Quantum mechanics of the fractional-statistics gas: Particle-hole interaction

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We compute the collective-mode spectrum of the fractional-statistics gas using the set of single particle-hole pairs of the Hartree-Fock ground state as a variational basis. We find that the particle-hole interaction reduces the energy of this mode to a finite value but does not cause it to disperse linearly to zero at long wavelengths, as would be expected of a superfluid. We attribute this to a flaw in the variational ground state, implicitly rectified by computing the response functions in the random-phase approximation. The formalism introduced in this paper provides the machinery for performing precise versions of such calculations.

I. INTRODUCTION

In a previous paper,[1] we discussed the Hartree-Fock solution for an ideal gas of particles obeying fractional statistics and made the case that it was consistent with superfluidity for special values of the statistics. However, we did not explicitly demonstrate the main features of a superfluid,[2,3] namely, an undamped linearly dispersing longitudinal collective mode and a Meissner effect. In this paper we begin our investigation of superfluidity by performing a variational study of the longitudinal collective mode based on single particle-hole excitations of the Hartree-Fock ground state. Despite the infinite energy necessary to make an isolated particle or hole reported in our previous work,[1] we find a collective-mode energy that is finite and comparable in magnitude to that of noninteracting electrons in a magnetic field. This is important because the finiteness of this gap was a key assumption of our previous description of the superfluid properties based on the random-phase approximation.[4] The central idea of this approach is that the inaccuracy of the Hartree-Fock ground state at long-distance scales is intimately associated with the presence of long-range interactions in the underlying equations of motion and can thus be accounted for by analogy with the Coulomb gas. The results presented here enable us to state this idea more precisely than was previously possible and to introduce the formalism we shall use to provide a detailed account of the random-phase approximation calculation.

By a fractional-statistics gas we mean a two-dimensional system of N spinless fermions described by a wave function $\Phi(r_1,\dots,r_N)$, satisfying the Schrödinger equation

$$\mathcal{H}\Phi=E\Phi\ , \tag{1.1}$$

with

$$\mathcal{H}=\frac{1}{2m}\sum_{i=1}^{N}\left[\mathbf{P}_i+\frac{e}{c}\mathbf{A}_i\right]^2\ , \tag{1.2}$$

where

$$\mathbf{A}_i=\sum_{j\neq 1}^{N}\mathbf{A}_{ij}=(1-\nu)\frac{hc/e}{2\pi}\hat{\mathbf{z}}\times\sum_{j\neq i}^{N}\frac{(\mathbf{r}_i-\mathbf{r}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^2}\ , \tag{1.3}$$

and ν is the fraction of the statistics.[5] By the Hartree-Fock solution, we mean a variational ground state of the form

$$\Phi(\mathbf{r}_1,\dots,\mathbf{r}_N)=\frac{1}{\sqrt{N!}}\sum_{\sigma}^{N!}\text{sgn}(\sigma)\varphi_{\sigma(1)}(\mathbf{r}_1)\cdots\varphi_{\sigma(N)}(\mathbf{r}_N)\ , \tag{1.4}$$

where σ denotes a permutation, $\text{sgn}(\sigma)$ is its sign, and the φ_i are single-particle orbitals. The variational minimum is achieved when the orbitals take the form

$$\varphi_{nk}(z)=\frac{(\frac{1}{2}z-2\partial_z^*)^n}{(2^n n!)^{1/2}}\frac{(\frac{1}{2}z^*-2\partial_z)^k}{(2^k k!)^{1/2}}\frac{e^{-(1/4)|z|^2}}{(2\pi)^{1/2}}\ , \tag{1.5}$$

where $z=x+iy$ denotes the position of a particle in the x-y plane, k indexes its angular momentum, and $n\geq 0$ is its Landau level.[1,6] In writing Eq. (1.5), we have used dimensionless units in which the effective cyclotron energy

$$\hbar\omega_c=\hbar\frac{eB}{mc}=2\pi(1-\nu)\frac{\hbar^2}{m}\rho=(1-\nu)E_F \tag{1.6}$$

and the corresponding magnetic length

$$a_0=\left[\frac{hc/e}{2\pi B}\right]^{1/2}=[2\pi(1-\nu)\rho]^{-1/2} \tag{1.7}$$

are set to unity. In the Hartree-Fock ground state, orbitals with $n<(1-\nu)^{-1}$ are occupied, and the rest are empty. The special values of ν that we associated with superfluidity are those for which $(1-\nu)^{-1}$ is an integer.[1,6]

The Hartree-Fock solution for the fractional-statistics gas is compatible with superfluidity in three ways. First, it is a liquid, as may be seen by its radial distribution function, which for $\nu=0$ is given by

$$\begin{aligned} g(\mathbf{r}_1-\mathbf{r}_2)&=\frac{N(N-1)}{\rho^2}\int d^2\mathbf{r}_3\cdots d^2\mathbf{r}_N \\ &\qquad\times|\Phi(\mathbf{r}_1,\ldots,\mathbf{r}_N)|^2 \\ &=1-e^{-(1/2)|\mathbf{r}_1-\mathbf{r}_2|^2}, \end{aligned} \tag{1.8}$$

where ρ denotes the particle density. Second, the total energy, given for $\nu=0$ by

$$E_{\rm HF}=\frac{1}{4}N\hbar\omega_c=\frac{\pi}{2}N\frac{\hbar^2}{m}\rho\ , \tag{1.9}$$

is proportional to the particle density ρ, and gives rise to a finite bulk modulus and thus, presumably, a finite sound speed.[1,6] Third, there is an absence of low-lying "fermionic" excitations, such as those present in a Fermi liquid, which would ordinarily cause the collective mode to decay.[7]

The set of particle-hole pair excitations is appropriate as a variational basis for this problem for several reasons. The collective mode of a superfluid appears, in principle, as a linearly dispersing pole in the density-density response function

$$\langle\rho_{-\mathbf{q}}\rho_{\mathbf{q}}\rangle_\omega=\sum_x|\langle x|\rho_{\mathbf{q}}|0\rangle|^2\left[\frac{1}{\hbar\omega-\mathcal{E}_x+i\eta}+\frac{1}{-\hbar\omega-\mathcal{E}_x-i\eta}\right], \tag{1.10}$$

where $|0\rangle$ is the ground state and $|x\rangle$ are the excited states of the system with energy $\mathcal{E}_x$ above the ground-state energy. Given that the Hartree-Fock ground state is a good approximation to $|0\rangle$, $\rho_{\mathbf{q}}$ acting on this ground state, which is a superposition of particle-hole pairs, should also be a good approximation to $\rho_{\mathbf{q}}|0\rangle$. This is particularly the case in the $\mathbf{q}\to 0$ limit, where the action of $\rho_{\mathbf{q}}$ on the ground state usually defines what we mean by a longitudinal sound wave.

Consideration of particle-hole pairs as a variational basis leads to an attractive interaction between a particle and a hole. The evaluation of this interaction is the main subject of this paper. The inclusion of this effect is important because the Hartree-Fock eigenvalues have logarithmically large contributions that cause the "bare" excitation energies to be formally infinite. For example, the variational energy of a single particle or hole in an orbital φ_{nk} is its Hartree-Fock eigenvalue, given in the case of $\nu=0$ by[1]

$$\begin{aligned} \varepsilon_n&=(-\tfrac{1}{2}E_R)\delta_{n0} \\ &\quad+\left[n+\frac{1}{4}-\frac{1}{4n(n+1)}+\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}+\frac{1}{2}E_R\right](1-\delta_{n0})\ , \end{aligned} \tag{1.11}$$

where

$$\begin{aligned} E_R&=\int_0^R dr\frac{1}{r}(1-e^{-(1/2)r^2}) \\ &\approx\ln(R)+\frac{1}{2}[\gamma-\ln(2)]\ , \end{aligned} \tag{1.12}$$

R is the sample radius, and $\gamma\approx 0.5772\ldots$ is Euler's constant. This large energy occurs because an isolated particle or hole is a charged vortex.[1,6] The total energy for a particle-hole pair is finite because, as illustrated in Fig. 1, they have opposite vorticities that cancel one another in the far-field limit. The way in which this attractive potential influences the motion of a particle-hole pair may be understood simply in terms of a classical model of oppositely charged particles in a uniform magnetic field $\mathbf{B}$, subject to an attractive logarithmic potential U. In this model each particle moves perpendicular to the attractive force $\mathbf{F}=-\nabla U$ exerted on it by the other particle and has a speed given by the classical drift velocity

$$\mathbf{V}=\frac{c}{e}\frac{\mathbf{B}\times\mathbf{F}}{B^2}\ . \tag{1.13}$$

In this paper we shall deal exclusively with the fraction $\nu=0$, which corresponds not to the fractional-statistics gas, but rather to Bose statistics. As discussed in Ref. 1, this is appropriate because the mathematics of $\nu=\frac{1}{2}$ and

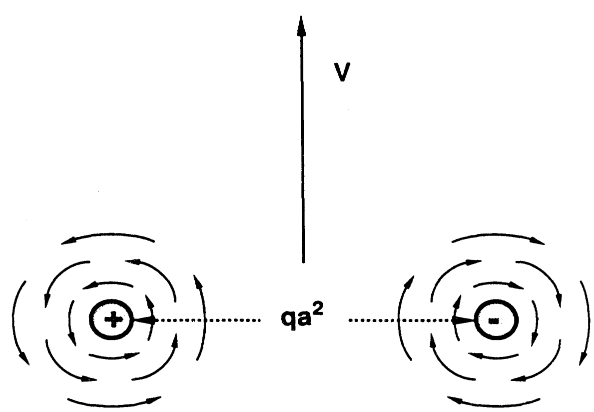

FIG. 1. The collective mode may be thought of classically as a pair of oppositely charged particles that attract each other by means of a logarithmic potential or as a pair of vortices with opposite handedness whose total vorticity is zero in the far-field limit. The particles drift with a constant velocity $\mathbf{V}$ at right angles to their separation vector, which is fixed and proportional to the magnitude of the wave vector $\mathbf{q}$.

$\nu=0$ are nearly identical, and the Bose gas is a test case for which the answer is known. It is reasonable to expect that the qualitative behavior is the same for any value of ν for which $(1-\nu)^{-1}$ is an integer, including the special case of interest, $\nu=\frac{1}{2}$.

II. PARTICLE-HOLE MATRIX ELEMENTS

The first step in evaluating the collective-mode energy is to obtain an expression for $\langle mp',0k'|\mathcal{H}|np,0k\rangle$, where $\mathcal{H}$ is the Hamiltonian of Eq. (1.2) and $|np,0k\rangle$ denotes the N-particle Hartree-Fock wave function containing a particle in φ_{np} and a hole in φ_{0k}, defined as in Eq. (1.5). To do this, it is useful to decompose the Hamiltonian into one-body, two-body, and three-body terms, in the manner

$$\mathcal{H}=\mathcal{H}_1+\mathcal{H}_{2a}+\mathcal{H}_{2b}+\mathcal{H}_3 , \tag{2.1}$$

where

$$\mathcal{H}_1=\frac{1}{2}\sum_{i=1}^{N}\mathbf{P}_i^2 , \tag{2.2}$$

$$\mathcal{H}_{2a}=\sum_{i=1}^{N}\sum_{j\neq i}^{N}\mathbf{A}_{ij}\cdot\mathbf{P}_i , \tag{2.3}$$

$$\mathcal{H}_{2b}=\sum_{i=1}^{N}\sum_{j\neq i}^{N}\mathbf{A}_{ij}\,\mathbf{A}_{ij} , \tag{2.4}$$

and

$$\mathcal{H}_3=\frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\sum_{k\neq i,j}^{N}\mathbf{A}_{ij}\cdot\mathbf{A}_{ik} . \tag{2.5}$$

We then have

$$\begin{aligned}\langle mp',0k'|\mathcal{H}|np,0k\rangle=&\delta_{mn}\delta_{kk'}\delta_{pp'}[E_{\rm HF}+(\varepsilon_n-\varepsilon_0)]-\int d1\int d2\left[\sum_{\sigma'}^{2!}\mathrm{sgn}(\sigma')\varphi^*_{0k}(\mathbf{r}_{\sigma'(1)})\varphi^*_{mp'}(\mathbf{r}_{\sigma'(2)})\right]\\&\times(\mathbf{A}_{12}\cdot\mathbf{P}_1+\tfrac{1}{2}|\mathbf{A}_{12}|^2)\left[\sum_{\sigma}^{2!}\mathrm{sgn}(\sigma)\varphi_{0k'}(\mathbf{r}_{\sigma(1)})\varphi_{np}(\mathbf{r}_{\sigma(2)})\right]\\&-\sum_{l=1}^{N}\int d1\int d2\int d3\left[\sum_{\sigma'}^{3!}\mathrm{sgn}(\sigma')\varphi^*_{0k}(\mathbf{r}_{\sigma'(1)})\varphi^*_{mp'}(\mathbf{r}_{\sigma'(2)})\varphi^*_{0l}(\mathbf{r}_{\sigma'(3)})\right]\\&\times\tfrac{1}{2}\mathbf{A}_{12}\cdot\mathbf{A}_{13}\left[\sum_{\sigma}^{3!}\mathrm{sgn}(\sigma)\varphi_{0k'}(\mathbf{r}_{\sigma(1)})\varphi_{np}(\mathbf{r}_{\sigma(2)})\varphi_{0l}(\mathbf{r}_{\sigma(3)})\right] ,\end{aligned} \tag{2.6}$$

where $E_{\rm HF}$ is the Hartree-Fock ground-state energy given by Eq. (1.9), ε_n is the single-particle energy given by Eq. (1.11), and σ and σ' are permutations of size 2 and 3. The first term of Eq. (2.6) is the result expected for a noninteracting particle-hole pair, while the remaining terms express the attractive interaction between the particle and hole.

The second step in evaluating the energy is to perform a unitary transformation that combines the states $|0k,np\rangle$ into "magnetoexciton"[8,9] wave functions of the form

$$|n\beta\rangle=\sum_{k,p}\left[\int d1\int d2\varphi^*_{np}(1)\varphi_{0k}(2)\psi_{n\beta}(1,2)\right]|np,0k\rangle , \tag{2.7}$$

with

$$\psi_{n\beta}(1,2)=\frac{(z_1-z_2-z_\beta)^n}{L(2\pi 2^n n!)^{1/2}}e^{-(1/4)(|z_1|^2+|z_2|^2+|z_\beta|^2)}\times e^{(1/2)(z_1^*z_2+z_1^*z_\beta-z_2z_\beta^*)} , \tag{2.8}$$

where $z_\beta=(iq_x-q_y)$ is the momentum $\mathbf{q}$ expressed as a complex number, and $L^2=\pi R^2$ is the sample area. This is done with the anticipation, motivated by previous studies of magnetic Hamiltonians,[9] that this momentum will be conserved. We find it helpful to write the matrix elements thus obtained in the manner

$$\langle m\alpha|\mathcal{H}|n\beta\rangle=\delta_{mn}\delta_{\alpha\beta}[E_{\rm HF}+(\varepsilon_n-\varepsilon_0)]+\sum_i\mathcal{E}^{(i)} . \tag{2.9}$$

Adopting the shorthand $A=(m,z_\alpha)$ and $B=(n,z_\beta)$, we have, for the 20 interaction terms,

$$\mathcal{E}^{(1)}=\int d1\lim_{3\to1}\int d2\,\psi_B(2,2)[\mathbf{A}_{12}\cdot(\mathbf{P}_1+\overline{\mathbf{A}}_1)]\times\psi^*_A(3,1) , \tag{2.10}$$

$$\mathcal{E}^{(2)}=\int d1\lim_{3\to1}\int d2\,\psi^*_A(2,2)[\mathbf{A}_{12}\cdot(\mathbf{P}_1+\overline{\mathbf{A}}_1)]\times\psi_B(1,3) , \tag{2.11}$$

$$\mathcal{E}^{(3)}=-\int d1\int d2\,\psi_B(2,1)[\mathbf{A}_{12}\cdot(\mathbf{P}_1+\overline{\mathbf{A}}_1)]\times\psi^*_A(2,1) , \tag{2.12}$$

$$\mathcal{E}^{(4)}=-\int d1\int d2\,\psi^*_A(1,2)[\mathbf{A}_{12}\cdot(\mathbf{P}_1+\overline{\mathbf{A}}_1)]\times\psi_B(1,2) , \tag{2.13}$$

$$\mathcal{E}^{(5)}=\int d1\int d2|\mathbf{A}_{12}|^2\psi^*_A(2,2)\psi_B(1,1) , \tag{2.14}$$

$$\mathcal{E}^{(6)}=-\int d1\int d2|\mathbf{A}_{12}|^2\psi^*_A(1,2)\psi_B(1,2) , \tag{2.15}$$

$$\mathcal{E}^{(7)}=-\int d1\int d2\int d3\,\mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(1,1)\times\psi^*_A(2,3)\psi_B(2,3) , \tag{2.16}$$

$$\mathcal{E}^{(8)}=-\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(1,3)\times\psi_A^*(1,3)\psi_B(2,2)\ , \quad (2.17)$$

$$\mathcal{E}^{(9)}=-\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(3,1)\times\psi_A^*(2,2)\psi_B(1,3)\ , \quad (2.18)$$

$$\mathcal{E}^{(10)}=-\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(1,2)\times\psi_A^*(3,3)\psi_B(2,1)\ , \quad (2.19)$$

$$\mathcal{E}^{(11)}=-\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(2,1)\times\psi_A^*(2,1)\psi_B(3,3)\ , \quad (2.20)$$

$$\mathcal{E}^{(12)}=-\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(3,2)\times\psi_A^*(1,1)\psi_B(2,3)\ , \quad (2.21)$$

$$\mathcal{E}^{(13)}=-\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(2,3)\times\psi_A^*(2,3)\psi_B(1,1)\ , \quad (2.22)$$

$$\mathcal{E}^{(14)}=\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(1,1)\times\psi_A^*(3,3)\psi_B(2,2)\ , \quad (2.23)$$

$$\mathcal{E}^{(15)}=\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(1,3)\times\psi_A^*(2,3)\psi_B(2,1)\ , \quad (2.24)$$

$$\mathcal{E}^{(16)}=\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(3,1)\times\psi_A^*(2,1)\psi_B(2,3)\ , \quad (2.25)$$

$$\mathcal{E}^{(17)}=\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(1,2)\times\psi_A^*(1,3)\psi_B(2,3)\ , \quad (2.26)$$

$$\mathcal{E}^{(18)}=\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(2,1)\times\psi_A^*(2,3)\psi_B(1,3)\ , \quad (2.27)$$

$$\mathcal{E}^{(19)}=\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(3,2)\times\psi_A^*(1,2)\psi_B(1,3)\ , \quad (2.28)$$

and

$$\mathcal{E}^{(20)}=\int d1\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi(3,2)\times\psi_A^*(3,1)\psi_B(2,1)\ , \quad (2.29)$$

where ψ is defined as in Eq. (2.8), $\mathbf{A}_{ij}$ is defined as in Eq. (1.3),

$$\Pi(z_1,z_2)=\sum_{k=0}^{\infty}\varphi_{0k}(z_1)\varphi_{0k}^*(z_2)=\frac{1}{2\pi}e^{-(1/4)(|z_1|^2+|z_2|^2)}e^{(1/2)z_1^*z_2} \quad (2.30)$$

is the projector for the lowest Landau level, and

$$\overline{\mathbf{A}}_1=\int d2\ \mathbf{A}_{12}\left[\sum_l|\varphi_l(2)|^2\right]=\tfrac{1}{2}\hat{\mathbf{z}}\times\mathbf{r}_1 \quad (2.31)$$

is the average vector potential generated by a uniform distribution of flux tubes.[1,6]

III. MOMENTUM CONSERVATION

Let us now explicitly evaluate the expressions for $\mathcal{E}^{(1)}$ through $\mathcal{E}^{(20)}$ and show them to be zero unless z_α equals z_β, thereby demonstrating that the momentum $\mathbf{q}$ to which z_α corresponds is a good quantum number. In what follows we shall make frequent use of the expressions

$$b=\tfrac{1}{2}|z_\alpha|^2=|\mathbf{q}|^2/2\ , \quad (3.1)$$

$$c=\frac{(-z_\alpha^*)^m(-z_\alpha)^n}{(2^m m!2^n n!)^{1/2}}\ , \quad (3.2)$$

$$f_n(b)=e^{-b}\sum_{k=0}^{n}\frac{b^k}{k!}\ , \quad (3.3)$$

and

$$E_b=\tfrac{1}{2}\int_0^b dx\frac{1}{x}(1-e^{-x})\ . \quad (3.4)$$

Note that E_b limits to $\frac{1}{2}[\ln(b)+\gamma]$ as $b\to\infty$. We shall also isolate the formally divergent quantities

$$I_1=\delta_{\alpha\beta}e^{-b}\frac{1}{2\pi}\int d^2z\frac{1}{|z|^2}e^{-(1/2)(z_\alpha^*z-z_\alpha z^*)}\ , \quad (3.5)$$

$$I_2=\delta_{\alpha\beta}e^{-b}\int_0^\infty dr\frac{1}{r}e^{-(1/2)r^2}\ , \quad (3.6)$$

and

$$I_3=\delta_{\alpha\beta}\delta_{mn}E_R\ , \quad (3.7)$$

where E_R is defined as in Eq. (1.12), as these cancel out of the final expression for the collective-mode energy. Note that some of the terms are defined only for $n\geq m$. The value of $\langle m\alpha|\mathcal{H}|n\beta\rangle$ for $n<m$ is the complex conjugate of the expression with the roles of m and n reversed, as appropriate for a Hermitian matrix. We have

$$\mathcal{E}^{(1)}=-\delta_{\alpha\beta}\frac{c}{2}e^{-b}\left[\frac{m}{b}-1\right]\ , \quad (3.8)$$

$$\mathcal{E}^{(2)}=-\delta_{\alpha\beta}\frac{c}{2}e^{-b}\left[\frac{n}{b}-1\right]\ , \quad (3.9)$$

$$\mathcal{E}^{(3)}=-\delta_{\alpha\beta}\frac{c}{2}e^{-b}\ , \quad (3.10)$$

$$\mathscr{E}^{(4)} \underset{n\geq m}{=} -\delta_{\alpha\beta}\delta_{mn}\frac{1}{2}-\delta_{\alpha\beta}\frac{c}{2}\left[\frac{m!}{b^m}f_m(b)-\frac{n!}{b^n}[1-f_{n-1}(b)]\right], \tag{3.11}$$

$$\mathscr{E}^{(5)}=cI_1\,, \tag{3.12}$$

$$\mathscr{E}^{(6)} \underset{n\geq m}{=} -cI_2-\delta_{\alpha\beta}\frac{c}{2}\left[n!e^{-b}\sum_{p=1}^{\infty}\frac{b^p}{(p+n)!p}-e^{-b}\sum_{k=1}^{n}\frac{1}{k}+\frac{1}{b}\sum_{k=0}^{m-1}\frac{k!}{b^k}f_k(b)\right], \tag{3.13}$$

$$\mathscr{E}^{(7)} \underset{n\geq m}{=} -I_3+\delta_{\alpha\beta}\delta_{mn}\left[E_b+\frac{1}{2}e^{-b}\sum_{p=0}^{n-1}\frac{b^p}{p!}\sum_{k=p+1}^{n}\frac{1}{k}\right]$$
$$-\delta_{\alpha\beta}(1-\delta_{mn})\frac{c}{2}\,\frac{1}{n-m}\left[\frac{m!}{b^m}f_{m-1}(b)+\frac{n!}{b^n}[1-f_{n-1}(b)]\right], \tag{3.14}$$

$$\mathscr{E}^{(8)}=\delta_{\alpha\beta}\frac{c}{4}e^{-b}\left[\frac{m}{b}-1\right], \tag{3.15}$$

$$\mathscr{E}^{(9)}=\delta_{\alpha\beta}\frac{c}{4}e^{-b}\left[\frac{n}{b}-1\right], \tag{3.16}$$

$$\mathscr{E}^{(10)}=-\delta_{\alpha\beta}\frac{c}{4}e^{-b}\left[\frac{1}{b}+\frac{1}{n+1}\right], \tag{3.17}$$

$$\mathscr{E}^{(11)}=-\delta_{\alpha\beta}\frac{c}{4}e^{-b}\left[\frac{1}{b}+\frac{1}{m+1}\right], \tag{3.18}$$

$$\mathscr{E}^{(12)}=-\frac{c}{2}(I_1-I_2)-\delta_{\alpha\beta}\frac{c}{4}e^{-b}\sum_{k=1}^{n}\frac{1}{k}, \tag{3.19}$$

$$\mathscr{E}^{(13)}=-\frac{c}{2}(I_1-I_2)-\delta_{\alpha\beta}\frac{c}{4}e^{-b}\sum_{k=1}^{m}\frac{1}{k}, \tag{3.20}$$

$$\mathscr{E}^{(14)}=\delta_{\alpha\beta}\frac{c}{2}e^{-b}\frac{1}{b}, \tag{3.21}$$

$$\mathscr{E}^{(15)}=\delta_{\alpha\beta}\frac{c}{4}e^{-b}, \tag{3.22}$$

$$\mathscr{E}^{(16)}=\delta_{\alpha\beta}\frac{c}{4}e^{-b}, \tag{3.23}$$

$$\mathscr{E}^{(17)} \underset{n\geq m}{=} \delta_{\alpha\beta}\frac{c}{4}\left[\frac{1}{n}\frac{n!}{b^n}[(1-\delta_{mn})-f_{n-1}(b)]+\frac{1}{n+1}\frac{m!}{b^m}f_m(b)\right], \tag{3.24}$$

$$\mathscr{E}^{(18)} \underset{n\geq m}{=} \delta_{\alpha\beta}\frac{c}{4}\left[-\frac{1}{m}\frac{m!}{b^m}f_{m-1}(b)-\frac{1}{m+1}\frac{n!}{b^n}[(1-\delta_{mn})-f_n(b)]\right], \tag{3.25}$$

$$\mathscr{E}^{(19)} \underset{n\geq m}{=} \delta_{\alpha\beta}\frac{c}{4}\left[\frac{m!}{b^m}e^b f_m(b)\frac{1}{b}\frac{n!}{b^n}[1-f_n(b)]+n!e^{-b}\sum_{p=1}^{\infty}\frac{b^p}{(p+n)!p}-e^{-b}\sum_{k=1}^{n}\frac{1}{k}+\frac{1}{b}\sum_{k=0}^{m-1}\frac{k!}{b^k}f_k(b)\right], \tag{3.26}$$

and

$$\mathscr{E}^{(20)}=\delta_{\alpha\beta}\frac{c}{4}e^{-b}\left[\frac{1}{b}+m!n!\sum_{p=1}^{\infty}\frac{b^p(p-1)!}{(p+m)!(p+n)!}\right]. \tag{3.27}$$

The presence of $\delta_{\alpha\beta}$ in each of these expressions shows that momentum is appropriately conserved.

IV. COLLECTIVE-MODE DISPERSION

Let us now use these expressions to evaluate the collective-mode dispersion relation. Since the momentum is a good quantum number, the desired energy $\mathscr{E}_{\mathbf{q}}$ is the lowest eigenvalue of the matrix

$$\mathscr{E}_{mn}=(\varepsilon_n-\varepsilon_0)\delta_{mn}+\sum_{i=1}^{20}\mathscr{E}_{mn}^{(i)}, \tag{4.1}$$

at fixed z_α. Substituting Eqs. (1.11) and (3.8)–(3.27) into this expression, we find that the matrix elements $\mathscr{E}_{mn}$ are finite and have the asymptotic limits

$$\mathcal{E}_{mn} \underset{q\to 0}{=} \delta_{mn} \left[n - \frac{3}{4} - \frac{1}{4}\delta_{n1} - \frac{1}{n(n+1)} + \sum_{k=1}^{n} \frac{1}{k} \right] + O(q^2) , \quad (4.2)$$

and

$$\mathcal{E}_{mn} \underset{q\to \infty}{=} \delta_{mn} \left[\ln(q) + n + \frac{1}{4} - \frac{1}{2}[\ln(2) - \gamma] - \frac{1}{4n(n+1)} + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} \right] + O(q^{-2}) . \quad (4.3)$$

This calculation therefore predicts that the collective mode possesses a gap of $\frac{1}{2}$ in units of the cyclotron energy (1.6) and grows like $\ln(q)$ for large q. For intermediate values of q, it is necessary to diagonalize $\mathcal{E}_{mn}$ numerically. The result of this calculation is plotted as the solid curve in Fig. 2. We find $\mathcal{E}_{\mathbf{q}}$ to increase quadratically with q from an energy gap of $\frac{1}{2}$ at $q=0$ as

$$\mathcal{E}_{\mathbf{q}} = 0.5 + 0.3675 q^2 + O(q^4) , \quad (4.4)$$

and then to roll over and grow logarithmically according to

$$\mathcal{E}_{\mathbf{q}} \cong \ln(q) + 1.567 + O(q^{-2}) . \quad (4.5)$$

Equations (4.4) and (4.5) are plotted as dashed and dotted curves in Fig. 2. $\mathcal{E}_{\mathbf{q}}$ differs from $\mathcal{E}_{11}$ mainly in the coefficient of q^2 at small q, but is otherwise nearly identical.

The behavior exhibited in Fig. 2 has the following physical interpretation. From Eq. (2.8) it may be seen that the separation $\mathbf{r}$ between the particle and the hole is related to the wave vector $\mathbf{q}$ by

$$\mathbf{r} = a_0^2 \hat{\mathbf{z}} \times \mathbf{q} , \quad (4.6)$$

with $\hat{\mathbf{z}}$ denoting the unit vector normal to the x-y plane and with a_0 defined as in Eq. (1.7), independent of the form of the attractive interparticle potential U.[8,9] We might therefore expect the slope of the energy dispersion curve to be given by Eqs. (1.13) and (4.6) as

$$\frac{1}{\hbar}\frac{d\mathcal{E}}{dq} = V = \frac{c}{eB}\frac{dU}{dr}\bigg|_{r\to q a_0^2} . \quad (4.7)$$

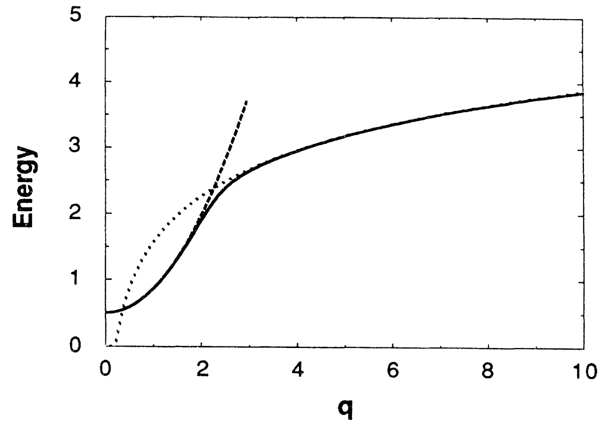

FIG. 2. The solid curve is the quantum-mechanical dispersion relation of the magnetoexciton for $\nu=0$. At $q=0$ it has an energy gap of $\frac{1}{2}$, and for $q<2$ it is well approximated by the dashed parabola [Eq. (4.4)]. For $q>3$ it is very nearly equal to Eq. (4.5), drawn as the dotted curve.

At long distances ($q \gg 1$), the particle and hole experience a logarithmic attraction, so that $d\mathcal{E}/dq \approx 1/q$, or

$$\mathcal{E}_{\mathbf{q}} \cong \ln(q) + \text{const} , \quad (4.8)$$

which agrees with Eq. (4.5). Since the circulations of the vortices corresponding to the particle and the hole vanish at the vortex cores,[1] $d\mathcal{E}/dq=0$ at short distances ($q \ll 1$), so that

$$\mathcal{E}_{\mathbf{q}} = \text{const} + O(q^2) , \quad (4.9)$$

consistent with Eq. (4.4).

V. COMPLEX INTEGRATION

We shall now discuss some of the algebraic steps leading to Eqs. (3.8)–(3.27).

We first consider the two-body terms $\mathcal{E}^{(1)}$ through $\mathcal{E}^{(6)}$. The integrals appearing in these expressions may be evaluated by changing variables to $z=z_1-z_2$ and $s=\frac{1}{2}(z_1+z_2)$. In every case the integral over the center-of-mass variable s gives $L^2\delta_{\alpha\beta}$. We begin with the four terms containing

$$\mathbf{A}_{12}\cdot(\mathbf{P}_1+\overline{\mathbf{A}}_1) = (1-\nu)\left[\frac{1}{z_1^*-z_2^*}\left[\frac{1}{4}z_1^* + \frac{\partial}{\partial z_1}\right] + \frac{1}{z_1-z_2}\left[\frac{1}{4}z_1 - \frac{\partial}{\partial z_1^*}\right]\right] . \quad (5.1)$$

Equation (2.10) may be written

$$\mathcal{E}^{(1)} = \frac{e^{-(1/4)(|z_\alpha|^2+|z_\beta|^2)}}{2\pi L^2(2^m m! 2^n n!)^{1/2}}(-z_\alpha^*)^m(-z_\beta)^n\left[\frac{1}{2} - \frac{m}{|z_\alpha|^2}\right] z_\alpha \int d1 \int d2\, e^{(1/2)(z_2^* z_\beta - z_2 z_\beta^*)} \frac{1}{z_1-z_2} e^{(1/2)(z_1 z_\alpha^* - z_1^* z_\alpha)}$$

$$= -\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m(-z_\alpha)^n}{(2^m m! 2^n n!)^{1/2}}\frac{1}{2}e^{-b}\left[\frac{m}{b} - 1\right]\frac{1}{\pi}\int d^2 z \frac{1}{z}\left[\frac{-\partial}{\partial z^*}\right] e^{(1/2)(z z_\alpha^* - z_\alpha z^*)} . \quad (5.2)$$

Equation (3.8) is obtained from this by integrating by parts and using the expression

$$\delta(\mathbf{r})=\nabla^2\left[\frac{1}{2\pi}\ln(r)\right]=4\left[\frac{\partial}{\partial z^*}\right]\left[\frac{\partial}{\partial z}\right]\left[\frac{1}{4\pi}\ln(z^*z)\right]=\frac{1}{\pi}\left[\frac{\partial}{\partial z^*}\right]\left[\frac{1}{z}\right]=\frac{1}{\pi}\left[\frac{\partial}{\partial z}\right]\left[\frac{1}{z^*}\right], \tag{5.3}$$

for the two-dimensional δ function. $\mathcal{E}^{(2)}$ is the Hermitian conjugate of $\mathcal{E}^{(1)}$. Equation (2.12) gives

$$\mathcal{E}^{(3)}=\frac{\delta_{\alpha\beta}}{2\pi(2^m m!2^n n!)^{1/2}}\int d^2z\frac{1}{z}\left[\frac{\partial}{\partial z^*}\right][(-z-z_\alpha)^n(-z^*-z_\alpha^*)^m e^{-(1/2)|z+z_\alpha|^2}] . \tag{5.4}$$

Equation (3.10) is obtained from this by integrating by parts and using Eq. (5.3). For $n\geq m$, Eq. (2.13) becomes

$$\mathcal{E}^{(4)}=\frac{-\delta_{\alpha\beta}}{(2^m m!2^n!)^{1/2}}\frac{1}{2\pi}\int d^2z\, e^{-(1/2)|z|^2}(z^*)^m z^n\left[\frac{n}{z}\frac{1}{z^*+z_\alpha^*}+\frac{z}{2}\frac{1}{z+z_\alpha}\right], \tag{5.5}$$

when integrated by parts using Eq. (5.3), transformed to the variables $z=z_1-z_2-z_\alpha$ and $s=\frac{1}{2}(z_1+z_2)$, and integrated on s. Equation (3.11) is obtained from this by writing $z=ru$, where $u=e^{i\theta}$, and performing the angular integral as a contour integral over u for $|u|=1$. Equation (3.12) is obtained from Eq. (2.14) by changing variables to $z=z_1-z_2$ and $s=\frac{1}{2}(z_1+z_2)$ and integrating on s. Finally, Eq. (2.15) becomes

$$\mathcal{E}^{(6)}=-\frac{\delta_{\alpha\beta}}{(2^m m!2^n n!)^{1/2}}\frac{1}{2\pi}\int d^2z\frac{1}{|z+z_\alpha|^2}e^{-(1/2)|z|^2}(z^*)^m z^n , \tag{5.6}$$

when transformed to the variables $z=z_1-z_2-z_\alpha$ and $s=\frac{1}{2}(z_1+z_2)$ and integrated on s. Performing the angular integration as a contour integral, we obtain the expression

$$\mathcal{E}^{(6)}=-\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m(-z_\alpha)^n}{(2^m m!2^n n!)^{1/2}}\frac{1}{2}\left[b^{-n}\int_0^b dx\frac{x^n e^{-x}}{b-x}+b^{-m}\int_b^\infty dx\frac{x^m e^{-x}}{x-b}\right], \tag{5.7}$$

with $x=r^2/2$. This is then manipulated using the identity

$$\frac{x^n}{b-x}=\frac{b^n}{b-x}-\sum_{k=0}^{n-1}x^k b^{n-1-k} , \tag{5.8}$$

to yield

$$\begin{aligned}\mathcal{E}^{(6)}=&-\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m(-z_\alpha)^n}{(2^m m!2^n n!)^{1/2}}\frac{1}{2\pi}\int d^2z\frac{1}{|z+z_\alpha|^2}e^{-(1/2)|z|^2}\\ &-\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m(-z_\alpha)^n}{(2^m m!2^n n!)^{1/2}}\frac{1}{2b}\left[-\sum_{k=0}^{n-1}\frac{k!}{b^k}[1-f_k(b)]+\sum_{k=0}^{m-1}\frac{k!}{b^k}f_k(b)\right] .\end{aligned} \tag{5.9}$$

The remaining integral is then performed by changing variables to $y=z+z_\alpha$ and expanding the exponential in powers of y and y^*. We obtain

$$-\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m(-z_\alpha)^n}{(2^m m!2^n n!)^{1/2}}\frac{1}{2\pi}\int d^2z\frac{1}{|z+z_\alpha|^2}e^{-(1/2)|z|^2}=-cI_2-\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m(-z_\alpha)^n}{(2^m m!2^n n!)^{1/2}}\frac{1}{2}e^{-b}\sum_{p=1}^{\infty}\frac{b^p}{p!p} . \tag{5.10}$$

Equation (3.13) is then obtained from this using the identities

$$\frac{1}{b^{k+1}}[1-f_k(b)]=e^{-b}\sum_{p=0}^{\infty}\frac{b^p}{(p+k+1)!} \tag{5.11}$$

and

$$\frac{1}{p!p}-\sum_{k=0}^{n-1}\frac{k!}{(k+p+1)!}=\frac{n!}{(p+n)!p} , \tag{5.12}$$

the latter of which may be generated by iterating the expression

$$\frac{1}{(p+k)!p}=\frac{1}{(p+k+1)!}+\frac{k+1}{(p+k+1)!p} , \tag{5.13}$$

beginning with $k=0$.

We shall next consider the first seven three-body terms $\mathcal{E}^{(7)}$ through $\mathcal{E}^{(13)}$. We shall make frequent use of the identity

$$\mathbf{A}_{12}\cdot\mathbf{A}_{13}=\frac{1}{2}(1-\nu)^2\left[\frac{1}{(z_1-z_2)(z_1-z_3)^*}+\frac{1}{(z_1-z_2)^*(z_1-z_3)}\right]. \tag{5.14}$$

Equation (2.16) becomes

$$\mathcal{E}^{(7)}=-\delta_{\alpha\beta}\frac{1}{8\pi^2(2^m m!2^n n!)^{1/2}}\int d^2y\, e^{-(1/2)|y|^2}(y^*)^m(y)^n\int d^2z\left[\frac{1}{(y+z)^*(z-z_\alpha)}+\frac{1}{(y+z)(z-z_\alpha)^*}\right], \tag{5.15}$$

when transformed to the variables $y=z_2-z_3-z_\alpha$ and $z=z_3-z_1+z_\alpha$ and integrated over z_1. Equation (3.14) is obtained from this by performing the angular integration over y and then over z using contour integration, taking care to work out the logarithmically infinite $z_\alpha=0$ contribution separately, and using the identity

$$\sum_{k=n}^{\infty}\frac{k!}{(k+p+1)!}=\frac{n!}{(p+n)!p}, \tag{5.16}$$

which follows from Eq. (5.12). Equation (2.17) becomes

$$\mathcal{E}^{(8)}=-\delta_{\alpha\beta}\frac{(-z_\alpha)^n}{8\pi^2(2^m m!2^n n!)^{1/2}}e^{-b}\int d^2z\, e^{-(1/2)|z|^2}(z^*-z_\alpha^*)^m e^{(1/2)zz_\alpha^*}\int d^2y\left[\frac{1}{(z-y)z^*}+\frac{1}{(z-y)^*z}\right]e^{(1/2)(y^*z_\alpha-yz_\alpha^*)}, \tag{5.17}$$

when transformed to the variables $y=z_2-z_3$ and $z=z_1-z_3$ and integrated on $s=\frac{1}{2}(z_1+z_3)$. Performing the angular integration over y using contour integration and reversing the order of integration, we obtain

$$\mathcal{E}^{(8)}=-\delta_{\alpha\beta}\frac{(-z_\alpha)^n}{8\pi^2(2^m m!2^n n!)^{1/2}}e^{-b}\int d^2z\, e^{-(1/2)|z|^2}(z^*-z_\alpha^*)^m e^{(1/2)zz_\alpha^*}2\pi\left[\frac{1}{z_\alpha z^*}-\frac{1}{z_\alpha^* z}\right]. \tag{5.18}$$

Equation (3.15) is obtained from this by expanding the first line of the integrand in powers of z^* and integrating. Equation (2.18) is the Hermitian conjugate of Eq. (2.17). Equation (2.19) becomes

$$\mathcal{E}^{(10)}=-\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m}{8\pi^2(2^m m!2^n n!)^{1/2}}e^{-b}\int d^2z\, e^{-(1/2)|z|^2}(-z-z_\alpha)^n e^{-(1/2)z_\alpha z^*}2\pi\left[\frac{1}{z_\alpha^* z}-\frac{1}{z_\alpha z^*}\right], \tag{5.19}$$

when transformed to the variables $y=z_3$ and $z=z_1-z_2$, integrated over y as in Eq. (5.17) and integrated on $s=\frac{1}{2}(z_1+z_2)$. Equation (3.17) is then obtained by expanding the first line of the integrand in powers of z and z^* and integrating. Equation (2.20) is the Hermitian conjugate of Eq. (2.19). Equation (2.21) divides naturally into two parts, corresponding to the two terms of Eq. (5.14) for $\mathbf{A}_{12}\cdot\mathbf{A}_{13}$. Exchanging the integration variables $2\leftrightarrow3$ in the first term, and then changing variables to $y=z_1-z_2$ and $z=z_2-z_3$ and integrating on $s=\frac{1}{2}(z_1+z_2)$, we obtain

$$\mathcal{E}^{(12)}=-\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m}{8\pi^2(2^m m!2^n n!)^{1/2}}e^{-b}\times\int d^2y\frac{1}{y^*}e^{(1/2)(yz_\alpha^*-y^*z_\alpha)}\int d^2z\frac{1}{z+y}e^{-(1/2)|z|^2}[(-z-z_\alpha)^n e^{-(1/2)z_\alpha z^*}+(z-z_\alpha)^n e^{(1/2)z_\alpha^* z}]. \tag{5.20}$$

Rewriting the denominator of the first term in the z integral as a sum of partial derivatives with respect to z^* and integrating by parts using Eq. (5.3), we find that this term gives zero upon integration over y. The second term in the z integral may be done as a contour integral. Equation (3.19) is obtained from this using the identity

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k}=-\sum_{k=1}^{n}\frac{1}{k}. \tag{5.21}$$

Equation (2.22) is the Hermitian conjugate of Eq. (2.21).

We consider finally the remaining three-body terms $\mathcal{E}^{(14)}$ through $\mathcal{E}^{(20)}$. Equation (2.23) becomes

$$\mathcal{E}^{(14)}=\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m(-z_\alpha)^n}{8\pi^2(2^m m!2^n n!)^{1/2}}e^{-b}\int d^2y\int d^2z\left[\frac{1}{y(y+z)^*}+\frac{1}{y^*(y+z)}\right]e^{-(1/2)(z_\alpha^* z-z_\alpha z^*)}, \tag{5.22}$$

when transformed to the variable $y=z_1-z_2$ and $z=z_2-z_3$ and integrated on $s=\frac{1}{2}(z_2+z_3)$. Equation (3.21) is obtained from this by performing the substitution

$$\left[\frac{1}{y(y+z)^*}+\frac{1}{y^*(y+z)}\right]e^{-(1/2)(z_\alpha^* z-z_\alpha z^*)}=\left[\frac{1}{y(y+z)^*}\left[\frac{-2}{z_\alpha^*}\frac{\partial}{\partial z}\right]+\frac{1}{y^*(y+z)}\left[\frac{2}{z_\alpha}\frac{\partial}{\partial z^*}\right]\right]e^{-(1/2)(z_\alpha^* z-z_\alpha z^*)}, \tag{5.23}$$

integrating by parts on z using Eq. (5.3) and then using the same trick to perform the y integral. Equation (2.24) becomes

$$\mathcal{E}^{(15)}=\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m(-z_\alpha)^n}{8\pi^2(2^m m!2^n n!)^{1/2}}\int d^2y\, e^{-(1/2)|y|^2}(-y)^n\int d^2z\left[\frac{1}{(y-z_\alpha)(y+z)^*}+\frac{1}{(y-z_\alpha)^*(y+z)}\right]e^{-(1/2)|z|^2}(z^*)^m e^{-(1/2)y^*z},$$

when transformed to the variables $y=z_1-z_2+z_\beta$ and $z=z_2-z_3-z_\alpha$ and integrated on $s=\frac{1}{2}(z_2+z_3)$. Equation (3.22) is obtained from this by transforming to the variable $w=z+y$, expanding in powers of w^*, and integrating. The contribution of the second term in large parentheses is zero. Equation (2.25) is the Hermitian conjugate of Eq. (2.24). Equation (2.26) divides naturally into two parts, corresponding to the two terms of Eq. (5.14) for $\mathbf{A}_{12}\cdot\mathbf{A}_{13}$. The first part contains $(z_1-z_2)^{-1}$, so that we may perform the z_2 integration by expressing $\Pi\psi_A^*\psi_B$ as a partial derivative with respect to z_2^*, integrate by parts, and use Eq. (5.3). We then transform to the variables $z=z_1-z_3$ and $s=\frac{1}{2}(z_1+z_3)$ and integrate on s. To evaluate the second piece, we use the change of variables $y=z_1-z_3-z_\alpha$, $z=z_3-z_2+z_\beta$, and $s=\frac{1}{2}(z_3+z_2)$, integrate on s, and then integrate over z using a contour integral. Equation (2.26) then becomes

$$\mathcal{E}^{(17)}=\delta_{\alpha\beta}\frac{(-1/4\pi)}{(2^m m!2^n n!)^{1/2}}\int d^2z\, e^{-(1/2)|z|^2}\frac{(z^*)^m z^{n-1}}{(z+z_\alpha)^*}+\delta_{\alpha\beta}\frac{1/[8\pi(n+1)]}{(2^m m!2^n n!)^{1/2}}\int d^2y\, e^{-(1/2)|y|^2}\frac{(y^*)^m y^{n+1}}{y+z_\alpha}. \quad (5.25)$$

Equation (3.24) is obtained from this using contour integration. Equation (2.27) is the Hermitian conjugate of Eq. (2.26). Equation (2.28) becomes

$$\mathcal{E}^{(19)}=\delta_{\alpha\beta}\frac{1/(8\pi^2)}{(2^m m!2^n n!)^{1/2}}\times\int d^2z\, e^{-(1/2)|z|^2}(-z^*)^m\int d^2y\, e^{-(1/2)|y|^2}(-y)^n\left[\frac{1}{(z-z_\alpha)(y-z_\alpha)^*}+\frac{1}{(y-z_\alpha)(z-z_\alpha)^*}\right]e^{(1/2)zy^*}, \quad (5.26)$$

when transformed to the variables $z=z_2-z_1+z_\alpha$ and $y=z_3-z_1+z_\beta$ and integrated on z_1. The z and y integrals are then evaluated by expanding the last exponential in powers of zy^* and using the identity

$$\int d^2y\, e^{-(1/2)|y|^2}\frac{y^n(y^*)^p}{y-z_\alpha}=-2\pi 2^p p!z_\alpha^{n-p-1}[1-f_p(b)-\Theta(n-p)], \quad (5.27)$$

with

$$\Theta(n)=\begin{cases}1, & n>0,\\ 0, & n\le 0,\end{cases} \quad (5.28)$$

obtained by contour integration. This gives

$$\mathcal{E}^{(19)}=\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m(-z_\alpha)^n}{(2^m m!2^n n!)^{1/2}}\frac{1}{4b}\sum_{p=0}^{\infty}\left[\frac{p!}{b^p}[1-f_p(b)-\Theta(n-p)][1-f_p(b)-\Theta(m-p)]+\frac{b^p}{p!}\frac{m!n!}{b^{m+n}}[1-f_n(b)-\Theta(p-n)][1-f_m(b)-\Theta(p-m)]\right], \quad (5.29)$$

which, in light of Eq. (5.12) and the identity

$$\sum_{p=0}^{\infty}\frac{p!}{b^p}[1-f_p(b)]^2=\sum_{p=0}^{\infty}\frac{p!}{b^p}\left[\frac{1}{p!}\int_0^b dx\, e^{-x}x^p\right]\left[\frac{1}{p!}\int_0^b dy\, e^{-y}y^p\right]=\int_0^b dx\int_0^b dy\, e^{-(x+y)}e^{xy/b}=be^{-b}\int_0^b dy\frac{e^y-1}{y}=be^{-b}\sum_{p=1}^{\infty}\frac{b^p}{p!p}, \quad (5.30)$$

is the same as Eq. (3.26). Following this same procedure for Eq. (2.29), we obtain

$$\mathcal{E}^{(20)}=\delta_{\alpha\beta}\frac{(-z_\alpha^*)^m(-z_\alpha)^n}{(2^m m!2^n n!)^{1/2}}\frac{1}{4b}\left[e^{-b}+b^{-n-m}\sum_{p=0}^{\infty}\frac{b^{-p}}{p!}(n+p)![1-f_{n+p}(x)](m+p)![1-f_{m+p}(x)]\right]. \quad (5.31)$$

Repeating also the procedure of Eq. (5.30), we obtain the identity

$$b^{-n-m}\sum_{p=0}^{\infty}\frac{b^{-p}}{p!}(n+p)![1-f_{n+p}(x)](m+p)![1-f_{m+p}(x)]=\int_0^b dx\int_0^b dy\, e^{-(x+y)}e^{xy/b}\left[\frac{x}{b}\right]^n\left[\frac{y}{b}\right]^m , \tag{5.32}$$

which may be transformed to the variable to $w=b-x$ and integrated over y to yield

$$be^{-b}\int_0^b dw\, e^{w}\left[1-\frac{w}{b}\right]^n\frac{m!}{w^{m+1}}[1-f_m(w)]=be^{-b}\sum_{p=1}^{\infty}b^p\frac{m!}{(p+m)!}\int_0^1 dx\,(1-x)^n x^{p-1} , \tag{5.33}$$

where we have used Eq. (5.11). Since Eq. (5.32) is manifestly invariant under the interchange $m\leftrightarrow n$, it follows from Eq. (5.33) that

$$\frac{(p+n)!}{n!}\int_0^1 dx(1-x)^n x^{p-1}=(p-1)! . \tag{5.34}$$

This result, in conjunction with Eqs. (5.31)–(5.33), gives Eq. (3.27).

VI. DISCUSSION

While it is significant that the particle-hole interaction discussed in this paper causes the collective mode to be finite, our most important result is the prediction of an energy gap at $\mathbf{q}\rightarrow 0$ for $\nu=0$, and thus presumably for $\nu=\frac{1}{2}$. We know this gap to be unphysical because it implies the existence of a preferred density for the particles. Furthermore, since $\rho_{\mathbf{q}}|0\rangle$ effectively defines compressional sound at long wavelengths, we are forced to ascribe the unphysical behavior to the ground state itself, rather than to the choice of excited state. While it is common[2] for variational fluid ground states to have wrong long-wavelength properties, it is completely unprecedented for them to produce a gap when used in a calculation of this kind. We must conclude, therefore, that this gap is a pathology of the formalism we have chosen, and that some new type of formal manipulation is required to eliminate it.

Let us now consider what might distinguish this problem from a traditional quantum fluid calculation. One difference is that the static structure factor, defined by

$$S_{\mathbf{q}}=1+\rho\int[g(\mathbf{r})-1]e^{-i\mathbf{q}\cdot\mathbf{r}}d\mathbf{r} , \tag{6.1}$$

where ρ is the particle density and $g(\mathbf{r})$ is the radial distribution function defined by Eq. (1.8), is forced by Fermi statistics to limit to 0 as $\mathbf{q}\rightarrow 0$. The use of any such structure factor in the Feynman-Bijl formula[10]

$$\mathcal{E}_{\mathbf{q}}\cong\frac{1}{2}\frac{\langle 0|[\rho_{-\mathbf{q}},[\mathcal{H},\rho_{\mathbf{q}}]]|0\rangle}{\langle 0|\rho_{-\mathbf{q}}\rho_{\mathbf{q}}|0\rangle}=\frac{1}{2}\frac{|\mathbf{q}|^2}{S_{\mathbf{q}}} , \tag{6.2}$$

will always produce a gap at $\mathbf{q}\rightarrow 0$ unless, as occurs with the Fermi sea,[7,11] $S_{\mathbf{q}}$ is nonanalytic at $q=0$. (Note that because the Hartree-Fock ground state is not exact, this formula gives a gap of 1 rather than the value of $\frac{1}{2}$ found by us numerically.) Thus the true behavior of a Bose fluid can be obtained in the Fermi representation only if the exchange-correlation hole is distended and equal in extent to the sample size. This, in turn, requires the presence of *long-range forces* in the Fermi representation. Since the fractional statistics interaction itself is very long ranged, we may identify the range of the interactions as another characteristic of this problem. The fractional statistics interaction is inherently "Coulombic" in nature. This may be seen most easily by observing that some of the three-body terms discussed in this paper are equivalent to logarithmic two-body terms. Still another feature distinguishing the problem is the presence of a fictitious magnetic field, which formally fixes the particle density and may thus be thought of as causing the gap. However, since the gap may also be accounted for by Eq. (6.2), the presence of this field must be related in a fundamental way to both the sum rule fixing $S_{\mathbf{q}}$ at $\mathbf{q}\rightarrow 0$ and the presence of long-range forces.

In light of these considerations, it is clear that the use of Hartree-Fock wave functions to calculate the density-density response function failed in this problem for the same reason that it fails[7,11] in an ordinary metal, namely, that the long-range properties of both the ground state and response functions are severely modified by the presence of long-range forces. It was this idea which motivated us to study the fractional-statistics gas in the random-phase approximation[4] and to show that this correctly accounted for all the superfluid properties. The next paper in this series is a detailed account of this study.

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