

Coefficients of the second viscosity in thin liquid-helium films

Chul-Won Jun

Department of Physics, College of Science, Kyung-Nam University, Ma-San 630-701, Kyung Nam, Korea

Chung-In Um

Department of Physics, College of Science, Korea University, Seoul 136-701, Korea

Thomas F. George*

Departments of Chemistry and Physics, 239 Fronczak Hall, State University of New York at Buffalo, Buffalo, New York 14260

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The coefficients of the second viscosity in thin liquid-helium films are evaluated explicitly as a function of temperature (T) via interactions between elementary excitations that include the anomalous excitation spectrum. For $T \gtrsim 0.3$ K the coefficients behave like those of the bulk liquid helium, while for $T \lesssim 0.3$ K the temperature variations of four coefficients of the second viscosity show a T^{-1} dependence, which is due to the three-phonon processes (3PP) originating from the anomalous excitation spectrum of two-dimensional liquid helium at low momenta and low temperatures. The T^{-1} dependence also appears in the first viscosity of bulk and thin liquid helium, which is based on the 3PP.

I. INTRODUCTION

Recently, the temperature variations of various sound propagations have been an important subject in the studies of liquid ^3He ,¹ ^4He ,² ^3He - ^4He mixtures,³ and solutions of ^3He in liquid ^4He ,⁴ because their properties are closely related to the excitation spectrum of liquid ^3He and ^4He , which can determine the transport coefficients^{5,6} through the kinetic equations. It is well known that at low temperatures and low pressures the thermal properties of superfluid helium are not determined mainly by the rotons but rather by the low-momentum acoustic phonons. These phonons do not possess the normal dispersion relation, but instead exhibit an anomalous excitation spectrum. In fact, these two cases are quite different in microscopic processes. The former contributes to four-phonon processes (4PP) while the latter is governed by three-phonon processes (3PP).

Recently, Andreev and Khalatnikov⁶ have evaluated the temperature variation of first sound, and Singh and Prakash⁷ have used the retarded single-particle Green's function for a weakly interacting Bose gas to obtain the first sound by using the wrong normal dispersion relation. However, in a recent paper⁸ we have developed the Landau-type elementary excitation spectrum, which is anomalous phononlike at low momenta and rotonlike at high momenta in two- and three-dimensional liquid helium. Starting with this elementary excitation, we have successfully derived and explained not only the various sounds⁹ and sound attenuations,¹⁰ but also thermal conductivity¹¹ and viscosity¹² in partly one-, two-, and three-dimensional liquid ^4He .

Regarding the evaluation of first and second sound,¹³ we have used a new approach, which takes into account a collision term in the Boltzmann equation, and have obtained first and second sound simultaneously. We reported that the second sound in thin helium films is approximately $2^{-1/2}$ times the first-sound velocity in the low-temperature and low-frequency limit. At low frequencies such that $\omega_s \tau \ll 1$, where ω_s is the sound frequency and τ is the characteristic time, we can make use of a hydrodynamical approach to sound propagation. However, for the opposite case of $\omega_s \tau \gg 1$, it is better to use the kinetic equations. In this collisionless region the first and second sound obtained in superfluid hydrodynamic equations are involved in the attenuation coefficients, which contain the four coefficients of second viscosity, i.e., ξ_1 , ξ_2 , ξ_3 , and ξ_4 . These four coefficients play a very important role in the investigation of sound attenuation in bulk liquid helium. In the bulk case, the attenuation of second sound depends on all four viscosity coefficients together. However, the term which generally contains the thermal conductivity strongly affects the attenuation.

Recently, we have adopted the Landau and Khalatnikov theory to derive the thermal conductivity¹¹ and viscosity¹² for three ranges of temperature, $T \lesssim 0.3$ K, $0.3 \lesssim T \lesssim 0.8$ K, and $T \gtrsim 0.8$ K, in which the scattering depends on the nature of interactions between elementary excitations. For $T \lesssim 0.3$ K it is important to note that the 3PP do not affect the thermal conductivity but the cause the viscosity to have a T^{-1} dependence, which was not proven by Landau and Khalatnikov's results. The 4PP and phonon-roton scattering govern the whole mechanism in the range of temperature $0.3 \lesssim T \lesssim 0.8$ K,

and for $T \gtrsim 0.8$ K the 5PP and the phonon-roton scattering contribute mostly to the transport coefficients.

The main purpose of the present paper is to evaluate the four coefficients ζ_1 , ζ_2 , ζ_3 , and ζ_4 of the second viscosity as a function of temperature by solving the superfluid hydrodynamics for the above temperature ranges through the theory of the kinetic phenomena developed by Khalatnikov,¹⁴ and especially for $T \lesssim 0.3$ K we will investigate the temperature variation of the four viscosity coefficients. Throughout this paper we will use the two-dimensional elementary excitation spectrum obtained by the microscopic ring-diagram approximation,⁸

$$E(p) = c_0 p (1 + \gamma p^2 - \delta p^4 + \dots), \quad (1.1)$$

$$E(P) = \Delta + \frac{(P - P_0)^2}{2\mu_0}, \quad (1.2)$$

where p and P are phonon and roton momenta, respectively, c_0 is the sound velocity at absolute-zero temperature, γ and δ are positive constants which can be determined by the potential parameters, and Δ , P_0 , and μ_0 are the roton parameters. Here, we have adopted a soft potential with a Lennard-Jones-type tail which helps to make a smooth connection between the attractive part and a soft repulsive core,⁸

$$\phi(r) = \begin{cases} V_0, & r \leq a \\ E_0 \left[\left(\frac{a}{r} \right)^{12} - \left(\frac{a}{r} \right)^6 \right], & r \geq a. \end{cases} \quad (1.3)$$

In this paper we define the liquid-helium film as two-dimensional—less than three atomic layers, namely one statistical layer of 3.6 Å—and neglect substrate effects. In Sec. II, to obtain the temperature variation of the viscosity coefficients in various processes, we will investigate the absorption and emission processes between elementary excitations through the collision integral equations, and we will solve the superfluid hydrodynamic equations to obtain four coefficients of the second viscosity in Sec. III. Finally, in Sec. IV we will give the results and discussion together with the thermodynamic properties in terms of a table and graphs.

II. ABSORPTION AND EMISSION OF PHONONS AND ROTONS

To describe the interactions between elementary excitations which cause liquid helium to make a transition from one state to another, we can consider generally three collision processes: (a) emission or absorption of phonons, (b) transformation of phonons into rotons and vice versa, and (c) emission or absorption of rotons. In addition, we have the 3PP, 4PP, and 5PP. In the case of (c) the energy of one roton would have to be at least 2Δ , such that it can decay into two rotons each with energy of about Δ . However, this kind of three-roton process⁶ is highly improbable, so that we may neglect it. Therefore, from now on, we only take into account the 3PP and 5PP and transformations of phonons into rotons and vice versa.

We consider first the 3PP

$$(\mathbf{p}_2 + \mathbf{p} \rightleftharpoons \mathbf{p}_1). \quad (2.1)$$

Since the total number of phonons traveling in a given direction is changed by the small-angle 3PP, the distribution function, which depends on the chemical potential α' , can be written as

$$n = [\exp(\alpha' + pc)/k_B T - 1]^{-1}. \quad (2.2)$$

The difference between Eq. (2.2) and the equilibrium distribution function n_0 can be expanded in a power series to give

$$\delta n = n - n_0 = -n_0(n_0 + 1)\alpha'. \quad (2.3)$$

The collision integral $J_{3PP}(n)$ and the kinetic coefficient Γ_{3PP} are related to each other through

$$\frac{1}{k_B T} \int J_{3PP}(n) \frac{d\mathbf{p}}{(2\pi\hbar)^2} = \alpha' \Gamma_{3PP}, \quad (2.4)$$

and the collision integral $J_{3PP}(n)$ obtained in the second quantization method¹¹ is given by

$$J_{3PP}(n) = -\frac{\pi c_0}{2\hbar\rho_0} (u+1)^2 \times \int p_1 p_2 \delta n(n_{p_{10}} - n_{p_{20}}) \delta(E_f - E_i) \frac{d^2\mathbf{p}}{(2\pi\hbar)^2}, \quad (2.5)$$

where u is Grüneisen's constant, and $n_{p_{i0}}$ represents the equilibrium distribution function for the phonons with momentum p_i . Performing the integration for Eq. (2.5) over momentum space, we obtain

$$\int J_{3PP}(n) p dp = \frac{213! \zeta(2) \zeta(3) (u+1)^2}{8\pi^2 \hbar^3 \rho_0} \alpha' \left(\frac{k_B T}{c_0} \right)^6. \quad (2.6)$$

Comparing Eqs. (2.4) and (2.6), we can easily obtain the kinetic coefficient Γ_{3PP} for the 3PP as

$$\Gamma_{3PP} = \frac{213! \zeta(2) \zeta(3) (u+1)^2}{16\pi^2 \rho_0 \hbar^5 c_0^6} (k_B T)^5. \quad (2.7)$$

As for the 4PP case, the direction of momenta of the colliding particles is not changed in the 5PP case. The calculation of the transition matrix elements by second-order perturbation theory is very complicated, and some terms contain vanishing denominators under the condition $\gamma=0$ in Eq. (1.1) and collinear scattering. The 5PP has maximum probability in small-angle scattering and leads to equilibrium for the phonons in the direction of motion. Instead of these tedious calculations using second-order perturbation theory, we make use of the kinetic coefficient given by Landau and Khalatnikov¹⁵ to obtain the Γ_{5PP} as¹¹

$$\Gamma_{5PP} = a T^7. \quad (2.8)$$

Here, a is constant which can be determined experimentally by the attenuation coefficient of ultrasonic waves.

Now we consider the transformations of phonons into

rotons and vice versa. When energetic phonons with energy on the order of Δ collide with rotons ($\mathbf{p}_1 + \mathbf{P}_2 \rightleftharpoons \mathbf{P}_3 + \mathbf{P}_4$), where \mathbf{p} and \mathbf{P} represent the phonon and roton momenta, respectively, the transformation can occur. Therefore, a phonon has at least a very large energy Δ , and the interactions between phonons and rotons are very similar to that of the scattering between rotons. Adopting Landau and Khalatnikov's assumption,¹⁵ we may take the interactions between a roton and phonon to be a δ -function potential,

$$V = V_0 \delta(\mathbf{r}_1 - \mathbf{r}_2), \quad (2.9)$$

where V_0 is an interaction constant, and \mathbf{r}_1 and \mathbf{r}_2 are the position vectors of the phonons and rotons, respectively.

The rate of change per unit time in the roton number due to the transformation of rotons into phonons is given by

$$\begin{aligned} \dot{N}_r = - \int \int \int [n_1 N_2 (N_3 + 1)(N_4 + 1) \\ - (n_1 + 1)(N_2 + 1)N_3 N_4] d\omega \frac{d\mathbf{p}_1 d\mathbf{P}_2}{(2\pi\hbar)^4}, \end{aligned} \quad (2.10)$$

where n_i and N_i are the distribution functions with momenta \mathbf{p}_i and \mathbf{P}_i , respectively. $d\omega$ is the differential decay rate for the transformation of phonons and rotons with momenta \mathbf{p}_1 and \mathbf{P}_2 into two rotons with momenta \mathbf{P}_3 and \mathbf{P}_4 and is given by

$$d\omega = \frac{2\pi}{\hbar} |V_{IF}|^2 \delta(E_1 + E_2 - E_3 - E_4) \frac{dP_3 dP_4}{(2\pi\hbar)^4}, \quad (2.11)$$

where $|V_{IF}|$ is the transition matrix element, and E_i and E_j are the phonon and roton energies, respectively. Combining Eq. (2.10) with (2.11), we get

$$\begin{aligned} \dot{N}_r = - \int \int \int [n_1 N_2 (N_3 + 1)(N_4 + 1) \\ - (n_1 + 1)(N_2 + 1)N_3 N_4] \\ \times \frac{2\pi}{\hbar} |V_{IF}|^2 \delta(E_1 + E_2 - E_3 - E_4) \\ \times \frac{d\mathbf{p}_1 d\mathbf{P}_2 d\mathbf{P}_3 d\mathbf{P}_4}{(2\pi\hbar)^8}. \end{aligned} \quad (2.12)$$

Now we construct the symmetrized pairwise plane wave over incoming and outgoing phonons and rotons as

$$\psi(\mathbf{p}_1, \mathbf{P}_2) = \frac{1}{\sqrt{2}} (e^{(i/\hbar)(\mathbf{p}_1 \cdot \mathbf{r}_1 + \mathbf{P}_2 \cdot \mathbf{r}_2)} + e^{(i/\hbar)(\mathbf{p}_1 \cdot \mathbf{r}_2 + \mathbf{P}_2 \cdot \mathbf{r}_1)}), \quad (2.13)$$

$$\psi(\mathbf{P}_3, \mathbf{P}_4) = \frac{1}{\sqrt{2}} (e^{(i/\hbar)(\mathbf{P}_3 \cdot \mathbf{r}_1 + \mathbf{P}_4 \cdot \mathbf{r}_2)} + e^{(i/\hbar)(\mathbf{P}_3 \cdot \mathbf{r}_2 + \mathbf{P}_4 \cdot \mathbf{r}_1)}),$$

and combining Eq. (2.13) with (2.11), we can obtain the transition matrix elements as

$$V_{IF} = 2V_0 \Omega^{-1/2} \int e^{(i/\hbar)(\mathbf{P}_3 + \mathbf{P}_4 - \mathbf{p}_1 - \mathbf{P}_2) \cdot \mathbf{r}_1} d\Omega_1. \quad (2.14)$$

Here, $d\Omega$ represents the area element. Making use of the δ -function identity together with Eq. (2.14), we obtain

$$\begin{aligned} \delta(\mathbf{P}_3 + \mathbf{P}_4 - \mathbf{p}_1 - \mathbf{P}_2) \\ = \frac{1}{(2\pi\hbar)^2} \int e^{(i/\hbar)(\mathbf{P}_3 + \mathbf{P}_4 - \mathbf{p}_1 - \mathbf{P}_2) \cdot \mathbf{r}_1} d\Omega_1, \end{aligned} \quad (2.15)$$

and performing the integration over momentum space, we get

$$\int |V_{IF}|^2 \frac{d\mathbf{P}_2}{(2\pi\hbar)^2} = 4|V_0|^2. \quad (2.16)$$

As mentioned earlier, from the fact that the roton momentum \mathbf{P} is almost the same as that of the roton parameter \mathbf{P}_0 , the expansion of the roton distribution function N as a function of chemical potential yields Eq. (2.12) as

$$\begin{aligned} \dot{N}_r = \frac{\mu_r - \mu_{ph}}{k_B T} \frac{16\pi^2}{\hbar} \frac{\Delta |V_0|^2}{c_0^2} - \frac{1}{(2\pi\hbar)^6} \\ \times \int \int N_{30} N_{40} d\mathbf{P}_3 d\mathbf{P}_4, \end{aligned} \quad (2.17)$$

where μ_r and μ_{ph} are the roton and phonon chemical potentials, respectively, and performing the integration over momentum space we arrive at

$$\dot{N}_r = \frac{\mu_r - \mu_{ph}}{k_B T} \frac{4\Delta |V_0|^2 N_r^2}{\hbar^3 c_0^2}, \quad (2.18)$$

where N_r is the number density of roton given by

$$N_r = \left[\frac{\mu k_B T}{2\pi} \right]^{1/2} \frac{P_0}{\hbar^2} e^{-\Delta/k_B T}. \quad (2.19)$$

The rate of change per unit time in the roton number can be expressed in terms of the kinetic coefficient Γ_{ph-r} as

$$\dot{N}_r = \Gamma_{ph-r} (\mu_r - \mu_{ph}). \quad (2.20)$$

Comparing Eqs. (2.20) and (2.18), we can deduce the kinetic coefficient Γ_{ph-r} for the transformation of rotons into phonons and vice versa as

$$\Gamma_{ph-r} = \frac{4\Delta |V_0|^2 N_r^2}{\hbar^3 c_0^2 k_B T}. \quad (2.21)$$

Since V_0 is not known experimentally, we can simply rewrite Eq. (2.21) as

$$\Gamma_{ph-r} = b e^{-2\Delta/k_B T}, \quad (2.22)$$

where b is a constant which contains several parameters of the elementary excitation and can be determined experimentally by the ultrasonic attenuation.

III. COEFFICIENTS OF TWO-DIMENSIONAL SECOND VISCOSITY

When liquid ^4He is in its equilibrium state, the distribution functions for phonons and rotons are expressed by

$$n_{0ph} = \left[\exp \left[\frac{E_p - \mathbf{p} \cdot (\mathbf{v}_s - \mathbf{v}_n)}{k_B T} \right] - 1 \right]^{-1}, \quad (3.1)$$

$$n_{0r} = \left[\exp \left(\frac{E_r - \mathbf{p} \cdot (\mathbf{v}_n - \mathbf{v}_s)}{k_B T} \right) \right]^{-1}. \quad (3.2)$$

However, due to the presence of the energy dissipation, nonequilibrium effects appear in energy and momentum conservations, superfluid flow, energy flux, and especially entropy which is not conserved but increases (entropy increases in order to determine the unknown dissipation coefficients). Taking into account all of these conditions, we can accumulate all these effects within the following two hydrodynamic equations which concern

$$\begin{aligned} \frac{\partial \mathbf{j}}{\partial t} + \nabla P + \mathbf{v}_s (\nabla \cdot \mathbf{j}) + (\mathbf{j} - \rho \mathbf{v}_n) (\nabla \cdot \mathbf{v}_n) \\ + (\mathbf{j} \cdot \nabla) \mathbf{v}_s + (\mathbf{v}_n \cdot \nabla) (\mathbf{j} - \rho \mathbf{v}_n) \\ = \nabla [\zeta_1 \nabla \cdot (\mathbf{j} - \rho \mathbf{v}_n) + \zeta_2 \nabla \cdot \mathbf{v}_n], \end{aligned} \quad (3.3)$$

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \left[\mu + \frac{\mathbf{v}_s^2}{2} \right] = \nabla [\zeta_3 \nabla \cdot (\mathbf{j} - \rho \mathbf{v}_n) + \zeta_4 \nabla \cdot \mathbf{v}_n]. \quad (3.4)$$

Here, \mathbf{j} and ρ are the momentum and mass density, P is the pressure, μ is the chemical potential of the liquid helium, and $\zeta_1, \zeta_2, \zeta_3,$ and ζ_4 are the coefficients of second viscosity. The coefficients ζ_1 and ζ_4 are equal according to the Onsager's reciprocity principle.

It is obvious that the coefficients of the second viscosity depend on the variations of the temperature and thermodynamic functions, and thus the number of phonons and rotons are changed by the various processes that we have mentioned earlier. Let N_{ph} and N_r be the number of phonons and rotons per unit area, respectively, and μ_{ph} and μ_r be their chemical potentials. When the system deviates slightly from its equilibrium state, N_r and N_{ph} are changed in time and try to return to their values in the equilibrium state. We assume that the nonequilibrium distribution function n deviates very slightly from equilibrium, i.e., small deviations of the density and entropy can be determined by the time derivative of the phonon and roton numbers, \dot{N}_{ph} and \dot{N}_r , in the expansion of the chemical potential. Since the phonons and rotons take part in normal motion with velocity \mathbf{v}_n , neglecting the quadratic effects and taking only linear terms in μ_{ph} and μ_r , we can obtain the following equations:

$$\dot{N}_r + N_r \nabla \cdot \mathbf{v}_n = -\gamma_{rr} \mu_r + \gamma_{\text{ph-r}} \mu_{\text{ph}}, \quad (3.5)$$

$$\dot{N}_{\text{ph}} + N_{\text{ph}} \nabla \cdot \mathbf{v}_n = \gamma_{\text{ph-r}} \mu_r - \gamma_{\text{pp}} \mu_{\text{ph}}. \quad (3.6)$$

The kinetic coefficients on the right-hand sides of Eqs. (3.5) and (3.6) are symmetric in the indices r and ph .

Since the rate of change per unit time in the density ρ and entropy S can be expressed by the continuity equations

$$\dot{\rho} + \nabla \cdot \mathbf{j} = 0, \quad (3.7)$$

$$\dot{S} + \nabla S \cdot \mathbf{v}_n = 0, \quad (3.8)$$

substitution of Eqs. (3.7) and (3.8) into (3.5) and (3.6) yields

$$\begin{aligned} \dot{N}_r = -\frac{\partial N_r}{\partial \rho} \nabla \cdot \mathbf{j} - \frac{\partial N_r}{\partial S} \nabla \cdot \mathbf{v}_n + \frac{\partial N_r}{\partial \rho} S \nabla \rho \cdot \mathbf{v}_n \\ - \frac{\partial N_r}{\partial \rho} \rho \nabla \cdot \mathbf{v}_n, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \dot{N}_{\text{ph}} = \frac{\partial N_{\text{ph}}}{\partial \rho} \nabla \cdot \mathbf{j} - \frac{\partial N_{\text{ph}}}{\partial S} S \nabla \cdot \mathbf{v}_n + \frac{\partial N_{\text{ph}}}{\partial \rho} \nabla \rho \cdot \mathbf{v}_n \\ - \frac{\partial N_{\text{ph}}}{\partial \rho} \nabla \cdot \mathbf{v}_n. \end{aligned} \quad (3.10)$$

Combining Eqs. (3.9) and (3.10) with Eqs. (3.5) and (3.6), we can express \dot{N}_r and \dot{N}_{ph} in terms of ρ and S by

$$\begin{aligned} -\frac{\partial N_r}{\partial \rho} \nabla \cdot (\mathbf{j} - \rho \mathbf{v}_n) + \left[N_r - \frac{\partial N_r}{\partial S} S - \frac{\partial N_r}{\partial \rho} \rho \right] \nabla \cdot \mathbf{v}_n \\ = -\gamma_{rr} \mu_r + \gamma_{\text{ph-r}} \mu_{\text{ph}}, \end{aligned} \quad (3.11)$$

$$\begin{aligned} -\frac{\partial N_{\text{ph}}}{\partial \rho} \nabla \cdot (\mathbf{j} - \rho \mathbf{v}_n) + \left[N_{\text{ph}} - \frac{\partial N_{\text{ph}}}{\partial S} S - \frac{\partial N_{\text{ph}}}{\partial \rho} \rho \right] \nabla \cdot \mathbf{v}_n \\ = \gamma_{\text{ph-r}} \mu_r - \gamma_{\text{pp}} \mu_{\text{ph}}. \end{aligned} \quad (3.12)$$

The kinetic coefficients $\gamma_{rr}, \gamma_{\text{ph-r}},$ and γ_{pp} (Refs. 15 and 16) may be replaced as $\Gamma_{3\text{RP}}, \Gamma_{5\text{PP}}, \Gamma_{\text{ph-r}},$ and $\Gamma_{3\text{PP}}$, which are the kinetic coefficients of the three-roton process (3RP), 5PP, phonon-roton interaction, and 3PP, respectively, as follows:

$$\begin{aligned} \gamma_{rr} &= \Gamma_{3\text{RP}} + \Gamma_{\text{ph-r}}, \\ \gamma_{r\text{-ph}} &= \gamma_{\text{ph-r}} = \Gamma_{\text{ph-r}}, \\ \gamma_{\text{pp}} &= \Gamma_{3\text{PP}} + \Gamma_{5\text{PP}} + \Gamma_{\text{ph-r}}. \end{aligned} \quad (3.13)$$

Since $\Gamma_{3\text{RP}}$ is very small compared to the other kinetic coefficients,¹⁶ we may neglect it. Substituting Eq. (3.13) into Eqs. (3.11) and (3.12) and solving for μ_{ph} and μ_r , we obtain

$$\begin{aligned} \mu_{\text{ph}} = \frac{1}{\Gamma_{3\text{PP}} + \Gamma_{5\text{PP}}} \left[\frac{\partial N}{\partial \rho} \nabla \cdot (\mathbf{j} \cdot \rho \mathbf{v}_n) \right. \\ \left. - \left[N - \frac{\partial N}{\partial S} S - \frac{\partial N}{\partial \rho} \rho \right] \nabla \cdot \mathbf{v}_n \right], \end{aligned} \quad (3.14)$$

$$\begin{aligned} \mu_r = \frac{1}{\Gamma_{3\text{PP}} + \Gamma_{5\text{PP}}} \left[\frac{\partial N}{\partial \rho} \nabla \cdot (\mathbf{j} - \rho \mathbf{v}_n) \right. \\ \left. - \left[N - \frac{\partial N}{\partial S} S - \frac{\partial N}{\partial \rho} \rho \right] \nabla \cdot \mathbf{v}_n \right] \\ + \frac{1}{\Gamma_{\text{ph-r}}} \left[\frac{\partial N_r}{\partial \rho} \nabla \cdot (\mathbf{j} - \rho \mathbf{v}_n) \right. \\ \left. - \left[N_r - \frac{\partial N_r}{\partial S} S - \frac{\partial N_r}{\partial \rho} \rho \right] \nabla \cdot \mathbf{v}_n \right], \end{aligned} \quad (3.15)$$

where $N = N_r + N_{\text{ph}}$.

In Eqs. (3.3) and (3.4) the pressure P and chemical po-

tential μ can be expressed in terms of the chemical potentials μ_{ph} and μ_r as

$$\begin{aligned}\nabla P &= -\nabla \left[\frac{\partial P}{\partial \mu_r} \mu_r + \frac{\partial P}{\partial \mu_{\text{ph}}} \mu_{\text{ph}} \right], \\ \nabla \mu &= -\left[\frac{\partial \mu}{\partial \mu_r} \mu_r + \frac{\partial \mu}{\partial \mu_{\text{ph}}} \mu_{\text{ph}} \right].\end{aligned}\quad (3.16)$$

Solving Eqs. (3.14)–(3.16) together with Eqs. (3.3) and (3.4) for the coefficients ξ_1 , ξ_2 , ξ_3 , and ξ_4 of the second viscosity, we obtain

$$\begin{aligned}\xi_1(T) &= -\frac{1}{\Gamma_{3\text{PP}} + \Gamma_{5\text{PP}}} \left[\frac{\partial P}{\partial \mu_r} + \frac{\partial P}{\partial \mu_{\text{ph}}} \right] \frac{\partial N}{\partial \rho} \\ &\quad - \frac{1}{\Gamma_{\text{ph-r}}} \frac{\partial P}{\partial \mu_r} \frac{\partial N_r}{\partial \rho},\end{aligned}\quad (3.17)$$

$$\begin{aligned}\xi_2(T) &= \frac{1}{\Gamma_{3\text{PP}} + \Gamma_{5\text{PP}}} \left[\frac{\partial P}{\partial \mu_r} + \frac{\partial P}{\partial \mu_{\text{ph}}} \right] \left[N - \frac{\partial N}{\partial S} S - \frac{\partial N}{\partial \rho} \rho \right] \\ &\quad + \frac{1}{\Gamma_{\text{ph-r}}} \frac{\partial P}{\partial \mu_r} \left[N_r + \frac{\partial N_r}{\partial S} S - \frac{\partial N_r}{\partial \rho} \rho \right],\end{aligned}\quad (3.18)$$

$$\begin{aligned}\xi_3(T) &= -\frac{1}{\Gamma_{3\text{PP}} + \Gamma_{5\text{PP}}} \left[\frac{\partial \mu}{\partial \mu_r} + \frac{\partial \mu}{\partial \mu_{\text{ph}}} \right] \frac{\partial N}{\partial \rho} \\ &\quad - \frac{1}{\Gamma_{\text{ph-r}}} \frac{\partial \mu}{\partial \mu_r} \frac{\partial N_r}{\partial \rho},\end{aligned}\quad (3.19)$$

$$\begin{aligned}\xi_4(T) &= \frac{1}{\Gamma_{3\text{PP}} + \Gamma_{5\text{PP}}} \left[\frac{\partial \mu}{\partial \mu_r} + \frac{\partial \mu}{\partial \mu_{\text{ph}}} \right] \left[N - \frac{\partial N}{\partial S} S - \frac{\partial N}{\partial \rho} \rho \right] \\ &\quad + \frac{1}{\Gamma_{\text{ph-r}}} \frac{\partial \mu}{\partial \mu_r} \left[N_r - \frac{\partial N_r}{\partial S} S - \frac{\partial N_r}{\partial \rho} \rho \right].\end{aligned}\quad (3.20)$$

Making use of the thermodynamic identities

$$dE_0 = TdS + \mu d\rho - N_r d\mu_r - N_{\text{ph}} d\mu_{\text{ph}}, \quad (3.21)$$

$$P = -E_0 + ST + \mu\rho, \quad (3.22)$$

we can transform Eqs. (3.17)–(3.20) into the following forms:

$$\begin{aligned}\xi_1(T) &= \frac{1}{\Gamma_{3\text{PP}} + \Gamma_{5\text{PP}}} \frac{\partial N}{\partial \rho} \left[N - \frac{\partial N}{\partial S} S - \frac{\partial N}{\partial \rho} \rho \right] \\ &\quad - \frac{1}{\Gamma_{\text{ph-r}}} \frac{\partial N_r}{\partial \rho} \left[N_r - \frac{\partial N_r}{\partial S} S - \frac{\partial N_r}{\partial \rho} \rho \right],\end{aligned}\quad (3.23)$$

$$\begin{aligned}\xi_2(T) &= \frac{1}{\Gamma_{3\text{PP}} + \Gamma_{5\text{PP}}} \left[N - \frac{\partial N}{\partial S} S - \frac{\partial N}{\partial \rho} \rho \right]^2 \\ &\quad + \frac{1}{\Gamma_{\text{ph-r}}} \left[N_r - \frac{\partial N_r}{\partial S} S - \frac{\partial N_r}{\partial \rho} \rho \right],\end{aligned}\quad (3.24)$$

$$\xi_3(AT) = \frac{1}{\Gamma_{3\text{PP}} + \Gamma_{5\text{PP}}} \left[\frac{\partial N}{\partial \rho} \right]^2 + \frac{1}{\Gamma_{\text{ph-r}}} \left[\frac{\partial N_r}{\partial \rho} \right]^2, \quad (3.25)$$

$$\begin{aligned}\xi_4(T) &= -\frac{1}{\Gamma_{3\text{PP}} + \Gamma_{5\text{PP}}} \frac{\partial N}{\partial \rho} \left[N - \frac{\partial N}{\partial S} S - \frac{\partial N}{\partial \rho} \rho \right] \\ &\quad - \frac{1}{\Gamma_{\text{ph-r}}} \frac{\partial N_r}{\partial \rho} \left[N_r - \frac{\partial N_r}{\partial S} S - \frac{\partial N_r}{\partial \rho} \rho \right] = \xi_1(T).\end{aligned}\quad (3.26)$$

Here ξ_1 and ξ_4 are shown to be equal as expected from Onsager's reciprocity symmetry relation for the kinetic coefficients.

As we have recently proven, near $T=0$ K the 3PP are the main influence on the first viscosity coefficient in two-dimensional (Ref. 11) and three-dimensional (Ref. 12) liquid helium, which was not shown by Landau and Khalatnikov, and in ultrasonic sound attenuation¹⁰ the 3PP also influence the coefficients of second viscosity near $T=0$ K. Therefore, to investigate the contribution of the 3PP to the coefficients of the second viscosity, we should consider the 3PP effects separately from other processes. To do this, we only keep the terms for the 3PP in Eqs. (3.3)–(3.12) and then obtain

$$\begin{aligned}\mu_{\text{ph}} &= \frac{1}{\Gamma_{3\text{PP}}} \left[\frac{\partial N_{\text{ph}}}{\partial \rho} \nabla \cdot (\mathbf{j} - \rho \mathbf{v}_n) \right. \\ &\quad \left. - \left[N_{\text{ph}} - \frac{\partial N_{\text{ph}}}{\partial S} S - \frac{\partial N_{\text{ph}}}{\partial \rho} \rho \right] \nabla \cdot \mathbf{v}_n \right].\end{aligned}\quad (3.27)$$

The pressure P and chemical potential μ in Eqs. (3.3) and (3.4) and Eq. (3.6), which depend only on the phonon chemical potential, can be written as

$$\nabla P = -\nabla \left[\frac{\partial P}{\partial \mu_{\text{ph}}} \mu_{\text{ph}} \right], \quad \nabla \mu = -\nabla \left[\frac{\partial \mu}{\partial \mu_{\text{ph}}} \mu_{\text{ph}} \right]. \quad (3.28)$$

Therefore, with the use of the thermodynamic identities Eqs. (3.21) and (3.22), we can obtain the contributions from the 3PP to the coefficients of the second viscosity in Eqs. (3.23)–(3.26), together with Eqs. (3.27) and (3.28), near $T=0$ K as

$$\xi_1(T) = -\frac{1}{\Gamma_{3\text{PP}}} \frac{\partial N_{\text{ph}}}{\partial \rho} \left[N_{\text{ph}} - \frac{\partial N_{\text{ph}}}{\partial S} S - \frac{\partial N_{\text{ph}}}{\partial \rho} \rho \right], \quad (3.29)$$

$$\xi_2(T) = \frac{1}{\Gamma_{3\text{PP}}} \left[N_{\text{ph}} - \frac{\partial N_{\text{ph}}}{\partial S} S - \frac{\partial N_{\text{ph}}}{\partial \rho} \rho \right]^2, \quad (3.30)$$

$$\xi_3(T) = \frac{1}{\Gamma_{3\text{PP}}} \left[\frac{\partial N_{\text{ph}}}{\partial \rho} \right]^2, \quad (3.31)$$

$$\xi_4(T) = -\frac{1}{\Gamma_{3\text{PP}}} \frac{\partial N_{\text{ph}}}{\partial \rho} \left[N_{\text{ph}} - \frac{\partial N_{\text{ph}}}{\partial S} S - \frac{\partial N_{\text{ph}}}{\partial \rho} \rho \right]. \quad (3.32)$$

Here ξ_1 and ξ_4 are equal and thus satisfy Onsager's reciprocity symmetric principle.

TABLE I. Theoretical Parameters.

n (\AA^{-2})	Δ/k_B (K)	q_0 (\AA^{-1})	μ_0	C_0 (m/s)
2.79×10^{-2}	4.12	1.02	$0.75m_{\text{He}}$	164.4

IV. RESULTS AND DISCUSSION

In the previous sections we have derived the kinetic coefficients for two temperature regions, and making use of these coefficients we have obtained the four coefficients of the second viscosity. For $T \lesssim 0.3$ K the contributions of the 3PP to the coefficients are most important, and for $T \gtrsim 0.3$ K we may neglect the contribution from the 3PP. Therefore, dropping the Γ_{3PP} terms in Eqs. (3.23)–(3.26), these coefficients of the two-dimensional (2D) second viscosity are reduced to those of the bulk case,¹⁷ except for differences between 2D and 3D thermodynamic dimensionality. Therefore, we will not write the coefficients here repeatedly for $T \gtrsim 0.3$ K.

To analyze concretely the four coefficients of the 2D second viscosity for $T \lesssim 0.3$ K, we use the following two-dimensional thermodynamic functions:⁸

$$S_{\text{ph}}(T) = \frac{k_B}{2\pi} \left[\frac{3\zeta(3)}{\hbar^2 c_0^2} (k_B T)^2 + \frac{5!\zeta(5)B_1}{c_0^5} (k_B T)^4 + \dots \right], \quad (4.1)$$

$$S_r(T) = \left[\frac{\mu_0 k_B T}{2\pi \hbar^2} \right]^{1/2} \frac{k_B P_0}{\hbar} \left[\frac{3}{2} + \frac{\Delta}{k_B T} \right] e^{-\Delta/k_B T}, \quad (4.2)$$

$$N_{\text{ph}}(T) = \frac{\zeta(2)}{2\pi} \left[\frac{k_B T}{\hbar c_0} \right]^2. \quad (4.3)$$

Here $B_1 = [1 + (3\pi n a^2/16)(\frac{3}{2}a - v_0)]/2c_0$, n is the number density of 2D liquid helium, and the other parameters are given in Eq. (1.3). To see the temperature variation of the coefficients numerically, we adopt the parameters which are deduced from the specific data of Bretz *et al.*¹⁸ To explain the various experimental data,^{9,10} we have made use of these parameters successfully, which are listed in Table I. With this choice, we have obtained the sound velocity $c_0 = 164.4$ m/s at absolute-zero temperature. However, from the analysis of the sound attenuation,¹⁰ we obtained $c_0 = 84.06$ m/s, which is very close to the experimental value of 76 ± 2 m/s given by Wusburn, Rutledge, and Mochel,¹⁹ and the parameters u , a , and b in Eqs. (2.7), (2.8), and (2.22) are assumed to be 1.8,²⁰ 1×10^{43} ,¹⁵ and 4×10^{49} ,²¹ as used by previous workers for the bulk case. With the use of the above parameters and those in Table I, the numerical expressions for the kinetic coefficients are given by

$$\Gamma_{3PP} = 1.24 \times 10^{39} T^5, \quad (4.4)$$

$$\Gamma_{5PP} = 1.0 \times 10^{43} T^7, \quad (4.5)$$

$$\Gamma_{\text{ph-r}} = 4 \times 10^{49} e^{-8.24/T}. \quad (4.6)$$

Through Eqs. (4.4)–(4.6) we see that the powers of the temperatures in the kinetic coefficients vary according to the interactions between elementary excitations.

Substituting Eqs. (1.4)–(4.4) into Eqs. (3.29)–(3.32), we obtain the temperature variation of the four coefficients of the 2D second viscosity as follows:

$$\zeta_1(T) = 4.32 \times 10^{-4} T^{-1} \left[1 - \frac{2[\frac{3}{2} + 4.12/T] + 3.59 \times 10^{-2} T^{3/2} e^{4.12/T}}{(\frac{3}{4} + 4.12/T + 16.97/T^2) + 7.19 \times 10^{-2} T^{3/2} e^{4.12/T}} + 3.6 \right], \quad (4.7)$$

$$\zeta_2(T) = 2.23 \times 10^{-13} T^{-1} \left[1 - \frac{2[(\frac{3}{2} + 4.12/T) + 3.59 \times 10^{-2} T^{3/2} e^{4.12/T}]}{(\frac{3}{4} + 4.12/T + 16.97/T^2) + 7.19 \times 10^{-2} T^{3/2} e^{4.12/T}} + 3.6 \right]^2, \quad (4.8)$$

$$\zeta_3(T) = 8.39 \times 10^5 T^{-1}, \quad (4.9)$$

$$\zeta_4(T) = \zeta_1(T). \quad (4.10)$$

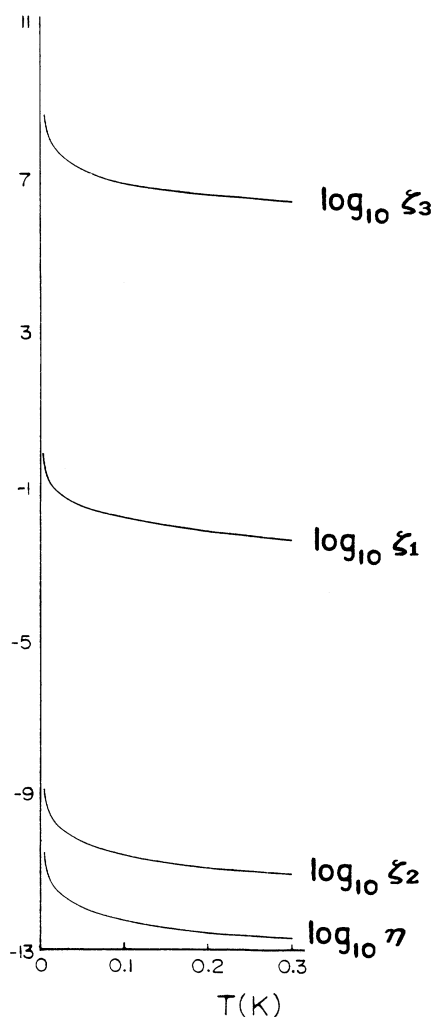


FIG. 1. Temperature variation of the coefficients ξ_1 , ξ_2 , ξ_3 , and ξ_4 of the second viscosity and the coefficient η of the first viscosity in thin helium film. The curves represent only the magnitude of these coefficients, where the units of ξ_1 ($=\xi_4$), ξ_2 ($=\eta$), and ξ_3 are cm^2/s , g/s , and $\text{cm}^4/\text{g s}$, respectively.

When T drops to less than 0.3 K, the second (fractional) term inside the large parentheses in Eqs. (4.7) and (4.8) approaches unity, and these equations are reduced to the following two equations, respectively:

$$\xi_1(T) = 1.555 \times 10^{-3} T^{-1}, \quad (4.11)$$

$$\xi_2(T) = 2.890 \times 10^{-12} T^{-1}. \quad (4.12)$$

All four coefficients have a T^{-1} dependence, which is the same as that of the coefficient of the first viscosity. Here, we note that there are generally 13 independent dissipative coefficients,^{22,23} but we only consider the above five coefficients. Moreover, we will not discuss the pressure and frequency variations of the coefficients of the second viscosity.

We have already shown that the 2D thermal conductivity and first viscosity of thin liquid-helium films¹¹ are all positive. We can easily confirm that the four coefficients of the second viscosity are all positive. Figure 1 illustrates the temperature dependence of the magnitudes of the four coefficients of the second viscosity and the coefficient of the first viscosity (bearing in mind that their dimensions are different). For $T \approx 0.05$ K we find that $\xi_1^2 < \xi_2 \xi_3$, where $\xi_1^2 \sim 10^{-3}$ and $\xi_2 \xi_3 \sim 10^{-2.7}$. However, as the temperature increases to about 0.1 K, the equality then holds, i.e., $\xi_1^2 \approx \xi_2 \xi_3$. Comparing order of magnitude for the four coefficients, we can write $\xi_3 > \xi_1 > \xi_2$, and all four coefficients have larger values than that of the coefficient of the first viscosity.

In conclusion, the coefficients of the second viscosity in two-dimensional thin helium films behave like those of bulk liquid helium for $T \gtrsim 0.3$ K, while for $T < 0.3$ K, the four coefficients of the second viscosity exhibit a T^{-1} dependence like that of the first viscosity, which is due to the 3PP originating from the anomalous excitation spectrum of two-dimensional liquid helium at low momenta and low temperatures.

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*To whom correspondence should be addressed.

¹K. Daly, J. Low Temp. Phys. **71**, 231 (1988); A. Holas and K. S. Singwi, Phys. Rev. B **40**, 167 (1989); **40**, 158 (1989); T. Kopp and P. Wölfle, Phys. Rev. Lett. **59**, 2979 (1987).
²M. V. Schwerdtner, G. Stamm, and D. W. Schmitt, Phys. Rev. Lett. **63**, 39 (1989); W. Fiszdon, Z. Peradzynski, and G. Stamm, Phys. Fluids A **1**, 881 (1989); J. A. Roth, T. P. Brosius, and J. D. Maynard, Phys. Rev. B **38**, 11 209 (1988); W. Fiszdon and M. V. Schwerdtner, J. Low Temp. Phys. **75**, 953 (1989).
³S. A. J. Wieggers, R. Jockemsen, C. C. Kranenburg, and G. Forssati, J. Low Temp. Phys. **71**, 69 (1988); J. P. Desideri, J. P. Laheurte, J. C. Noiray, and J. P. Romagnan, Phys. Rev. B **35**, 6725 (1987); R. B. Bowley, J. Low Temp. Phys. **63**, 291

(1985).

⁴I. N. Ademenko, E. Ya Rudowskii, V. I. Tsygnok, and V. K. Chagovets, J. Low Temp. Phys. **71**, 261 (1988); M. J. Lea, P. W. Retz, and P. Fozooni, *ibid.* **66**, 325 (1987).
⁵T. McMullen, J. Low Temp. Phys. **51**, 33 (1983); A. S. Sachrajda, D. F. Brewer, and W. S. Truscott, *ibid.* **56**, 617 (1984); W. Zhang and J. Kurkijärvi, *ibid.* **73**, 483 (1989); M. Grabinski and M. Liu, *ibid.* **75**, 271 (1989); J. R. Owers-Bradley, P. C. Main, R. J. Church, T. M. M. Hampson, G. McHale, and R. M. Bowley, Phys. Rev. Lett. **61**, 1619 (1988).
⁶A. Andreev and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **44**, 2053 (1963) [Sov. Phys.—JETP **17**, 1384 (1963)].
⁷K. K. Singh and J. Prakash, Phys. Rev. B **17**, 1253 (1978).
⁸A. Isihara and C. I. Um, Phys. Rev. B **19**, 5725 (1979); A.

- Isihara, S. T. Choh, W. H. Kahng, and C. I. Um. *Physica B* **100**, 74 (1980); C. I. Um and W. H. Kahng, *J. Korean Phys. Soc.* **13**, 57 (1980); C. I. Um and A. Isihara, *Phys. Rev. B* **23**, 4498 (1981).
- ⁹C. I. Um, W. H. Kahng, H. G. Oh, and A. Isihara, *Phys. Rev. B* **28**, 2509 (1983); C. I. Um, H. G. Oh, W. H. Kahng, and S. T. Choh, *J. Korean Phys. Soc.* **15**, 61 (1982); C. I. Um, W. H. Kahng, K. H. Yeon, and A. Isihara, *Phys. Rev. B* **29**, 5203 (1984); C. I. Um, W. H. Kahng, E. H. Whang, and S. K. Hong, *ibid.* **39**, 6537 (1989).
- ¹⁰C. I. Um, W. H. Kahng, S. W. Nam, A. Isihara, and S. T. Choh, *Physica B* **100**, 191 (1982); *J. Korean Phys. Soc.* **14**, 46 (1981); C. I. Um, C. W. Jun, W. H. Kahng, and S. T. Choh, *Jpn. J. Appl. Phys.* **26**, 26 (1987); A. Isihara, C. I. Um, C. W. Jun, and S. T. Choh, *Phys. Rev. B* **37**, 7348 (1988); C. I. Um, G. Oh, W. H. Kahng, and A. Isihara *ibid.* **33**, 7550 (1986); C. I. Um and H. G. Oh, *J. Korean Phys. Soc.* **18**, 269 (1985).
- ¹¹C. I. Um, C. W. Jun, W. H. Kahng, and T. F. George, *Phys. Rev. B* **38**, 8838 (1988); C. I. Um, C. W. Jun, W. H. Kahng, and A. Isihara, *Jpn. J. Appl. Phys.* **26**, Suppl. 26-3, 297 (1987); C. I. Um, C. W. Jun, W. H. Kahng, and S. T. Choh, *J. Korean Phys. Soc.* **21**, 41, 51, 209 (1988).
- ¹²C. I. Um, C. W. Jun, W. H. Kahng, and T. F. George, *Phys. Rev. B* **38**, 8834 (1988); C. I. Um, C. W. Jun, and W. H. Kahng, *J. Korean Phys. Soc.* **21**, 213 (1988).
- ¹³H. G. Oh, C. I. Um, W. H. Kahng, and A. Isihara, *Phys. Rev. B* **34**, 6151 (1986); H. G. Oh, C. I. Um, and W. H. Kahng, *J. Korean Phys. Soc.* **19**, 50 (1986).
- ¹⁴I. M. Khalatnikov, *Zh. Eksp. Teor. Fiz.* **23**, 8 (1952); K. H. Benemann and J. B. Ketterson, *The Physics of Liquids and Solid Helium* (Wiley, New York, 1976), Part 1, Chap. 1; S. J. Putterman, *Superfluid Hydrodynamics* (North-Holland, Amsterdam, 1974), Chap. 2.
- ¹⁵L. D. Landau and I. M. Khalatnikov, *Zh. Eksp. Teor. Fiz.* **19**, 637 (1949); **19**, 709 (1949).
- ¹⁶I. M. Khalatnikov, *Zh. Eksp. Teor. Fiz.* **20**, 243 (1950).
- ¹⁷I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965), Chap. 21.
- ¹⁸M. Bretz, J. G. Dash, D. C. Hickernell, E. D. Mclean, and D. E. Vilches, *Phys. Rev. A* **8**, 1589 (1973); C. I. Um, W. H. Kahng, H. G. Oh, and S. T. Choh, *J. Korean Phys. Soc.* **17**, 2, 129 (1984).
- ¹⁹T. E. Wusburn, J. E. Rutledge, and J. M. Mochel, *Phys. Rev. Lett.* **34**, 183 (1975).
- ²⁰R. Jones and P. D. Taylor, *J. Phys. C* **15**, 6709 (1989).
- ²¹J. Wilks, *The Properties of Liquid and Solid Helium* (Oxford University, Oxford, 1967), Chap. 7, p. 159.
- ²²A. Clark, Jr., Ph.D. thesis, Massachusetts Institute of Technology, 1963.
- ²³See S. J. Putterman, in Ref. 14, Chap. 2 and Appendix VI.