Superconductivity in a very high magnetic field

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An external magnetic field does not destroy superconductivity. As the field increases, the diamagnetic pair breaking is eliminated and the Abrikosov flux lattice crosses over into a new quantum limit, characterized by a transition temperature that is an increasing function of the field, the virtual absence of the Meissner effect, and a supercurrent flow along the field direction. The transition temperature remains finite in an arbitrarily strong external field as long as both spin states are present. Such a superconducting state in a very high magnetic field can occur irrespective of the nature of the ground state at low fields. We study various properties of this new state and discuss the relevance of our results for experimental work in high magnetic fields.

I. INTRODUCTION

In a classic paper[1] Abrikosov has shown that in type-II superconductors, the Meissner effect is incomplete and part of the external magnetic-field flux penetrates into the superconductor, resulting in a lattice of quantized vortex lines. This theory and its more microscopic extensions[2] lead to a critical temperature which decreases monotonically to zero as a function of the external field H_{c2} (the dashed line in Fig. 1). T_c is identically zero when $H_{c2}(T=0)=\phi_0/2\pi\xi_0^2$, where ϕ_0 is the elementary flux and ξ_0 is the coherence length at zero temperature. This theory, which in this paper we call the Abrikosov-Gorkov (AG) theory, has been applied, rather successfully, to numerous known type-II systems.[3]

Recently, there has been renewed interest in the circumstances under which the AG theory is not appropriate. The AG curve in Fig. 1 is a consequence of a commonly employed *semiclassical* approximation, in which the Landau-level structure of the electron system is completely neglected. The proper inclusion of the Landau levels leads to several consequences. The H_{c2} curve or, more precisely, the $T_c(H)$ curve develops oscillations similar in origin to the familiar De Haas–Van Alphen oscillations in the normal state.[4] This leads to a reentrant superconducting behavior, with interesting properties.[5] Furthermore, a new regime arises at very high fields, in which the superconducting transition temperature becomes enhanced by the external field.[6–8] This enhancement is most significant in the "quantum" limit in which only the lowest Landau level is occupied: For typical type-II materials, like Pb or Nb alloys, the experimental realization of this limit would require enormous magnetic fields (thousands of teslas!) and is completely unrealistic. However, there is a large class of systems in which the quantum limit can be reached with available laboratory fields. These systems are low-carrier-density semiconductors and semimetals (LCDSS's), many of which are superconductors in zero field.[9] In the quantum limit, a quasi-one-dimensional nature of the particle dispersion leads to a variety of possible instabilities somewhat reminiscent of a true one-dimensional case: spin-density wave (SDW),[10] charge-density wave (CDW), valley-density wave (VDW),[11] etc. We argue here that a superconducting instability can also compete as a candidate for the ground state of the system even in these very strong external fields.

In this paper we present a theoretical study of the influence of the Landau-orbit quantization on the properties of superconductors in a very strong external magnetic field. It is first necessary to define what is meant by a "very strong field." Here we will use this term to describe the situation where only several of the Landau levels are occupied by carriers (the number of the occupied Landau levels we denote by n_c). Thus n_c is of order unity, and the correct quantized motion of the carriers has to be included from the outset: Various semiclassical approximations fail in this limit. Of course, in the familiar AG theory such a very high-field limit (VHFL) does not exist, and the superconducting state is destroyed long before one gets into this regime. This is a consequence of the complete neglect of Landau orbits in the AG theory (n_c is taken to infinity), which leads to a depression of the superconducting transition temperature T_c as the field increases. The AG theory is an excellent approximation for almost all cases of interest. However, as we will discuss below, if n_c is of order unity, a *qualitatively* different situation arises in which T_c becomes a rapidly *increasing* function of H. This new behavior is a result of the van-

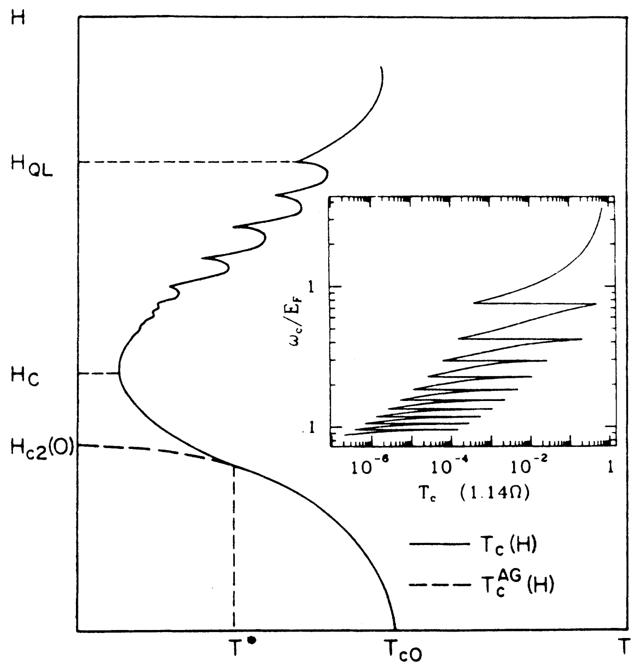

FIG. 1. Full line is an artist's rendering of the $T_c(H)$ curve for a BCS-like type-II superconductor. The dashed line is the AG result. The physical significance of H_C, H_{QL}, and T^* is explained in the text. The inset shows T_c evaluated from (2.3) in the high-field limit including up to 11 Landau levels [we have chosen $N(0)V=0.6$ and have used the standard Coulomb cutoff to round off divergencies in the density of states (see text for discussion of this point)]. Actually, for LCDSS's in the high-field limit, $T_c(H)$ shown here represents a lower bound for a BCS superconductor. Further enhancement of the coupling constant will arise due to a reduction in the average Coulomb repulsion when $k_F^{-1} \simeq l$ (Ref. 6). However, see Sec. IV for discussion of strong-coupling effects.

ishing of the diamagnetic pair breaking in the VHFL. This is an aspect of superconductivity that has not been much explored in the past, and here we discuss some of the properties of this new regime, as well as the question of the circumstances in which this new state might be observed. The paper is organized as follows: In Sec. II we evaluate the $T_c(H)$ curve in the VHFL and discuss the evolution of $T_c(H)$ from the semiclassical, low-field regime to the VHFL. Section III discusses the effects of Pauli pair breaking, disorder, etc., on $T_c(H)$, with an emphasis on the VHFL. Discussion of strong-coupling corrections and numerical results for T_c for several simple models is given in Sec. IV. In Sec. V we turn to the question of the behavior below $T_c(H)$: We present a systematic scheme which leads to the *exact* solution of the nonlinear Ginsburg-Landau (GL) equations in the quantum limit (QL), when $n_c=1$ (to our knowledge, this is the first example of the exact solution of the GL theory in an external magnetic field). Section VI contains discussion of the experimental relevance of our results and conclusions.

II. TRANSITION TEMPERATURE

In this section we study the orbital effect of an external field on the transition temperature of a superconductor, i.e., the diamagnetic pair breaking (DPB). In the AG theory of type-II superconductivity, DPB arises through the impossibility of having a uniform order parameter in the presence of a field penetrating into the interior of a superconductor. The nonuniformity is required in order to accommodate zeros of the order parameters in the cores of the quantized field vortices. We start from the partition function Z for a BCS superconductor which, in an arbitrarily strong uniform magnetic field $\mathbf{H}$, can be written as

$$Z=\int \mathcal{D}\psi_\sigma(\mathbf{r},\tau)\mathcal{D}\bar{\psi}_\sigma(\mathbf{r},\tau)\mathcal{D}\mathbf{a}(\mathbf{r})\exp\left[-\int_0^\beta d\tau\, L(\tau)\right], \tag{2.1}$$

where $\int \mathcal{D}\psi_\sigma(\mathbf{r},\tau)\mathcal{D}\bar{\psi}_\sigma(\mathbf{r},\tau)$ denotes the functional integration over Grassman variables $\psi_\sigma(\mathbf{r},\tau)$ and $\bar{\psi}_\sigma(\mathbf{r},\tau)$, $\mathbf{a}(\mathbf{r})$ is the fluctuating part of the vector potential, and

$$L(\tau)=\sum_\sigma \int d^3r \left[\bar{\psi}_\sigma(\mathbf{r},\tau)\partial_\tau\psi_\sigma(\mathbf{r},\tau)+\frac{1}{2m}D_{\mathbf{r}}^*\bar{\psi}_\sigma(\mathbf{r},\tau)D_{\mathbf{r}}\psi_\sigma(\mathbf{r},\tau) + \frac{V}{2}\bar{\psi}_\sigma(\mathbf{r},\tau)\bar{\psi}_{-\sigma}(\mathbf{r},\tau)\psi_{-\sigma}(\mathbf{r},\tau)\psi_\sigma(\mathbf{r},\tau)-g\mu_B\sigma H\bar{\psi}_\sigma(\mathbf{r},\tau)\psi_\sigma(\mathbf{r},\tau)-\frac{[\mathbf{H}+\mathbf{b}(\mathbf{r})]^2}{8\pi}\right]. \tag{2.2}$$

In Eq. (2.2), $D_{\mathbf{r}}\equiv\partial_{\mathbf{r}}-(ie/c)[\mathbf{A}(\mathbf{r})+\mathbf{a}(\mathbf{r})]$, $\mathbf{H}\equiv\nabla\times\mathbf{A}(\mathbf{r})$, $\mathbf{b}(\mathbf{r})\equiv\nabla\times\mathbf{a}(\mathbf{r})$, and V is the BCS attractive interaction. Since we first want to concentrate on the DPB in this section, we ignore the Pauli pair breaking (PPB) and assume that the g factor of the carriers equals zero. We will discuss PPB in the following section. Also, to keep our discussion at a simple and more general level, we have assumed that V is a weak function of the magnetic field. This assumption needs to be reexamined for each specific physical situation. In low-carrier-density semimetals and semiconductors, V represents an electron-phonon and electron-electron interaction the form of which in the VHFL has been discussed in Ref. 6.

In the BCS theory one approximates (2.1) by self-consistently evaluating $\langle\psi_\sigma(\mathbf{r},\tau)\psi_{-\sigma}(\mathbf{r},\tau)\rangle$ and $\langle\mathbf{a}(\mathbf{r})\rangle$, where $\langle\cdots\rangle$ denotes the expectation value. T_c is determined from the spectrum of the following integral equation:

$$\frac{1}{V}\Delta(\mathbf{r})=\beta^{-1}\sum_{\omega_\nu}\int d^3r' G^H_\sigma(\mathbf{r},\mathbf{r}';\omega_\nu)\times G^H_{-\sigma}(\mathbf{r},\mathbf{r}';-\omega_\nu)\Delta(\mathbf{r}')\ , \tag{2.3}$$

where

$$\Delta(\mathbf{r})\equiv V\beta^{-1}\sum_{\omega_\nu}\langle\bar{\psi}_{\omega_\nu,\sigma}(\mathbf{r})\bar{\psi}_{-\omega_\nu,-\sigma}(\mathbf{r})\rangle\ ,$$

$\omega_\nu=(2\nu+1)\pi/\beta$ are the Matsubara frequencies, and $G^H_\sigma(\mathbf{r},\mathbf{r}';\omega_\nu)$ is the Green's function in the presence of a uniform magnetic field $\mathbf{H}$ (we use symmetric gauge[12]).

For the zero-field case, Eq. (2.3) yields the familiar expression $T_{c0}=1.14\Omega\exp[-1/N(0)V]$, where Ω is some weak-coupling cutoff frequency and $N(0)$ is the density of states at the Fermi level. If $\mathbf{H}\neq 0$, however, the analytic solution of (2.3) is not known. To evaluate $T_c(H)$, or equivalently $H_{c2}(T)$, and to study the nature of the flux lattice below H_{c2}, one routinely uses the semiclassical phase integral approximation (SCPIA), originally due to Gor'kov.[2] The Green's function $G^H_\sigma(\mathbf{r},\mathbf{r}';\omega_\nu)$ is approximated by

$$G^{H=0}_\sigma(\mathbf{r}-\mathbf{r}';\omega_\nu)\exp\left[\frac{ie}{c}\int_{\mathbf{r}'}^{\mathbf{r}}d\mathbf{s}\cdot\mathbf{A}(\mathbf{s})\right]\ ,$$

where the path of integration between $\mathbf{r}'$ and $\mathbf{r}$ is a straight line. This approximation is a *crucial* step in obtaining the standard $H_{c2}(T)$ curve, and it amounts to a neglect of Landau levels. The SCPIA becomes accurate if the bending of the semiclassical paths by a magnetic field is negligible over the range of $G^{H=0}(\mathbf{r}-\mathbf{r}',\omega_\nu)$. The latter is given by $v_F/2|\omega_\nu|$, while the radius of a semiclassical path equals l^2k_F, where v_F and k_F are the Fermi velocity and wave vector, and $l=\sqrt{c/eH}$. Thus the condition for the validity of SCPIA reads $l^2k_F\gg v_F/2\pi T$ or, equivalently, $\omega_c\ll 2\pi T$.

The first hint that something interesting may be happening is that the SCPIA *necessarily* breaks down at sufficiently low temperatures. One can define a temperature T^*, below which the standard Gor'kov calculation is not reliable, from $\omega_c(H_{c2}(T^*))=2\pi T^*$. It follows that $T^*\sim T_{c0}^2/E_F\ll T_{c0}$. If $T_c(H)<T^*$, we have to devise an entirely different scheme for solving (2.3). For great majority of superconducting systems, T^* is a very low temperature and is in the 1-mK range. Thus it would appear that there is little practical interest in studying this very-low-temperature region. However, in high-temperature oxide superconductors or in Nb-Sn systems, T^* may be sufficiently high to allow for the observation of these initial deviations from the AG theory.[5,8] To obtain some physical insight, we first consider the limit opposite to SCPIA, i.e., the situation of a very strong field $\omega_c\gg 2\pi T$. The extreme example of this is the situation where only a *single* Landau level is occupied. This happens for

$$H>H_{\rm QL}=2mcE_F/(3\sqrt{2})^{2/3}e\sim(E_F/T_{c0})^2H_{c2}(0)\ ,$$

where $H_{c2}(0)$ refers to the standard result. In this limit the integral equation (2.3) reads

$$\frac{1}{V}\Delta(z,z^*)=\beta^{-1}\sum_{\omega_\nu}\sum_{k_z}\frac{1}{\omega_\nu^2+(\varepsilon_{k_z}+\frac{1}{2}\omega_c)^2}\int d^2R'\frac{1}{(2\pi l^2)^2}\exp\left[-\frac{|z|^2}{2}-\frac{|z'|^2}{2}+zz'^*\right]\Delta(z',z'^*)\ , \tag{2.4}$$

where $\int d^2R'\equiv\int dx'\int dy'$. We have also neglected the Zeeman splitting ($g=0$) and assumed that Δ is uniform along $\mathbf{H}$.

Equation (2.4) shows a remarkable decoupling of the (x,y) plane from the ζ axis, which is a consequence of an infinite degeneracy of Landau levels. There is an infinitely degenerate manifold of solutions, given by $\Delta(z,z^*)=\Delta f(z)\exp(-|z|^2/2)$, where $f(z)$ is an arbitrary holomorphic function, while $T_c(H>H_{\rm QL})=1.14\Omega\exp[-2\pi l^2/N_1(0)V]$, where $N_1(0)=m/2\pi k_F(H)$ is a one-dimensional (1D) density of states at the Fermi level.[13] [The approximation of the constant density of states in the k_z integration of Eq. (2.4) is accurate for weak coupling, i.e., $\Omega\ll E_{F0}$, where E_{F0} is the Fermi energy for the lowest Landau level. As H increases, one eventually gets into a strong-coupling regime where $\Omega>E_{F0}$. This situation is discussed in Sec. IV.] The above result seems puzzling initially, since $T_c(H>H_{\rm QL})$ is *comparable* to $T_c(H=0)$ [T_c's for $H=2.13H_{\rm QL}$ and 0 are equal if one assumes $V(H_{\rm QL})\cong V(H=0)$] and it *grows* with the field: $2\pi l^2/N_1(0)\propto H^{-2}$ as shown in Fig. 2. (Of course, T_c cannot grow indefinitely, and the strong-coupling effects will renormalize it back to zero for $H\gg H_{\rm QL}$. The simplest source of reduction in T_c will be the change in the cutoff frequency from Ω to E_{1F}, where E_{1F} is the quasi-one-dimensional Fermi energy, once $\Omega<E_{1F}$. This situation is also depicted in Fig. 2.) However, this new behavior is just a manifestation of the complete breakdown of SCPIA. Physically, the magnetic field influences superconductivity by frustrating the order parameter which is manifested by a nonuniform configuration of $\Delta(z,z^*)$ and the penetration of a magnetic flux into the superconductor. The resulting cost in kinetic energy is always *overestimated* in SCPIA. Thus, for $T<T^*$, SCPIA leads to an *unphysical* result that T_c vanishes for some finite field. When the Landau-level structure is properly included, the nonuniformity in $\Delta(z,z')$ becomes progressively *less* costly as H increases.

It is very important to understand how the superconductivity evolves from the QL (and the VHFL) toward the low-field regime and the familiar superconducting state. One possibility, which we consider here for simplicity, is that in the low-field limit the system is a type-II superconductor. In that case there must be a continuous $T_c(H)$ curve joining the low- and high-field limits of the BCS superconductivity (continuous only in some average sense as will become clear below). But it is perfectly pos-

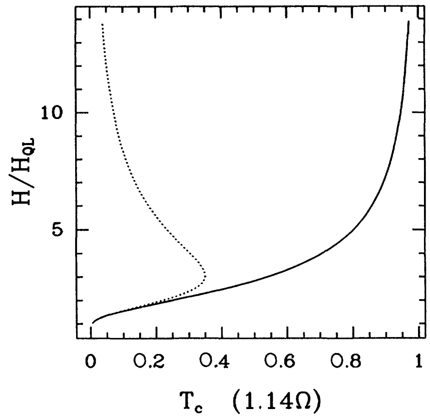

FIG. 2. $T_c(H)$ of a BCS superconductor in the QL. $\lambda(H=H_{\rm QL})$ is set to 0.18. Note the rapid rise of T_c as H increases. Such a rise is unphysical for $\Omega > E_{1F}$ when Ω has to be replaced by E_{1F} as the cutoff in the BCS formula. This is illustrated by the dashed line. We have chosen Ω to be $\sim 10\%$ of the 3D Fermi energy.

sible for a VHFL superconductor to behave as a type-I superconductor in low fields. In fact, a VHFL superconductor may not be a low-field superconductor at all, but instead have a ground state of completely different symmetry (SDW, CDW, Fermi liquid, etc.). If this is the case, various phase transitions will take place as the external field increases, resulting ultimately in a VHFL superconducting state. Study of such transitions is obviously a very complex problem, involving detailed understanding of the interacting electron systems in a varying field. We thus restrict ourselves to a simple case of a BCS problem with a weak attractive interaction $V(H)$ and assume the low-field state to be of type II. To find the solution of Eq. (2.3) when the number of occupied Landau levels $n_c > 1$, we note that the kernel has two types of terms: *diagonal,* for which the Landau-level index is the same for both the advanced and retarded Green's functions, and *off diagonal,* where the Landau level indices differ. Only the diagonal terms possess a Cooper singularity, and for $H \le H_{\rm QL}$, when n_c is not "too large," one expects that neglecting the off-diagonal terms will be a good approximation. We call this the quantum limit approximation (QLA). Also, it is easy to demonstrate that $\Delta(z,z^*)=\Delta f(z)\exp(-|z|^2/2)$, $f(z)$ being an arbitrary holomorphic function, remains the solution of (2.3) for *any H*. In QLA we find

$$T_c^{\rm QLA}=1.14\Omega\exp\left[-\frac{2\pi l^2}{V}\left[\sum_{n=0}^{n_c} N_{1n}(0)\frac{(2n)!}{2^{2n}(n!)^2}\right]^{-1}\right], \tag{2.5}$$

where $N_{1n}(0)$ is the 1D density of states for the nth Landau level. In the above equation it was assumed that $\Omega \ll \omega_c$ and that Fermi level is $\sim\Omega$ away from the singularities in the density of states. Clearly, this approximation becomes increasingly unreliable as one moves to lower fields. $T_c^{\rm QLA}(H)$ displays an oscillatory behavior reflecting the Landau level structure and has a monotonically *increasing* trend (on the average) as a function of H. Consequently, the QLA can never recover the AG low-field limit. The inset of Fig. 1 shows $T_c^{\rm QLA}(H)$. As the number of Landau levels increases (H decreases), $T_c^{\rm QLA}$ decreases rapidly. As the Fermi level crosses each Landau level, the density of states in Eq. (2.5) diverges. This is an unphysical divergence, and it can be removed either by a more accurate evaluation of the BSC equation for T_c (in other words, without making a constant density of states approximation once $E_{Fn} < \Omega$, where E_{Fn} is the quasi-one-dimensional Fermi energy for the nth Landau level) or by more physical effects of strong-coupling renormalization and thermal and/or disorder broadening of the density of states. Effectively, the Landau level crossing the Fermi energy is "turned off," and it does not contribute to T_c. In Fig. 1 the Coulomb "cutoff" $\mu^*=\mu/[1+\mu\ln(E_{Fn}/\Omega]$ was included to smooth out the oscillations ($\mu=0.15$). To obtain the crossover from the QLA to the SCPIA, it is necessary to include the off-diagonal terms. Although these terms do not have the Cooper singularity, their number grows as n_c^2 as opposed to n_c for the diagonal terms, when n_c becomes large. At some field H_C the off-diagonal terms will become dominant, leading to a crossover from T_c increasing with H to a smooth transition to the AG curve, as depicted in Fig. 1. With off-diagonal terms present, $T_c(H)$ cannot be written in a closed form and has to be evaluated numerically. The results are summarized in Fig. 1 in the full line. It is, of course, clear that this full line is an artist's rendering of the overall trend in $T_c(H)$ and that there will be rapid oscillations throughout this region. We find that as long as n_c is less than ~ 25 or so, the QLA is a very good approximation to the numerical $T_c(H)$: This is because an off-diagonal term contributes only if the energy difference between two Landau levels ($n\omega_c$) is less than Ω. For larger n_c, once $\omega_c \ll \Omega$, the off-diagonal terms become increasingly more important. This now signals the breakdown of the QLA as one moves from the VHFL down to low fields ($n_c \gg 1$) and indicates that some form of a quasiclassical approximation should be appropriate. This crossover has been investigated in some detail in Ref. 14 and more recently in Ref. 5 using quasiclassical approximation for the Landau levels. Using a rough approximation for the off-diagonal terms and defining H_C from $dT_c/dH=0$, we find $H_C \sim (E_F/T_{c0})H_{c2}(0)$. Throughout the crossover region T_c is typically extremely small ($<10^{-10}\Omega$), and only the high-field limit [in which $T_c^{\rm QLA}(H)$ is perfectly appropriate] will typically have observational significance. However, it has been argued in Refs. 8 and 5 that in type-II superconductors with very high upper critical field, the crossover region may be observable. In such systems the temperature T^* is a sizable fraction of $T_c(H=0)$, and one may be able to reach the crossover region with available fields and still have observable transition temperature. One should emphasize that T_c remains finite at all fields, a result obtained previously in Ref. 14. This result is valid for an ideal system in the absence of Zeeman splitting and disorder. We now turn to a discussion of these two perturbations in the VHFL.

III. EFFECTS OF PAULI PAIR BREAKING AND DISORDER

The discussion of Sec. II ignored the Zeeman splitting. The effective g factor can indeed be zero if we consider the intervalley pairing in multivalley semiconductors and semimetals.[6,7] But even in those cases there can be a contribution from the spin-singlet channel, and it is therefore important to understand how the Pauli pair breaking affects the results of Sec. II. Naively, one might expect that PPB will simply wipe out the VHFL. We know that in the low-field limit the PPB leads to the Pauli critical field H_p, which is simply obtained by comparing the Zeeman energy with T_c. If the Zeeman splitting is larger, then no spin-singlet superconductivity is possible (this is well known as the Chandrasekhar-Clogston limit). One can go slightly above the Chandrasekhar-Clogston limit if a superconducting state with a finite linear momentum of the Cooper pairs is introduced to recover the Cooper singularity.[15] Unfortunately, the region of stability of the Fulde-Ferrell state is very narrow since one can truly have a Cooper singularity only at a single point in the phase space. Consequently, the critical field is still of order $T_c/g\mu_B$, but the numerical factor in front is larger by a few percent. However, in the VHFL there is a qualitatively new possibility: Because of the quasi-one-dimensional nature of the electronic dispersion relation, one can choose to pair electrons with momenta along the field axis k_z and $-k_z+q_0$, where q_0 is chosen so as to offset the Zeeman splitting. The important difference is that now fully one-half of all the phase space, available for pairing, contributes to the Cooper singularity as opposed to a fraction of measure zero in the Fulde-Ferrell state.[16] As a result, superconductivity exists at arbitrary strong fields as long as both spin species are present. While for a Zeeman splitting much larger than the thermal energy T_c is somewhat reduced, it is still definitely observable, and thus the PPB is dramatically decreased. Since the above discussion is very important in the light of experimental observation of superconductivity in the VHFL, we now illustrate it in detail.

The most interesting region is the QL. When Zeeman splitting is present, we have a situation depicted in Fig. 3. If we consider the uniform (along the field) superconducting state of Sec. II and include the Zeeman splitting, we obtain the following weak-coupling equation for T_c:

$$\frac{1}{V}=\frac{1}{2\pi l^2}\left[\frac{2N_{1\uparrow}N_{1\downarrow}}{N_{1\uparrow}+N_{1\downarrow}}\right]\ln\left[\frac{1.14\Omega}{\max(T,A)}\right], \qquad (3.1)$$

where $N_{1\uparrow(\downarrow)}=m/2\pi k_{F\uparrow(\downarrow)}$, $A=2k_{F\uparrow}k_{F\downarrow}g\mu_B H/\pi(k_{F\uparrow}+k_{F\downarrow})^2$, and $\max(T,A)\approx(T^2+A^2)^{1/2}$. It is assumed throughout that $A\ll\Omega$, which will be true in a realistic case. There is no reduction in the coupling constant in the above equation since $2N_{1\uparrow}N_{1\downarrow}/(N_{1\uparrow}+N_{1\downarrow})=N_1(g=0)$ and thus $\lambda(g\neq 0)=\lambda(g=0)$. This fact (which, of course, is true only in weak coupling) is, however, not of much help here since the Cooper singularity has been cut off by A, the pair-breaking parameter of the Zeeman splitting. Obviously, T_c will be reduced rather rapidly with increasing g: Equation (3.1) is easily solved

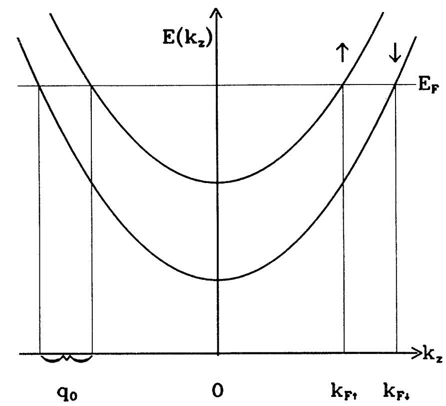

FIG. 3. Quasi-one-dimensional spin-up and -down bands in the QL. Zeeman splitting is assumed to be small compared to the cyclotron frequency.

and gives $T_c^2(g)\approx T_c^2(g=0)-A^2$. $T_c(g)$ in the uniform state is suppressed to zero for $A=T_c(g=0)$, and thus the Zeeman splitting of the order of the thermal energy will destroy the uniform superconducting state (this result is similar to what occurs in the low-field limit).

Figure 3 seems to suggest that one should consider a nonuniform superconducting state of the type $\Psi(\mathbf{r})=\Psi f(z)\exp(-|z|^2/2)\exp(iq_0\zeta)$, where, for $H>H_{\rm QL}$, $q_0=2g\mu_B meH^2/\pi^2 nc$. By solving Eq. (2.4) (with Zeeman splitting included), one finds

$$\frac{1}{V}=\frac{1}{4\pi l^2}\left[\frac{2N_{1\uparrow}N_{1\downarrow}}{N_{1\uparrow}+N_{1\downarrow}}\right]\times\left[\ln\left[\frac{1.14\Omega}{T}\right]+\ln\left[\frac{1.14\Omega}{\max(T,2A)}\right]\right]. \qquad (3.2)$$

In this case half of the available electronic states still contributes to the logarithmic singularity. Combined with the fact that the coupling constant is unchanged, as discussed above, this state appears to be far more promising. The reason is as follows: For $A\ll T_c(0)$ the uniform state still has a higher transition temperature. But for $A>T_c(0)$, when the uniform state is destroyed, the nonuniform state has a transition temperature given by $T_c(g)=1.14\Omega\exp(-1/\bar{\lambda})$, where

$$\bar{\lambda}=\frac{\lambda(g=0)}{1+\frac{1}{2}\lambda(g=0)\ln[\max(T,2A)/T]},$$

and thus, in the weak-coupling limit ($\lambda\ll 1$), $T_c(g)\simeq T_c(g=0)$. Therefore, the nonuniform superconducting state can exist even for Zeeman splitting considerably larger than the thermal energy, and the transition temperature will still be of the order of $T_c(g=0)$. This is a very important result from a practical viewpoint since it demonstrates that the VHFL spin-singlet superconductivity could in principle be observable in physical systems with finite g factors. One must emphasize, however, that all the reasoning and approximations used above implicitly assume that the range of fields in the QL over which *both* spin states are occupied is reasonably large; obviously, no spin-singlet superconductivity can exist if one of the spin states is completely depopulated. This condition immediately rules out a great majority of standard superconductors which have effective masses of the order of the bare electron mass and g factors of order 2, since in that case the QL, strictly speaking, *does not exist.* The region of H for which both spin species are present and only the lowest Landau level is occupied will be either very narrow or nonexistent, and the VHFL superconductivity will be destroyed even for the nonuniform state. This is not of great consequence since most of the standard superconductors are already ruled out as candidates for the VHFL superconductivity by virtue of their too high electronic densities, which would require enormous fields in order to reach the QL. We want to emphasize that it is important to have small g factors so as to have a wide region in the QL where both spins are present. In many LCDSS's the effective g factors are quite low, and such materials would be best suited for the VHFL superconductivity.

As n_c increases, the situation becomes quite complicated. The PPB will start suppressing T_c very rapidly since now one cannot choose a single wave vector which would restore the Cooper singularity for all occupied Landau levels. One can still argue, however, that as long as $\Omega \ll \omega_c$ and E_{Fn} is away from the singularities in the density of states, T_c will remain finite, although considerably depressed. An interesting situation would arise if the g factor is very close to $2/m_c$, where m_c is the effective cyclotron mass. In this case the Zeeman splitting would be very close to the cyclotron splitting, making the nth spin-up Landau level nearly degenerate to the $(n+1)$th spin-down one. T_c would then be revived again, although it would still be less than for the $g=0$ case. This may be the situation in $SrTiO_3$, a well-known low-carrier-density superconductor.

The presence of impurities will affect the VHFL superconductivity in several ways. In addition to simple pair breaking (which is obviously present in a realistic, $g \neq 0$ nonuniform case), thermal and quenched disorder will broaden sharp features in the density of states and may change the effective interaction, particularly the Coulomb repulsion. Here we will consider the effect of pair breaking only; the broadening of the density of states can simply be included phenomenologically and leads to suppression of T_c for $n_c > 1$, since it flattens out the jumps in the density of states, but has little effect on the QL. A study of the effect of disorder on the effective electron-phonon and electron-electron interaction is a very complicated subject, even for ordinary, zero-field superconductivity, and we will not consider it in this paper. Finally, probably the most serious problem encountered by experimentalists searching for other quasi-one-dimensional instabilities in the QL, like SDW, CDW, etc., is the "magnetic freezeout." This dramatic loss of carriers (due to deepening of local impurity levels in high fields) will affect the superconducting state in a similar way. Recently, however, there has been considerable experimental progress in minimizing the effect of disorder and producing high-mobility 3D samples, particularly in so-called wide parabolic quantum wells (WPQW).[17] It appears likely that advances in artificially structured materials will soon lead to systems where the electron-impurity interactions will be negligible.

The pair breaking can be included by considering the effect of disorder on the kernel in Eq. (2.3). We first consider the realistic $g \neq 0$ case and use the Born approximation to derive the following expression for $T_c^{\text{dis}}(H, g \neq 0)$, valid for weak disorder, $1/\tau E_F \ll 1$, where $1/2\tau$ is the scattering rate due to disorder (defined in zero field) and E_F is the zero-field Fermi energy (we only give the expression for the QL, for $g \neq 0$, since the calculation is very involved when several Landau levels are included):

$$\ln\left[\frac{T_c^{\text{dis}}\max(T_c^{\text{dis}},2A)}{T_c\max(T_c,2A)}\right]=\pi T\sum_{\omega_n} D_1(\omega_n,1/2\tau)\left[1-\frac{1}{2\tau_{\text{QL}}}[D_1(\omega_n,1/2\tau)+D_2(\omega_n,1/2\tau)]\right]^{-1}$$
$$+\pi T\sum_{\omega_n}\left[D_2(\omega_n,1/2\tau)\left[1-\frac{1}{2\tau_{\text{QL}}}[D_1(\omega_n,1/2\tau)+D_2(\omega_n,1/2\tau)]\right]^{-1}\right.$$
$$\left.-D_1(\omega_n,0)-D_2(\omega_n,0)\right], \tag{3.3}$$

where

$$D_2(\omega_n,1/2\tau)=\frac{1}{(d_1^2+d_2^2)^{1/4}}[\exp(-i\delta/2)\Theta(\cos(\delta/2))-\exp(-i\delta/2)\Theta(-\cos(\delta/2))],$$

and

$$d(\omega_n,1/2\tau)\equiv d_1+id_2\equiv(d_1^2+d_2^2)^{1/2}\exp(i\delta)=\omega_n^2+\omega_n\eta[1/4\tau_\uparrow+1/4\tau_\downarrow+v_{F\uparrow}/4v_{F\downarrow}\tau_\downarrow+v_{F\downarrow}/4v_{F\uparrow}\tau_\uparrow+i(v_{F\uparrow}+v_{F\downarrow})q_0]$$
$$+\eta[1/4\tau_\uparrow\tau_\downarrow+(v_{F\uparrow}/2\tau_\downarrow-v_{F\downarrow}/2\tau_\uparrow)^2/4v_{F\uparrow}v_{F\downarrow}$$
$$-v_{F\uparrow}v_{F\downarrow}q_0^2+i(v_{F\uparrow}/2\tau_\downarrow+v_{F\downarrow}/2\tau_\uparrow)q_0],$$

with

$$1/2\tau_{QL}=[2N_{1\uparrow}N_{1\downarrow}/(N_{1\uparrow}+N_{1\downarrow})]\pi W^2/4\pi l^2 \ .$$

W^2 is the rms value for the random potential, $1/2\tau_{\uparrow,\downarrow}$ are the corresponding single-particle scattering rates, and $\eta=[1+(v_{F\uparrow}-v_{F\downarrow})^2/4v_{F\uparrow}v_{F\downarrow}]^{-1}$. $D_1(\omega_n,1/2\tau)$ is obtained from $D_2(\omega_n,1/2\tau)$ by setting $q_0=0$.

To get some feeling as to what happens below the QL, we give the result for $g=0$:

$$\ln\frac{T_c^{\rm dis}}{T_c}=-\psi\left[\frac{1}{2}+\left[\frac{1}{2\tau E_F}\right]\left[\frac{E_F}{2\pi T_c^{\rm dis}}\right]\sum_{n=0}^{n_c}\frac{N_{1n}(0)}{2\pi l^2 N_{3D}(0)}\left[2-\frac{(2n)!}{2^{2n}(n!)^2}\right]\right]+\psi\left\{\frac{1}{2}\right\}, \tag{3.4}$$

where T_c is given in (2.5). In the above equations the Born approximation should be reasonable in the QL as long as the field is not too high, making k_F^{-1} too long. When this occurs one must go beyond the Born approximation, which is quite a complicated issue. Furthermore, as the field becomes very strong, l grows shorter and eventually becomes less than a typical range of the impurity potential, resulting in reduced pair breaking. Below the QL the Born approximation is more suspect since the density of states oscillates faster and faster as the field decreases. The pair breaking increases, and disorder will eventually suppress T_c to zero, resulting in a crossover region where superconductivity does not occur even for $T=0$. Eventually, there will be a reentrant transition to the low-field regime.

Several qualitative points should be emphasized here: In LCDSS's the pair breaking due to disorder should be a minor problem and will not prevent the observation of superconductivity in the QL (although it may become a serious problem for lower fields) since such materials can very often be made extremely pure. One may think that this will be at odds with our assumption of type-II superconductivity in the low-field limit since high-purity LCDSS superconductors are likely to be type I in this limit. But the assumption of type-II superconductivity in the low-field regime was purely for convenience and simplicity in demonstrating continuous evolution of superconductivity from the low- to the high-field limit. In fact, there is nothing in the physics discussed here that requires a VHFL superconductor to be a low-field type-II superconductor. Such a VHFL superconductor could certainly be a type-I field superconductor: In this situation there will be a reentrant behavior with type-I superconductivity being destroyed at a thermodynamic critical field and reappearing as VHFL superconductivity at some much higher field. Probably the most interesting would be the situation in which a particular material is not even a superconductor at low fields and has some other type of ground state, but where superconductivity is induced by the application of a very strong field. Which of these situations will occur in a particular system will depend on the details of the electron-phonon and electron-electron interactions[6] and is clearly a difficult problem. We must leave it for future study.

IV. MODEL CALCULATIONS

In Sec. III we presented some of the qualitative features of the transition temperature in the VHFL superconductivity. Here we present several model calculations which illustrate these qualitative features. First, we specify the type of system where VHFL superconductivity can be a realistic possibility. As already mentioned before, LCDSS's are probably the most suitable materials. We consider two examples, Ge and GaAs. We emphasize that we are not predicting VHFL superconductivity in Ge and GaAs. We simply use their material parameters as an illustration of what range of these parameters will be favorable for superconductivity. The model based on Ge parameters we call model I, while the corresponding model for GaAs is called model II. Models I and II represent systems where the quantum limit can be reached with reasonable laboratory fields and where the existence of a relatively wide region between H_{QL} and the spin-depopulation field H_d is well established. In fact, in n-doped Ge, a spin-density wave is expected to exist in the QL.[18] This is due to a modest g factor of ~ 1.6 and large valley anisotropy. The conduction band consists of four equivalent valleys which are ellipsoids of revolution about the $\langle 111 \rangle$ crystal axes. The longitudinal mass m_l along the $\langle 111 \rangle$ axes is 1.64 in units of the electron mass, while the transverse mass m_t is only 0.08. Thus, for a field along one of the $\langle 111 \rangle$ directions, the cyclotron mass $m_c \approx m_t$, leading to the effective g factor $g^*=gm_t=0.12$. The small effective g factor results in both spin states being present considerably above H_{QL} and, combined with strong valley anisotropy, favors SDW's relative to other high-field quasi-one-dimensional instabilities like CDW's and VDW's.[11] Since general conditions regarding the availability of both spin states are similar for SDW and VHFL spin-singlet superconductivity, we take Ge parameters for our model I. We will also assume that **H** is applied along the $\langle 111 \rangle$ axis of one of the valleys and that Ge is simultaneously being subjected to an uniaxial stress along the same direction. The stress leads to a situation in which only this one valley is occupied with $H_{QL}\approx 5.6$ T for a carrier density 6×10^{17} cm^{-3}. One should remark that in our calculations density does not appear explicitly, and one can use our results for various carrier concentrations by scaling appropriate quantities. Furthermore, Ge is rich with experimental possibilities since, in the unstressed case, all four valleys will be degenerate and an intervalley pairing in the spin-triplet case becomes a possibility (which we will not investigate in detail in this paper). Similarly, in GaAs the cyclotron mass is ~ 0.07, which, combined with $g=0.32$, leads to $g^*\sim 0.02$, providing an example of a system with very small Zeeman splitting (relative to the cyclotron frequency). Finally, we choose $m_c=0.075$, $m_l=1.0$, and $g=1.5$ as our model III, illustrating the values of parameters that can be found in LCDSS's which will be favorable for VHFL superconducting state.

The formulas of Sec. III have been obtained using a simple weak-coupling BCS approximation. While this approximation is expected to be qualitatively correct, one of the important characteristics of the QL is a rapid rise in the density of states (and coupling constants) with increasing field ($\propto H^2$). This is accompanied by a corresponding drop in the Fermi energy. Thus the conditions for the validity of the weak-coupling BCS theory will be violated at high fields. Consequently, we use here a strong-coupling theory, based on a solution of the two-square-well model for kernels of electron-phonon and electron-electron interactions. This results in a familiar reduction of the coupling constant due to the effects of a Coulomb pseudopotential μ^* and the quasiparticle renormalization factors:

$$\lambda_{SC}=\frac{\sqrt{Z_\uparrow Z_\downarrow}}{\lambda-\mu^*}, \tag{4.1}$$

where $Z_{\uparrow(\downarrow)}$ are the spin-up (-down) quasiparticle renormalization factors, λ is the electron-phonon coupling constant, and

$$\mu^*=\frac{\mu}{1+\mu\ln(\sqrt{E_{F\uparrow}E_{F\downarrow}}/\Omega)}.$$

In the above we are restricting our consideration to the lowest Landau level. The cutoff of frequency integration is equal to Ω. The above form is applicable for $\Omega\ll\sqrt{E_{F\uparrow}E_{F\downarrow}}$. In the opposite limit, $\Omega\gg\sqrt{E_{F\uparrow}E_{F\downarrow}}$, which occurs as the field increases, the same two-square-well model leads to

$$\lambda_{SC}=\frac{\sqrt{Z_\uparrow Z_\downarrow}}{\lambda^*-\mu}, \tag{4.2}$$

where now

$$\lambda^*=\frac{\lambda}{1-\lambda\ln(\sqrt{E_{F\uparrow}E_{F\downarrow}}/\Omega)},$$

and the cutoff in frequency space is replaced by $\sqrt{E_{F\uparrow}E_{F\downarrow}}$. The solution in this limit should not be taken too seriously since many of the physical concepts of the standard Eliashberg theory fail for a Fermi energy comparable or much smaller than typical phonon energy.[19] Still, the solution in this limit does have a qualitative feature of the vanishing retardation effect and the corresponding rapid loss of effective attraction. We therefore use this form to illustrate the drop of the transition temperature which should be a qualitative feature of the $\Omega\gg\sqrt{E_{F\uparrow}E_{F\downarrow}}$ limit. (Here we do not consider the possibility of various bipolaronic instabilities which may occur for low Fermi energies.)

The results are plotted in Fig. 4. We have used $\lambda=0.6$ and $\mu=0.1$ for $H=H_{QL}$ in all cases, as well as $\Omega\sim10\%$ of the 3D Fermi energy. T_c is calculated for both the uniform and nonuniform states of Sec. III. Several qualitative features are apparent from our results. First, the uniform state gives $T_c=0$ for models I and III (for model II, with its very small effective g factor, there is a region where the uniform state is competitive). Thus the nonuniform state is essential in obtaining a finite transition temperature. Furthermore, all transition temperatures are

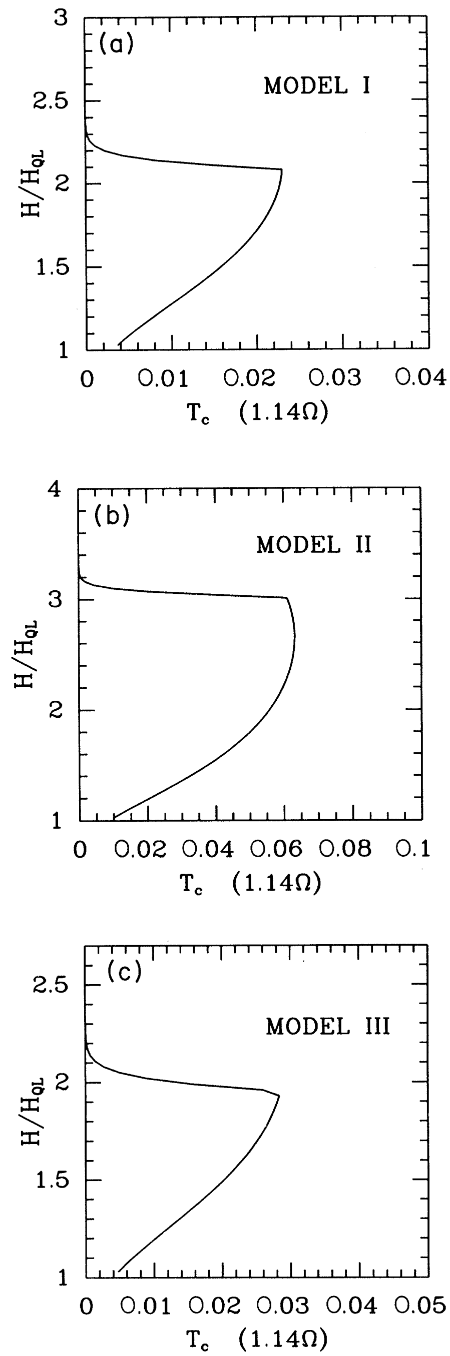

FIG. 4. $T_c(H)$ in strong coupling, for a nonuniform state of Sec. III. $T_c(H)$ for a uniform state is zero for models I and III, while it is smaller for model II. (a) Model I: $m_t=0.08$, $m_l=1.64$, and $g=1.6$; (b) model II: $m_t=m_l=0.07$ and $g=0.32$; (c) model III: $m_t=0.075$, $m_l=1.0$, and $g=1.5$. For further discussion, see text.

uniformly suppressed relative to weak coupling. As the coupling constant increases ($\lambda \propto H^2$), the quasiparticle renormalization factors reduce T_c considerably, particularly for higher fields. One should not look for high-temperature superconductivity in the VHFL, at least not in LCDSS's. In these systems T_c's are likely to be in the $\sim$10-mK to $\sim$1-K range. The break present in all $T_c(H)$ curves occurs for $\Omega \sim \sqrt{E_{F\uparrow}E_{F\downarrow}}$ and will disappear in a more realistic calculation including modifications of the Eliashberg theory for a Fermi energy comparable to the average phonon frequency. Finally, while the reduction in T_c due to Zeeman splitting is not negligible and is more significant than in weak coupling, the nonuniform state still leads to T_c's which are of the same order as $T_c(g=0)$. For models I–III we find that $T_c(g)$ is typically $\sim$30–80% of $T_c(g=0)$. This qualitatively confirms the analysis of Sec. III and leads to the conclusion that the Zeeman effect (PPB) does not destroy the nonuniform superconducting state in the QL. All examples studied here (models I–III) indicate that if $T_c(g=0)$ is itself observable, $T_c(g)$ will be too.

V. NATURE OF THE SUPERCONDUCTING STATE

The nature of the new superconducting state in the VHFL regime can be investigated using a Ginsburg-Landau expansion in the magnitude of the order parameter $\Psi(\mathbf{r})$. Again, it is instructive to first consider the QL with $n_c=1$. The total free energy can be written as

$$F=F_s+F_b+F_{s-b} \; , \tag{5.1}$$

where

$$F_s=\alpha(T)\int d^2r_1 d^2r_2 \Psi^\dagger(\mathbf{r}_1)K_2(\mathbf{r}_1,\mathbf{r}_2)\Psi(\mathbf{r}_2) + \frac{\beta(T)}{2}\int d^2r_1 d^2r_2 d^2r_3 d^2r_4 \Psi^\dagger(\mathbf{r}_1)\Psi(\mathbf{r}_2) \times K_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)\Psi^\dagger(\mathbf{r}_3)\Psi(\mathbf{r}_4) \; , \tag{5.2}$$

$$F_b=\int d^3r \frac{[\mathbf{H}+\mathbf{b}(\mathbf{r})]^2}{8\pi} \; , \tag{5.3}$$

and

$$F_{s-b}=\gamma(T)\int d^2r_1 d^2r_2 \Psi^\dagger(\mathbf{r}_1)K_3(\mathbf{r}_1,\mathbf{r}_2)\Psi(\mathbf{r}_2) \; . \tag{5.4}$$

In these equations, $\alpha(T)$, $\beta(T)$, and the order parameter $\Psi(\mathbf{r})$ are all defined as the standard GL quantities for a 1D superconductor, and

$$\gamma(T)\equiv\frac{16\pi^2 VT^3}{7\zeta(3)n}\sum_{k,\omega_\nu}\frac{\xi_k}{(\omega_\nu^2+\xi_k^2)^2} \; , \tag{5.5}$$

where n is the density of electrons.

The kernels K_2, K_3, and K_4 are found to be

$$K_2(\mathbf{r}_1,\mathbf{r}_2)=\frac{1}{(2\pi l^2)^2}\exp\left[-\frac{z_1^*z_1}{2}-\frac{z_2^*z_2}{2}+z_1z_2^*\right] \; , \tag{5.6}$$

$$K_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)=\frac{1}{(2\pi l^2)^4}\exp\left[-\frac{z_1^*z_1}{2}-\frac{z_2^*z_2}{2}-\frac{z_3^*z_3}{2}-\frac{z_4^*z_4}{2}+\frac{(z_1+z_3)(z_2^*+z_4^*)}{2}\right] \; , \tag{5.7}$$

and

$$K_3(\mathbf{r}_1,\mathbf{r}_2)=\frac{1}{(2\pi l^2)^3}\exp\left[-\frac{z_1^*z_1}{2}-\frac{z_2^*z_2}{2}+\frac{z_1z_2^*}{2}\right] \times\int d^3r_3\exp\left[-\frac{z_3^*z_3}{4}+\frac{z_1z_3^*}{2}\right]h(z_3,z_3^*) \times\exp\left[-\frac{z_3^*z_3}{4}+\frac{z_3z_2^*}{2}\right] \; , \tag{5.8}$$

where

$$h(z,z^*)\equiv\frac{e}{2imc}\left[\left[a^*,\frac{\partial}{\partial z^*}\right]_+ + \left[a,\frac{\partial}{\partial z}\right]_+\right] + \frac{e^2}{2mc^2}(A^*a+Aa^*) \; ,$$

where $a(A)\equiv a_x(A_x)+ia_y(A_y)$.

All the kernels entering the GL free energy are fully nonlocal, and no gradient expansions are possible since the order parameter varies over the same length scale as the kernels. The quadratic kernel K_2 *projects* $\Psi(\mathbf{r})$ to the "lowest bosonic Landau level"; i.e., $K_2(\mathbf{r}_1,\mathbf{r}_2)$ is proportional to the Green's function of charge-$2e$ bosons restricted to the lowest Landau level. Thus, if we consider contributions to K_2 coming only from the lowest electronic Landau level, all $\Psi(\mathbf{r})$ have to be of the form $f(z)\exp(-z^*z/2)$ and have the same T_c, while all other functional forms of the order parameter (coming from higher bosonic Landau levels) do not contribute at all. This simple situation illustrates the important physical point already mentioned in previous sections: The orbital pair-breaking effect of the magnetic field is eliminated, and there is no frustration characterizing the low-field superconducting state. The electronic wave functions constrained to the lowest Landau level naturally produce the order parameter describing charge-$2e$ bosons (Cooper pairs) in their corresponding lowest Landau level. This remains true in the quartic term as well. One can easily see that K_4 also acts as a projection operator by rewriting the quartic part of (5.2) as

$$\propto \int d^2r \exp(-2z^*z/2)\int d^2r_1 \exp(-z_1^*z_1/2+z_1z^*)\Psi^\dagger(\mathbf{r}_1)\int d^2r_2 \exp(-z_2^*z_2/2+z_2^*z)\Psi(\mathbf{r}_2)$$
$$\times\int d^2r_3 \exp(-z_3^*z_3/2+z_3z^*)\Psi^\dagger(\mathbf{r}_3)\int d^2r_4 \exp(-z_4^*z_4/2+z_4^*z)\Psi(\mathbf{r}_4)\;, \quad (5.9)$$

which is clearly nonzero only for the above "holomorphic" form of $\Psi(\mathbf{r})$. There is an important consequence of this projection property: First, we note that K_4 and F_{s-b} select $\Psi(\mathbf{r})$ which minimizes F. Finding such a configuration involves a variation of F with respect to $\Psi(\mathbf{r})$ and $a(\mathbf{r})$. For $T_c \ll E_F$ the coupling between $\Psi(\mathbf{r})$ and $a(\mathbf{r})$, given by $\gamma(T)$, is of order T_c/E_F, which translates to order $(T_c/E_F)^2$ for F_{s-b}. Thus, in weak coupling, we can ignore the Meissner effect, and the minimization of F reduces to minimization of F_s with respect to $\Psi(\mathbf{r})$ at a fixed external $\mathbf{H}$. (It is clear that this is an excellent approximation since the external field in the VHFL will be far stronger than any field that can be created by the motion of Cooper pairs.) Consequently, we can immediately conclude that $\Psi(\mathbf{r})=\Psi_0 f(z)\exp(-z^*z/2)$ must be of the form of the *exact* solution of the nonlinear GL equation. This equation reads

$$0=\frac{\delta F_s}{\delta\Psi^\dagger(\mathbf{r})}=\alpha(T)\int d^2r_2 K_2(\mathbf{r},\mathbf{r}_2)\Psi(\mathbf{r}_2)+\beta(T)\int d^2r_2 d^2r_3 d^2r_4 \Psi(\mathbf{r}_2)K_4(\mathbf{r},\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)\Psi^\dagger(\mathbf{r}_3)\Psi(\mathbf{r}_4)\;. \quad (5.10)$$

There are many possible solutions of Eq. (5.10), depending on a choice of $f(z)$. From variational calculations in the low-field limit, it is known that a triangular lattice gives particularly low free energy. So we can simply take a variational Abrikosov solution (which *is* confined to the lowest bosonic Landau level) for a triangular vortex lattice[20] and check whether this is a solution of Eq. (5.10). After some algebra one finds that indeed it is! In fact, every function of the above form with a periodic $|\Psi(\mathbf{r})|$ such that there is a flux quantum per each zero is an exact solution of (5.10).[21] Therefore, we have now obtained an exact solution of the nonlinear GL equation which is likely to be the absolute minimum of the mean-field free energy. Furthermore, the inspection of the higher-order terms in the GL expansion of the mean-field free energy reveals that, in the QL, all the kernels (of order six and higher) act as projectors in a similar way. Thus the form $\Delta(\mathbf{r})=\Psi_0 f(z)\exp(-z^*z/2)$, with a proper periodicity, is the *exact* solution of the full BCS mean-field theory in the QL at any temperature! Similarly, the above form represents the exact solution of a BCS theory in two dimensions in the QL. In the 2D case, however, there is no weak-coupling parameter T_c/E_F, and the validity of the mean-field approximation is questionable. We will study such solutions in more detail in our future publications. If several Landau levels are occupied, the GL free energy can be found in a similar fashion. The kernels K_2 and K_4 are not projectors any more, and the exact solution for $\Psi(\mathbf{r})$ will have a contribution from higher bosonic Landau levels. However, we can still find an exact solution of the GL equation. This will be discussed elsewhere.[21] As H decreases toward the low-field limit, the vortex lattice (in the GL region) simply expands, keeping the area of the elementary hexagonal plaquette equal to $2\pi l^{*2}$. In the semiclassical limit ($n_c \gg 1$), the effect of Landau quantization on the GL expansion has been studied in Ref. 22. We should also mention that we have not studied the GL expansion for the nonuniform state in detail, but this expansion should be well defined and lead to a continuous transition, at least for small g factors.

While the coupling between the order parameter and $\mathbf{a}(\mathbf{r})$ can be ignored in the above discussion, it is still important conceptually. This coupling can be treated perturbatively in the above exact solution of GL equations.[21] The magnetic field induced by superconductivity itself is very small compared to the external field, and one can think of the VHFL superconductivity as an extreme case of type-II behavior in which the penetration depth becomes very large (of course, one should caution against pushing this analogy too far since it may not be appropriate in all circumstances). In this sense the VHFL superconductor will have electromagnetic properties similar to those of an extreme type-II superconductor having a very short coherence length in the x-y plane ($\sim k_F^{-1}\sim l$) and a very long one ($\sim v_F/T_c \gg k_F^{-1}$) along the field direction. At the mean-field level there will be superflow along the z axis, and no potential drop will occur for infinitesimal current. Transport properties in the x-y plane will be a "hybrid" between the extreme type-II superconductor and a Hall effect, and will be studied in more detail elsewhere. Whether the critical currents will be finite is related to the issue of fluctuations from the mean-field solution. The fluctuations are not one dimensional, as one may naively think, but have a three-dimensional character[23] and should again be similar to an extreme type-II superconductor with a large anisotropy in coherence lengths. Thermal fluctuations of a vortex lattice, in extreme type-II superconductors, have been a subject of considerable interest recently in relation to high-temperature superconductors.[24–26] We expect similar results to hold in the VHFL with the vortex lattice melting in the neighborhood of the mean-field transition point and vortex lattice solidification taking place at some lower temperature. This solidification transition should also be influenced by quantum fluctuations, which, due to the separation of vortices in the x-y plane being very small, should be more significant than in the ordinary low-field-limit type-II systems (where they are completely negligible). However, one should point out that the quantum fluctuations are still restrained by the long coherence length along the field direction. Thus we do not expect quantum fluctuations to produce much more than a quantitative correction to the qualitative behavior outlined above.

VI. DISCUSSION AND CONCLUSIONS

In this paper we have studied superconducting state in a very high magnetic field. Using the full quantummechanical treatment of the problem, with the Landau-level quantization included from the beginning, we have demonstrated that a new type of superconductivity exists at very high fields, comparable or higher than the quantum limit field. In this limit there is no diamagnetic pair breaking, and the transition temperature can actually be enhanced by the field. The VHFL superconductivity can be a ground state of the system even if the low-field-limit ground state is not superconducting. We have studied the influence of Zeeman splitting and disorder on this new state, and have shown that both effects, while reducing T_c, will not preclude the observation of VHFL superconductivity. Of course, as we have already emphasized, typical superconducting materials like Al, Pb, Nb, etc., will not be candidates for the VHFL superconductors. In such systems the field required to reach the quantum limit is enormous, and the effect of the Zeeman splitting will practically eliminate the superconducting state. The most interesting condensed-matter systems appear to be low-carrier-density semiconductors and semimetals (for example Ge, GeTe, SnTe, GaAs, Bi, $SrTiO_3$, etc.), some of which are superconductors in zero field. Here the VHFL can be reached with available fields (either in the laboratory or through explosive techniques), and effective g factors are often low enough to allow for a wide region of fields in the quantum limit where both spin states are occupied. The multivalley Fermi surface of many of the above materials will also be helpful in raising T_c to an observable level.[6,7,9] In LCDSS's the very-high-field superconductivity will compete with other high-field instabilities like SDW, CDW, VDW, and alike.[27,28] In this sense one is reminded of the one-dimensional "g-ology" (in zero field) where similar instabilities compete for the ground state. Our paper establishes that even in the VHFL the superconducting instability is a viable candidate for a portion of the phase diagram.

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