Lee and D. S. Fisher, Phys. Rev. Lett. **47**, 882 (1981).
[10]L. Schweizer, B. Kramer, and A. MacKinnon, J. Phys. C **17**, 4111 (1984).
[11]T. Martin and S. Feng, Phys. Rev. Lett. **64**, 1971 (1990).
[12]P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985).
[13]B. L. Altshuler, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 530 (1985) [JETP Lett. **41**, 648 (1985)].
[14]M. A. Kastner *et al.*, Phys. Rev. Lett. **60**, 2535 (1988).
[15]J. M. Kinaret and P. A. Lee, Phys. Rev. B **42**, 11 768 (1990).
[16]A. B. Fowler, A. Hartstein, and R. A. Webb, Physica B **117&118B**, 66 (1983).
[17]A. M. Chang *et al.*, Solid State Commun. **67**, 769 (1988).
[18]J. A. Simmons *et al.*, Phys. Rev. Lett. **63**, 1731 (1989).