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Power-law temperature dependence of the inelastic-scattering rate in disordered superconductors

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We present a theory of the quasiparticle inelastic lifetime τ_{in} in disordered superconducting films. We find that both the Coulomb and the electron-phonon contribution to τ_{in}^{-1} are enhanced by disorder, and that for reasonably strong electron-phonon coupling the latter is dominant. In contrast to clean superconductors, the scattering rate is larger than the recombination rate at all temperatures. This leads to a power-law temperature dependence of τ_{in}^{-1}, in agreement with experimental observations. The theory quantitatively accounts for the magnitude, disorder dependence, and temperature dependence of τ_{in} measured in recent experiments.

There has been much interest in recent years in the properties of disordered superconductors. Initially, interest had been focused on the degradation of T_c with increasing disorder,[1] but recently increasing effort has been put into trying to understand the effects of disorder on the superconducting state at temperatures $T < T_c$. An important tool for investigating microscopic interactions in a material, and how they are influenced by disorder, is the quasiparticle (QP) inelastic lifetime τ_{in}. This is especially true in superconductors close to thermal equilibrium, where the rate $\Gamma_{in}=(2\tau_{in})^{-1}$ is directly measurable since $\Gamma_{in}\neq 0$ cuts off various singularities predicted by BCS theory.[2] One of these is the square-root singularity in the superconducting density of states (DOS), which is directly measured in a tunneling experiment. The main contributions to Γ_{in} are scattering and recombination processes due to Coulomb and electron-phonon (e-ph) interactions. In clean superconductors, the e-ph contribution is by far the dominant one, and for temperatures $T \gtrsim 0.1\Delta$, with Δ the gap, the recombination process is dominant over scattering. As a result, the T dependence of Γ_{in} at readily accessible temperatures is exponential.[2] This has been confirmed by experiment.[3]

In a disordered superconductor, two major changes in Γ_{in} are observed. First, the values of Γ_{in} are strongly enhanced, and Γ_{in} increases with increasing disorder.[4-6] Second, the T dependence of Γ_{in} is much weaker than exponential, and can be fitted well by a power law.[6,7] The enhancement is usually qualitatively explained[4] in analogy to the well-known enhancement of the Coulomb scattering rate in disordered normal metals.[8] A theoretical examination of this suggestion[9] showed that, while there is indeed a strong disorder enhancement of the Coulomb rate in analogy to the normal-metal case, the T dependence is always exponential. This suggests that a major contribution to the observed Γ_{in} is not of Coulombic origin. However, a calculation of the e-ph contribution in the presence of disorder[10] yielded a result which decreases with increasing disorder. This leaves us with the situation that neither the disorder dependence nor the temperature dependence of Γ_{in} observed in experiment can be explained by current theory.

It is the purpose of this paper to report the result of a calculation which considers both the Coulomb and the e-ph contribution to Γ_{in}, and which takes into account all known mechanisms by which disorder influences Γ_{in}. Specifically, we have included the effects of the correlation gap contribution Y to the self-energy, which has been shown to be of crucial importance in disordered superconductors,[11,12] and all vertices have been renormalized both by the usual diffusion pole-ladder diagrams[8,13] and by the Cooper propagator or crossed-ladder diagrams.[14] Neither the self-energy piece Y nor the Cooper-propagator renormalization have been considered before in the present context, and a combination of them turns out to give the leading contribution to Γ_{in} in disordered superconductors.

We find that the scattering rates are larger than the recombination rates for both the e-ph and the Coulomb interaction. For a superconducting film of thickness d, we find for the sum of the scattering rates,

$$\tau_s^{-1}=\frac{\Delta}{(Z')^2}\left[1+\frac{6}{\pi}(4\lambda-\tfrac{3}{2})\hat{\rho}\right]\hat{\rho}^{1/2}\frac{9\pi^2\sqrt{6}}{5}\Gamma(\tfrac{7}{2})\zeta(\tfrac{7}{2})\lambda\left[\frac{c_L}{c_T}\right]^4\left[\frac{\Delta}{\omega_D}\right]^2\left[\frac{\Delta}{E_F}\right]^{3/2}F(d)\left[\frac{T}{\Delta}\right]^{7/2}\left[1+\frac{T}{\Delta}\frac{d}{\xi_0}\frac{v_F}{c_T}\frac{7}{2\pi^2}\frac{\zeta(\frac{9}{2})}{\zeta(\frac{7}{2})}\right]$$
$$+\frac{\Delta}{Z'}\hat{\rho}\frac{3\pi^{7/2}}{4\sqrt{2}(1+\pi)}\frac{\Delta}{E_F}G(d)\left[\frac{T}{\Delta}\right]^{1/2}e^{-\Delta/T}. \tag{1a}$$

Here the first and the second term correspond to the e-ph and the Coulomb contribution, respectively, and we use units such that $k_B=\hbar=1$. ω_D is the Debye frequency, and λ is the usual e-ph coupling constant for a corresponding material with no disorder.[11] Z' is the usual renormalization constant of Eliashberg theory, and $\xi_0=v_F/\pi\Delta$ is the Pippard coherence length. v_F, c_T, c_L are the Fermi velocity and the transverse and longitudinal speed of sound, respectively, and E_F denotes the Fermi energy. The functions F and G are given by

$$F(d)=\frac{\xi_0}{d}\frac{\sinh x+\sin x}{\cosh x-\cos x}, \tag{1b}$$

$$G(d)=\frac{\xi_0}{d}\frac{4}{\pi-1}\left[\arctan\frac{\tan(x\sqrt{\pi}/2)}{\tanh(x\sqrt{\pi}/2)}-\frac{1}{\pi}\arctan\frac{\tan(x/2)}{\tanh(x/2)}\right], \tag{1c}$$

with $x=(d/\xi)(3/2\pi)^{1/2}$, where $\xi=(\xi_0/k_F)^{1/2}\hat{\rho}^{-1/2}$ is the dirty-limit coherence length. Finally, $\hat{\rho}=\rho/\rho_M$, where ρ is the extrapolated residual resistivity, and ρ_M is the Mott number. In a jellium model, $\rho_M=3\pi^2/e^2k_F$, with k_F the Fermi wave number. $\hat{\rho}$ is the disorder parameter of our theory. It enters in Eq. (1) both explicitly, and through the coherence length ξ. We note, however, that in general Δ, k_F, c_L, c_T, and ω_D will depend on disorder as well. These quantities appear as parameters in our calculation. They have to be determined by a separate theory, or by experiment.

Let us briefly explain the origin and the physical meaning of the various terms in Eq. (1). A complete account of the theory will be given elsewhere.[15] The QP decay rate is given by the imaginary part of the self-energy. Traditionally, the energy- and frequency-dependent self-energy $\Sigma(\varepsilon,\omega)$ is separated[16] into the anomalous self-energy W and the normal self-energy S. The latter is further decomposed into parts which are odd and even, respectively, in frequency, $S(\varepsilon,\omega)=\omega Z(\varepsilon,\omega)+Y(\varepsilon,\omega)$. In a clean superconductor the even part, Y, is a constant which simply renormalizes the Fermi energy and can be neglected. In the presence of disorder Y develops a nonanalytic energy dependence at the Fermi level and must be kept.[11,12] For a QP at the gap edge the decay rate is

$$\Gamma_{\rm in}=\Delta Z''/Z'-W''/Z'-Y''(\Delta+Y')/\Delta Z'^2. \tag{2}$$

Here we denote real parts by $Z'\equiv\mathrm{Re}Z$, etc., and imaginary parts by $Z''\equiv\mathrm{Im}Z$, etc. We have also gone on shell, $Z''\equiv Z''(\Delta,\Delta)$, etc. and we have used $W'/Z'=\Delta$. In a normal metal, Y' gives rise to the correlation gap in the DOS.[17] In a superconductor, Y' provides an efficient mechanism for the degradation of T_c.[11] Here we also have to deal with the imaginary part Y''. In a normal metal, both Y'' and $\omega Z''$ contribute to the standard result[8,13] (though normally in a normal metal Σ is not decomposed into ωZ and Y). In a superconductor, Y'' has not been considered before.

In order to calculate the three self-energy pieces Z, Y, and W we use the model and the general method developed in Ref. 11. The only difference is that here we use the dynamically screened Coulomb interaction which was calculated in Ref. 18, while Ref. 11 used a static interaction. Apart from this generalization, we consider the various self-energy parts as derived there. Reference 11 then proceeded to calculate the real part of the self-energy in order to determine the transition temperature. Here we also calculate the respective imaginary parts which determine $\Gamma_{\rm in}$.

Both the Coulomb and the e-ph interaction contribute to the self-energies. From simple energy-conservation considerations it is obvious that all Coulomb contributions as well as the e-ph recombination rate must have an exponential temperature dependence proportional to $e^{-\Delta/T}$ or $e^{-2\Delta/T}$. In contrast, QP scattering by phonon absorption will have a power-law T dependence. Nevertheless, it is not clear *a priori* that the exponential dependence will not dominate at realistic temperatures, as the example of clean superconductors shows.

We start with the Coulomb contributions. $\omega Z''$ and W'' have been considered in Ref. 9, so we concentrate on Y. If we supplement Ref. 11 by a dynamical Coulomb interaction, we obtain

$$Y_c''=\frac{-1}{2}\sum_{\mathbf{q}}\int d\varepsilon\int\frac{d\omega}{\pi}V_c''(\mathbf{q},\omega)\int d\omega' G''(\varepsilon,\omega')\delta(\Delta-\omega-\omega')[n(\omega)-f(\omega')+1][R_c''(\mathbf{q},\Delta-\varepsilon)-R_c''(\mathbf{q},\Delta+\varepsilon)]. \tag{3}$$

Here V_c denotes the Coulomb propagator,[18] and G is the normal Gorkov Green's function. For G we use a BCS Green's function, i.e., we neglect the effects of Y on the right-hand side of Eq. (3) and of all similar expressions. It can be shown[15] that the inclusion of Y'' in G in a self-consistent fashion does not lead to qualitative changes. n and f are Bose and Fermi distribution functions, respectively, and R_c is the Coulomb vertex function. The detailed form of the latter depends on how the Coulomb vertex is renormalized. We have considered renormalizations by both diffusion ladder diagrams [diffusion-propagator renormalization (DPR)] and Cooper-propagator renormalization (CPR) as explained in Ref. 11. Both renormalization procedures yield a result which is much smaller than the Coulomb contributions to $\omega Z''$ and W'' which were considered in Ref. 9. If we repeat the calculation of Ref. 9 for a film of finite thickness, we thus obtain the leading Coulomb contribution, viz. the second term in Eq. (1a).

The e-ph contributions to the self-energy can be written in a very similar form. For $\omega Z''$ and W'' we find[15] that DPR gives the leading contribution, and for thick films, $d\gg\xi$, we recover the result of Ref. 10. The e-ph contribution to Y'' can be written

$$Y_{e\text{-ph}}''=\int d\varepsilon\int d\nu[n(\nu)+f(\Delta+\nu)]G''(\varepsilon,\Delta+\nu)\times[\alpha^2F^F(\Delta+\varepsilon,\nu)-\alpha^2F^F(\Delta-\varepsilon,\nu)]. \tag{4}$$

Here $\alpha^2F^F(\varepsilon,\nu)$ is the generalized Eliashberg function defined in Ref. 11. Its detailed form depends again on whether DPR or CPR is used to renormalize the e-ph vertex. In this case we find[15] that CPR yields a strong disorder enhancement of Y'', in contrast to the other self-energy pieces. By doing the integrals one obtains the power-law temperature dependence in the first term in Eq. (1a). The exponent is $\frac{9}{2}$ for three-dimensional (3D) systems and $\frac{7}{2}$ for 2D systems. In a film of finite thickness d, a combination of $T^{9/2}$ and $T^{7/2}$ appears. The function $F(d)$ describes the crossover from 3D to 2D behavior. The disorder dependence of $Y''_{e\text{-ph}}$ is $\hat{\rho}^{(D-2)/2}$ for $D=2,3$. In the decay rate, Eq. (2), Y'' is multiplied by $(\Delta+Y')$. This is the term in large parenthesis in the first term in Eq. (1a), and Y' has been taken from Ref. 11 (the Coulomb contribution to Y' has been calculated in a perfect screening approximation). Combining our results, we obtain Eq. (1a).

Let us summarize. We have found that for disordered superconductors (i) for reasonable e-ph coupling strength the e-ph contribution to Γ_{in} is larger than the Coulomb contribution [this can be seen by putting numbers in Eq. (1)], (ii) the scattering rate is larger than the recombination rate at all temperatures below T_c, and (iii) the contribution of Y'' is dominant over the combined contributions of $\omega Z''$ and W''. The second feature is in sharp contrast to the case of clean superconductors. Together with the first one it gives rise to a power-law T dependence of the QP decay rate. The disorder dependence of the rate, $\hat{\rho}^{D/2}$ for $D=2,3$, is due to a strong disorder enhancement of Y. At first sight the diffusion enhancement of Y seems to violate the well-known insensitivity of the e-ph coupling to diffusive electron dynamics,[13] which is due to screening and shows, e.g., in Pippard's results for the sound attenuation.[19] However, this insensitivity holds only for the usual ladder renormalization of the e-ph vertex. The Cooper-propagator renormalization mentioned earlier does lead to a strong enhancement. This has remarkable consequences for the sound attenuation,[20] and the enhancement of Y'' is a manifestation of the same physics in the electronic self-energy. The enhancement is also present in Y', which is important for the theory of T_c degradation put forward in Ref. 11. For technical reasons, the Cooper-propagator mechanism is ineffective in the imaginary parts of W and Z.[15] We conclude that the disorder enhancement of the e-ph vertex which leads to an enhancement of the sound attenuation also is responsible for a large quasiparticle scattering rate. As a result, Γ_{in} in a superconducting film is proportional to $R_\Box T^{7/2}$ (with $R_\Box=\rho/d$ the sheet resistance) in 2D and to $\rho^{3/2}T^{9/2}$ in 3D.

We now compare our result, Eqs. (1), with experimental data. For the temperature dependence of Γ_{in}, the only data available to date are those of Ref. 6 on amorphous InO_x films. These are relatively thick films, so from Eq. (1a) we expect a power-law T dependence with an exponent somewhat less than 4.5 in the temperature range of the experiment. The figure shows a comparison between Eq. (1) and the experiment. For the fit we have used values of Δ as given by the experimentalists. For the Fermi energy we have assumed $E_F=91$ and 63 meV for samples 1 and 2, respectively, which is close to the value estimated in Ref. 6. We note that in this experiment the disorder was controlled by the amount of oxide, so one expects E_F to decrease with increasing ρ. For λ and ω_D we have used values for clean bulk In, viz. $\lambda=0.8$ and $\omega_D=108$ K. c_T in a-InO_x is not known either, and we have used again the value for clean bulk In, $c_T=710$ m/s. Finally, we need the dimensionless residual resistivity $\hat{\rho}$. Reference 6 gives $\rho_{4.2}$, the resistivity at 4.2 K. We assume a T-dependence of the conductivity $\sigma(T)=\sigma_0+aT^b$, where $\sigma_0=1/\rho$ is the residual conductivity, and a and b are constants. The metal-insulator transition occurs at $\rho_{4.2}=9$ mΩ cm.[6] This gives the following relation between $\hat{\rho}$ and $\rho_{4.2}$:

$$\hat{\rho}=(\rho_{4.2}/\rho_M)(1-\rho_{4.2}/9\ \text{m}\Omega\,\text{cm})^{-1}. \tag{5}$$

For ρ_M we choose 1.85 mΩ cm, which is reasonable for a material with such a low electron concentration. With these values the theory yields the solid lines shown in Fig. 1. We see that the theory accounts well for the magnitude and the temperature dependence of the observed rate. We note that a T dependence of T^3 would give a slightly better fit to the data.[6] However, one has to keep in mind that Eq. (1) is strictly valid only to lowest order in T/Δ. Higher-order corrections cannot be obtained without information about the phonon spectrum. For this reason, the low-temperature data for sample 1 should be given more weight in comparing theory and experiment. In this low-T region, our theory actually gives a better fit than a T^3 law.

With respect to the disorder dependence of Γ_{in}, Ref. 6 is not quite conclusive because only two samples were considered and because of the likely disorder dependence of E_F. The latter problem should be absent in the experiment of Ref. 5 on quench-condensed Sn films. These were thin films, $d<\xi$, and $R_\Box$ was controlled by varying the film thickness only, so E_F is expected to be constant. Also, Δ was observed to depend very weakly on $R_\Box$. The observed linear dependence of Γ_{in} on $R_\Box$ is in agreement with our result. The prefactor (at fixed T) depends again on many parameters which are not well known. If we use values for clean bulk Sn for all parameters (including E_F), the theoretical rate is smaller than the experimental one by a factor of 10. We note, however, that a deviation of, e.g., c_L/c_T from its clean value by a factor of 1.78 would account for this discrepancy.

Reference 4 investigated thick $(d>\xi)$ granular Al films with resistivities up to 10^5 $\mu\Omega$ m. This corresponds to values of $\hat{\rho}$ as large as 200, and the samples are very close to the metal-insulator transition. The mechanism for the breakdown of superconductivity in this region is currently not understood, and our theory is not expected

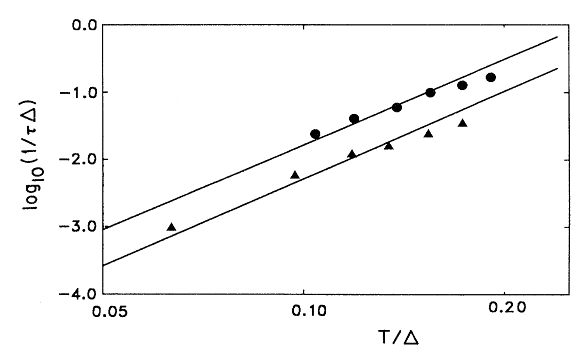

FIG. 1. Triangles and circles are experimental data for samples 1 and 2, respectively, of Ref. 6. The solid lines represent the theory, Eq. (1). Parameters have been chosen as explained in the text.

to apply there. From a theory for Γ_{in} in normal metals[21] we expect Eq. (1) to hold for $\hat{\rho} \lesssim 10$, and Γ_{in} to saturate for larger $\hat{\rho}$. The latter behavior is indeed observed in Ref. 4, and the two data points at relatively small ρ are consistent with the $\rho^{3/2}$ behavior given by Eq. (1). We note, however, that Ref. 4 observed a substantial rate at a temperature of only 60 mK. The magnitude of this rate cannot be understood with the present theory. The only possible explanation we can think of is nonequilibrium effects, though there is no experimental evidence for this.[7]

In conclusion, we have presented a theoretical expression for the dominant contribution to the quasiparticle decay rate Γ_{in} in disordered superconducting films of arbitrary thickness. We have found that the leading contribution is given by the electron-phonon scattering rate. This leads to a power-law temperature dependence of Γ_{in}. A comparison with recent experiments shows very good agreement with respect to temperature dependence and disorder dependence of Γ_{in}. Agreement with respect to the absolute value of Γ_{in} is reasonable except for the experiment of Ref. 4. The reason for the latter discrepancy is unknown.

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[1]See, e.g., *Superconductivity in d- and f-band metals,* edited by A. Suhl and M. B. Maple (Academic, New York, 1980); D. Belitz, in *High Temperature Superconductivity,* edited by J. W. Lynn (Springer, New York, 1990).

[2]S. B. Kaplan, C. C. Chi, D. N. Langenberg, J. J. Chang, S. Jafarey, and D. J. Scalapino, Phys. Rev. B **14**, 4854 (1976).

[3]R. C. Dynes, V. Narayanamurti, and J. P. Garno, Phys. Rev. Lett. **41**, 1509 (1978).

[4]R. C. Dynes, J. P. Garno, G. B. Hertel, and T. P. Orlando, Phys. Rev. Lett. **53**, 2437 (1984).

[5]A. E. White, R. C. Dynes, and J. P. Garno, Phys. Rev. B **33**, 3549 (1986).

[6]D. S. Pyun and T. R. Lemberger, preceding paper, Phys. Rev. B **43**, 3732 (1991).

[7]R. C. Dynes (private communication).

[8]A. Schmid, Z. Phys. **271**, 251 (1974).

[9]T. P. Devereaux and D. Belitz, J. Low Temp. Phys. **77**, 319 (1989).

[10]M. Yu. Reizer and A. V. Sergeyev, Zh. Eksp. Teor. Fiz. **90**, 1056 (1986) [Sov. Phys. JETP **63**, 616 (1986)].

[11]D. Belitz, Phys. Rev. B **35**, 1636 (1987); **35**, 1651 (1987); **40**, 111 (1989).

[12]D. A. Browne, K. Levin, and K. A. Muttalib, Phys. Rev. Lett. **58**, 156 (1987).

[13]A. Schmid, Z. Phys. **259**, 421 (1973); B. Keck and A. Schmid, J. Low Temp. Phys. **24**, 611 (1976).

[14]See, e.g., P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[15]T. P. Devereaux and D. Belitz (unpublished).

[16]See, e.g., J. R. Schrieffer, *Theory of Superconductivity* (Benjamin, Reading, MA, 1983), cf. also Ref. 11.

[17]B. L. Altshuler and A. G. Aronov, Solid State Commun. **30**, 115 (1979).

[18]D. Belitz, S. De Souza-Machado, T. P. Devereaux, and D. W. Hoard, Phys. Rev. B **39**, 2072 (1989).

[19]A. B. Pippard, Philos. Mag. **46**, 1104 (1955).

[20]A. Houghton and H. Won, J. Phys. C **18**, L507 (1985); B. G. Kotliar and T. V. Ramakrishnan, Phys. Rev. B **31**, 8188 (1985); T. R. Kirkpatrick and D. Belitz, *ibid.* **34**, 2168 (1986).

[21]D. Belitz and K. I. Wyoskinski, Phys. Rev. B **36**, 9333 (1987).