Magneto-optical absorption by electrons in the presence of parabolic confinement potentials

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(Received 28 June 1990)

We study theoretically the absorption of long-wavelength perturbation by a many-electron system with externally imposed parabolic confinement potentials and a uniform magnetic field. It is shown that the absorption lines are at exactly the same frequencies as if there were only one electron. The general formulas for the frequencies and line-intensity ratios are found, and explicit analytical expressions are shown in some examples.

I. INTRODUCTION

There has been considerable recent interest in the physics of an electron gas under a magnetic field in the absence or presence of confinement potentials. Examples that have already been experimentally realized include (i) a two-dimensional electron gas confined at a heterojunction, say, between GaAs and $Ga_{1-x}Al_xAs$, where the motion in the third direction is effectively frozen out[1] (the much stronger confinement potential which "freezes" out the motion in a particular direction is simply assumed to be so strong that the intersubband transitions associated with this confinement can be ignored as is assumed throughout this paper). (ii)quantum wells, where electronic motion is free in two directions but experiences a finite confinement potential in the third direction;[2] (iii) quantum "dots" or "coins" where the electrons effectively have no degree of freedom in the z direction, but experience confinement potentials for motions in the other two;[3] (iv) "flat" quantum "wires" where again the z-direction motion has been frozen, while it is free in the, say, x direction, and subject to a finite confinement potential in the y direction.[4] More examples are sure to come in the future.

There are many physical properties of these systems in which one may be interested. An important example includes the collective mode frequencies one would observe by optical absorption. If one takes the somewhat idealized situation of a pure electron gas with only electron-electron interaction in the form

$$U=\sum_{i,j} u(\mathbf{r}_i-\mathbf{r}_j)\ , \tag{1.1}$$

where $\mathbf{r}_i$ denotes the electron positions, in the absence of confinement potentials the celebrated Kohn's theorem[5] tells us that the only observed frequency is the cyclotron frequency $\omega_c=|e|B/mc$ with the same effective mass m that enters in the one-particle dispersion; i.e., the same frequency as if one has only *one* electron. This is applicable to case (i) above. Recently it has been shown by Brey, Johnson, and Halperin[6] (BJH) that this reduction to a single-particle problem holds also for the case of parabolic quantum wells [case (ii)]. It is natural to ask whether this statement is actually more general, i.e., given an electron gas with interparticle interaction (1.1), in an external *parabolic* confinement potential, but otherwise of *arbitrary* unequal strength in the three directions, and under a magnetic field in an arbitrary direction, do the optical absorption frequencies always correspond to that of a *single* particle in the same external potential and magnetic field?

Recently Li *et al.*[7] pointed out that this is indeed true. There we have also shown the explicit forms of frequencies and the raising and lowering operators for some particular cases. It is the purpose of this paper to show the details of the proof of the statement and moreover give a systematic procedure of finding the relevant operators and frequencies, and thus also the ratio of line intensities.

We shall first give a proof of the theorem in its entire generality. It should be noted that this proof is *independent* of the construction of the raising and lowering operators. It is simply based on the separation of center-of-mass motion and the relative coordinates in parabolic confinement potentials. This proof, however, gives us no hint as to what the actual observed frequencies are. Thus we shall also present another argument, though in effect equivalent to the first one, which has an obvious classical analog and is "constructive" (in the sense used in mathematical literature), i.e., it actually instructs us how to obtain the observed frequencies, the raising and lowering operators (and the "conserved" quantities), and hence the oscillator strengths and the line intensities of absorptions. This proof incidentally also tells us some useful properties of the raising and lowering operators. We shall give the frequency for the general case, and apply this procedure to the parabolic well [case (ii)], an *asymmetric* quantum coin [case (iii)], a "*cylindrical*" quantum wire (i.e., free motion along the z direction, but with equal confinement potentials along the x and y directions), and a general "quantum droplet" with the magnetic field in an arbitrary direction.

II. THE THEOREM, ITS PROOF, THE RAISING AND LOWERING OPERATORS AND THEIR PROPERTIES

Consider an electron gas confined by parabolic potentials with electron-electron interaction given by (1.1). The Hamiltonian is given by

$$H=\sum_{i=1}^{N}\frac{1}{2m}\pi_i^2+\tfrac{1}{2}m(\omega_x^2x_i^2+\omega_y^2y_i^2+\omega_z^2z^2)+\sum_{(i,j)}U(\mathbf{r}_i-\mathbf{r}_j)\ . \tag{2.1}$$

Here $i=1,2,\ldots,N$ denotes the electrons and (i,j) denotes all pairs, $\mathbf{r}_i\equiv(x_i,y_i,z_i)$, and we have parametrized the confinement potentials by ω_x^2, ω_y^2, and ω_z^2. π_i is the canonical momentum

$$\pi_i=\frac{1}{i}\nabla_i-\frac{e}{c}\mathbf{A}(\mathbf{r}_i)\ , \tag{2.2}$$

with $\mathbf{A}$ the vector potential characterizing the external uniform magnetic field B. $e<0$ the electronic charge. (We have put $\hbar=1$.) We shall show the following: Under a uniform external time-dependent perturbation (e.g., the optical absorption to a good approximation), the only frequencies one observes are identical to those one would observe for a single particle in the parabolic potential under the same magnetic field.

We shall first give the simplest proof. Define the center of mass and relative coordinates and momentum

$$\mathbf{R}\equiv\mathbf{R}^{(1)}\equiv\frac{1}{N}\sum_{i=1}^{N}\mathbf{r}_i\ , \tag{2.3a}$$

$$\Pi\equiv\Pi^{(1)}\equiv\sum_{i=1}^{N}\pi_i\ , \tag{2.3b}$$

$$\begin{aligned}&X^{(2)}\equiv x_1-x_2\ ,\\ &X^{(3)}\equiv x_1+x_2-2x_3\ ,\\ &X^{(N)}\equiv x_1+x_2+\cdots+x_{N-1}-(N-1)x_N\ ,\end{aligned} \tag{2.4}$$

and similarly for $Y^{(2)},\ldots,Y^{(N)}$, $Z^{(2)},\ldots,Z^{(N)}$, and $\Pi^{(2)},\ldots,\Pi^{(N)}$. Noticing the identity

$$\sum_{i=1}^{N}x_i^2=\frac{1}{N}(\sum_{i=1}^{N}x_i)^2+\frac{1}{N}\sum_{1\le i,j\le N}(x_i-x_j)^2\ ,$$

and similarly for y_i, z_i, and π_i, one can rewrite H as

$$H=H_{\text{C.M.}}+H_{\text{rel}}\ , \tag{2.5}$$

where

$$H_{\text{C.M.}}\equiv\left[\frac{1}{2Nm}\Pi^2+\frac{Nm}{2}(\omega_x^2X^2+\omega_y^2Y^2+\omega_z^2Z^2)\right] \tag{2.6}$$

involves the center of mass only and is of a form similar to the single-electron Hamiltonian. H_{rel}, of a complicated form, involves the relative coordinates and momenta only. Notice now *due to the uniformity of the external magnetic field,* $\mathbf{A}(\mathbf{r}_i)$ is a *linear* functional of $\mathbf{r}_i$ and hence

$$\Pi=\frac{1}{i}\nabla_R-\frac{Ne}{c}\mathbf{A}(\mathbf{R}) \tag{2.7}$$

involves $\mathbf{R}$ and ∇_R alone. It is easy to show that the center-of-mass momentum and position on the one hand commute with the relative coordinates and momenta on the other, and hence

$$[H_{\text{C.M.}},\ H_{\text{rel}}]=0 \tag{2.8}$$

and a complete set of eigenstates of the Hamiltonian can be chosen to be the product states

$$|\Psi\rangle=|\Psi_{\text{C.M.}}\rangle\,|\Psi_{\text{rel}}\rangle\ . \tag{2.9}$$

Thus we have successfully separated the center-of-mass motion and the relative motion. Notice that our center-of-mass particle has mass Nm, charge Ne, and is subject to parabolic potentials N times that of the single particle. Thus the eigenfrequencies of $|\Psi_{\text{C.M.}}\rangle$ are identical with those of a single electron. Under a uniform time-dependent perturbation, say from a long-wavelength light, the coupling is given by the perturbation

$$H_i=-\frac{e}{mc}\sum_i\pi_i\cdot\mathbf{A}_1\ , \tag{2.10}$$

where $\mathbf{A}_1$ characterizes the light waves with $\mathbf{A}_1$ independent of $\mathbf{r}_i$. H_1 can be rewritten as

$$H_1=-\frac{e}{mc}\Pi\cdot\mathbf{A}_1 \tag{2.11}$$

and is thus independent of the relative coordinates and momenta. Operating H_1 on (2.8) thus only operates on $|\Psi_{\text{C.M.}}\rangle$, and moreover in a way identical to the case as if one perturbs a single electron in the well with the incident light. Thus one only observes the same frequencies and with the same intensity ratios among lines as if one had only one electron in the confinement potential with the same magnetic field. (The overall intensity, however, in increased.) This completes the proof.

We now present the second argument showing the separation of the center-of-mass degree of freedom. We consider the commutator of π_i and $\mathbf{r}_i$ with the Hamiltonian H in (2.1). It is easy to show that

$$\begin{aligned}&[\pi_{ix},\ \pi_{iy}]=\frac{ie}{c}B_z\ ,\\ &[\pi_{ix},x_i]=-i\ ,\end{aligned} \tag{2.12}$$

and similar equations for the other directions, whereas operators of different electrons commute. Notice that the right-hand sides of (2.12) are *constants.* We have the equations

$$\begin{aligned}-i[\pi_{ix},H]&=\frac{e}{mc}(\pi_{iy}B_z-\pi_{iz}B_x)\\ &\quad-m\omega_x^2x_i-\frac{\partial}{\partial x_i}\sum_jU(\mathbf{r}_i-\mathbf{r}_j)\ ,\\ -i[x_i,H]&=\pi_{ix}/m\ ,\end{aligned} \tag{2.13}$$

and similarly for the other directions.

We now transform these equations to the basis $\mathbf{R}^{(1),\ldots,(N)}$, $\Pi^{(1),\ldots,(N)}$. The center-of-mass coordinates and momenta (2.3) equations are obtained by summing (2.13) over i. Notice that the gradient terms of U cancel pairwise. The rest of the terms are *linear* in the momentum or position coordinates. One gets

$$-i[\Pi_x, H]=\frac{e}{mc}(\Pi\times\mathbf{B})_x-Nm\omega_x^2X \ ,$$

and similarly for the other directions (2.14)

$$-i[\mathbf{R}, H]=\Pi/(Nm) \ ,$$

which are independent of the relative coordinates. Notice that the right-hand sides of (2.14) are linear.

The equations of motion for the relative coordinates, which will not be shown here, can be found by forming appropriate linear combinations of (2.13) and easily seen to be independent of Π and $\mathbf{R}$. Thus we have shown the desired separation. Recall that in the Heisenberg representation of quantum mechanics, the left-hand sides of (2.14) are just the time derivatives. From (2.14) we thus see that the time derivatives of the center-of-mass coordinates and momenta obey the same equations as a single electron except the replacements $m\rightarrow Nm$, $e\rightarrow Ne$. Under the uniform external perturbation H_1 we have already seen that [Eq. (2.11)] that only the center-of-mass motion is affected. This completes the proof.

It should be noticed that (2.13) and (2.14) have a close classical analog. Equation (2.13) is simply the classical equations of motion of the ith electron, provided that one identifies π_i with $m\mathbf{v}_i$ where $\mathbf{v}_i$ is the velocity of the ith electron, and the left-hand sides as $m\dot{v}_{ix}$ and $\dot{x}_i$. The classical equations of motion of center of mass is then (2.14) with the analogous replacements.

Under an external perturbing electric field $\mathbf{E}(t)$ the classical equations of motion are

$$m\dot{\mathbf{v}}_{ix}=\frac{e}{c}(\mathbf{v}_i\times\mathbf{B})_x-m\omega_x^2x_i$$
$$-\sum_j\frac{\partial}{\partial x_i}U(\mathbf{r}_i-\mathbf{r}_j)+eE_x(t) \ , \qquad (2.15)$$

and similarly for the other directions. Notice the electric field term is independent of i. The center-of-mass equation of motion is then

$$(Nm)\mathbf{V}_x=\frac{(Ne)}{c}(\mathbf{V}\times\mathbf{B})_x-Nm\omega_x^2X+NeE_x(t) \ ,$$
$$\dot{X}\equiv V_x \ , \qquad (2.16)$$

and similarly for the other directions. Here $\mathbf{V}$ is the center-of-mass velocity. Notice that (2.16) does not involve any relative coordinates. It can also be seen easily that the electric field $\mathbf{E}(t)$ drops out of the relative coordinates equations of motion. The above proves our statement in classical mechanics.

The success of the above argument relies on the fact that the Lorentz force and the restoring forces on the electrons are both linear [cf. (2.13) and (2.14)] and the electric field acts on all electrons identically. It is interesting that the statistics of the particles involved plays no role in the argument.

Equation (2.14) actually instructs us how to obtain the relevant eigenfrequencies, raising and lowering operators, and hence the intensities of absorption. Writing any general operator as

$$\hat{O}\equiv\xi\cdot\Pi+\lambda\cdot N\mathbf{R} \ . \qquad (2.17)$$

Then the operations on operators $\hat{O}$ [equivalent to the time derivative $i(d/dt)$], defined by $-[H,\hat{O}]=[\hat{O},H]$, then forms a *linear* map onto the same linear space [see Eq. (2.14)]. First we shall treat the case with ω_x, ω_y, and ω_z all nonzero, deferring the other cases to Sec. III below. In this case it is more convenient to use the "basis" operators (Π_x/m, Π_y/m, Π_z/m, ω_xNX, ω_yNY, and ω_xNZ), and with

$$\hat{O}=\tilde{\xi}_x\frac{\Pi_x}{m}+\tilde{\xi}_y\frac{\Pi_y}{m}+\tilde{\xi}_z\frac{\Pi_z}{m}+\tilde{\lambda}_x\omega_xNX$$
$$+\tilde{\lambda}_y\omega_yNY+\tilde{\lambda}_z\omega_zNZ \qquad (2.18)$$

the linear map is then described by the matrix [by (2.14); this matrix multiplies the column vector $(\tilde{\xi}_x,\tilde{\xi}_y,\tilde{\xi}_z,\tilde{\lambda}_x,\tilde{\lambda}_y,\tilde{\lambda}_z)^t$]

$$M=\begin{pmatrix} 0 & -i\omega_c\cos\theta & i\omega_c\sin\theta\sin\psi & -i\omega_x & 0 & 0 \\ i\omega_c\cos\theta & 0 & -i\omega_c\sin\theta\cos\psi & 0 & -i\omega_y & 0 \\ -i\omega_c\sin\theta\sin\psi & i\omega_c\sin\theta\cos\psi & 0 & 0 & 0 & -i\omega_z \\ i\omega_x & 0 & 0 & 0 & 0 & 0 \\ 0 & i\omega_y & 0 & 0 & 0 & 0 \\ 0 & 0 & i\omega_z & 0 & 0 & 0 \end{pmatrix} . \qquad (2.19)$$

Here (θ,ψ) are the angles in spherical coordinates defining the direction of the magnetic field. Notice that M is Hermitian, moreover

$$M^*=-M \ . \qquad (2.20)$$

Thus there exist six real eigenvalues for M, and moreover if ω is an eigenvalue, so is $-\omega$ with a complex-conjugate eigenvector. The eigenvalues are easily found to satisfy

$$\omega^6-\omega^4(\omega_c^2+\omega_x^2+\omega_y^2+\omega_z^2)$$
$$+\omega^2[\omega_c^2(\omega_x^2\sin^2\theta\cos^2\psi+\omega_y^2\sin^2\theta\sin^2\psi+\omega_z^2\cos^2\theta)$$
$$+(\omega_x^2\omega_y^2+\omega_z^2\omega_x^2+\omega_y^2\omega_z^2)]-\omega_x^2\omega_y^2\omega_z^2=0 \ . \qquad (2.21)$$

Thus all eigenvalues are nonzero (we have assumed all ω_x, ω_y, and ω_z are nonzero), which we shall denote by $\pm\omega_{1,2,3}$, with $\omega_1 \geq \omega_2 \geq \omega_3 > 0$. The corresponding eigenvectors can be chosen to form a unitary transformation U, which diagonalizes M

$$U^\dagger M U = \Omega \ , \tag{2.22}$$

where Ω is the diagonal matrix

$$\Omega \equiv \begin{pmatrix} \omega_1 & & & & & \\ & -\omega_1 & & & & \\ & & \omega_2 & & & \\ & & & -\omega_2 & & \\ & & & & \omega_3 & \\ & & & & & -\omega_3 \end{pmatrix} \tag{2.23}$$

and U is formed by the eigenvectors

$$U \equiv \begin{pmatrix} \tilde{\xi}_x^{(1)} & \tilde{\xi}_x^{(1)*} & \tilde{\xi}_x^{(2)} & \tilde{\xi}_x^{(3)*} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\lambda}_x^{(1)} & \tilde{\lambda}_x^{(1)*} & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix} , \tag{2.24}$$

where the superscripts identifies the eigenfrequencies (do not confuse them with the electron labels or relative coordinates labels). The operators $\tilde{a}^{\pm}$, $\tilde{b}^{\pm}$, and $\tilde{c}^{\pm}$, defined by

$$\begin{aligned} \tilde{a}^- &\equiv \tilde{\xi}_x^{(1)} \frac{\Pi_x}{m} + \cdots + \tilde{\lambda}_x^{(1)} \omega_x NX + \cdots \ , \\ \tilde{a}^+ &\equiv \tilde{\xi}_x^{(1)*} \frac{\Pi_x}{m} + \cdots + \tilde{\lambda}_x^{(1)*} \omega_x NX + \cdots \end{aligned} \tag{2.25}$$

satisfy

$$[H, \tilde{a}^{\pm}] = \pm\omega_1 \tilde{a}^{\pm} \tag{2.26}$$

with similar equations for $\tilde{b}^{\pm}$ and $\tilde{c}^{\pm}$, are thus then the raising and lowering operators associated with H. $\tilde{a}^-$ and $\tilde{a}^+$ are Hermitian conjugate of each other and we shall write them simply as $\tilde{a}$ and $\tilde{a}^+$ when convenient. Classically the quantities in (2.21) with Π/m replaced by $N\mathbf{V}$ satisfy $(d/dt)\tilde{a}^{\mp} = \mp i\omega_1 \tilde{a}^{\mp}$ etc., and the eigenfrequencies are identical with those quantum mechanically.

The unitary transformation can also be viewed as a linear transformation of the operators

$$(\tilde{a}, \tilde{a}^+, \tilde{b}, \tilde{b}^+, \tilde{c}, \tilde{c}^+) = \left(\frac{\Pi_x}{m}, \ldots, \omega_x NX, \ldots \right) U \ , \tag{2.27}$$

with the inverse transformation

$$\left(\frac{\Pi_x}{m}, \ldots, \omega_x NX, \ldots \right) = (\tilde{a}, \tilde{a}^+, \tilde{b}, \tilde{b}^+, \tilde{c}, \tilde{c}^+) U^\dagger \ . \tag{2.28}$$

Equation (2.28) shows that Π is a linear combination of $\tilde{a}^{\pm}$, $\tilde{b}^{\pm}$, and $\tilde{c}^{\pm}$, and thus an electromagnetic wave can only cause transitions at $\pm\omega_{1,2,3}$. Equation (2.28) will also be useful in obtaining the oscillator strengths, as we shall see below.

The above observations actually provide a lot of useful properties of the Hamiltonian and the associated raising and lowering operators, to which we shall now turn. Noticing the form (2.6) of the Hamiltonian for the center of mass, we see that, for any given state $|\Psi\rangle$,

$$\langle \Psi | H_{\text{C.M.}} | \Psi \rangle = \frac{1}{2Nm} \langle \Pi_x \Psi | \Pi_x \Psi \rangle + \cdots + \frac{Nm}{2} \omega_x^2 \langle X\Psi | X\Psi \rangle + \cdots \geq 0 \ ,$$

and thus there must be a state $|G\rangle$ which has the lowest eigenvalue E_G: $H_{\text{C.M.}}|G\rangle = E_G|G\rangle$ (the uniqueness of this state will not concern us at this moment). Since $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ are lowering operators [see (2.26)].

$$\tilde{a}|G\rangle = \tilde{b}|G\rangle = \tilde{c}|G\rangle = 0 \ , \tag{2.29}$$

for otherwise they would be eigenstate(s) of still lower energy.[8] Notice now that the commutators among $\tilde{a}^{\pm}$, $\tilde{b}^{\pm}$, and $\tilde{c}^{\pm}$ are *c-numbers*, since each of them is a linear combination of Π and $\mathbf{R}$, which obey commutation analogous to Eq. (2.12). It is then easy to show that, provided $\omega_1 > \omega_2 > \omega_3$ thus barring accidental degeneracies,

$$[\tilde{a}^{\pm}, \tilde{b}^{\pm}] = [\tilde{b}^{\pm}, \tilde{c}^{\pm}] = 0 \ . \tag{2.30}$$

For consider $[\tilde{a}, \tilde{b}^+]|G\rangle$. Since $[\tilde{a}, \tilde{b}^+]$ is a *c*-number,

$$H[\tilde{a}, \tilde{b}^+]|G\rangle = [\tilde{a}, \tilde{b}^+] H|G\rangle = E_G[\tilde{a}, \tilde{b}^+]|G\rangle \ .$$

On the other hand, using (2.26) and (2.29)

$$H[\tilde{a}, \tilde{b}^+]|G\rangle = H(\tilde{a}\tilde{b}^+ - \tilde{b}^+\tilde{a})|G\rangle = (E_G - \omega_1 + \omega_2)[\tilde{a}, \tilde{b}^+]|G\rangle \ ,$$

which is possible only if $[\tilde{a}, \tilde{b}^+]|G\rangle = 0$. But $[\tilde{a}, \tilde{b}^+]$ is a *c* number, hence $[\tilde{a}, \tilde{b}^+] = 0$. The same proof applies to $[\tilde{a}, \tilde{c}^+]$, $[\tilde{b}, \tilde{c}^+]$, $[\tilde{a}, \tilde{c}]$, $[\tilde{a}, \tilde{b}]$, and $[\tilde{b}, \tilde{c}]$. Taking the Hermitian conjugates completes the proof of (2.30). When degeneracies exist, one can always first consider slightly different parameters ω_c and $\omega_{x,y,z}$ so that the degeneracies are lifted. The operators then satisfy (2.30). Then one lets the parameters approach the original value, and (2.30) is still guaranteed: i.e., one can always, for the case with degeneracies, *construct* the raising and lowering operators such that (2.30) is still satisfied.

Notice, however, that $[\tilde{a}, \tilde{a}^+]$, $[\tilde{b}, \tilde{b}^+]$, and $[\tilde{c}, \tilde{c}^+]$ are not zero in general. We have no general proof for this, but we shall verify in a case by case basis that this is correct. (See Sec. III.) Notice that

$$[\tilde{a}, \tilde{a}^+] = 0 \ \text{ if and only if } \ \tilde{a}^+|G\rangle = 0 \ ,$$

for

$$\langle \tilde{a}^+ G | \tilde{a}^+ G \rangle = \langle G | \tilde{a}\tilde{a}^+ | G \rangle = \langle G | [\tilde{a}, \tilde{a}^+] | G \rangle = [\tilde{a}, \tilde{a}^+] \langle G | G \rangle \ . \tag{2.31}$$

We shall proceed assuming that $[\tilde{a},\tilde{a}^{+}]$, $[\tilde{b},\tilde{b}^{+}]$, and $[\tilde{c},\tilde{c}^{+}]$ are nonzero (and thus positive), i.e., $\tilde{a}^{+}|G\rangle$, $\tilde{b}^{+}|G\rangle$, and $\tilde{c}^{+}|G\rangle$ are eigenstates of H with energies $E_G+\omega_{1,2,3}$, respectively.

Now using (2.28), $H_{\text{C.M.}}$ can be rewritten as

$$H_{\text{C.M.}}=N^{-1}m(\tilde{a}^{+}\tilde{a}+\tilde{b}^{+}b+\tilde{c}^{+}\tilde{c}) + \frac{N^{-1}m}{2}([\tilde{a},\tilde{a}^{+}]+[\tilde{b},\tilde{b}^{+}]+[\tilde{c},\tilde{c}^{+}]) \ . \quad (2.32)$$

Operating this on $|G\rangle$, we thus get

$$E_G=+\frac{N^{-1}m}{2}([\tilde{a},\tilde{a}^{+}]+[\tilde{b},\tilde{b}^{+}]+[\tilde{c},\tilde{c}^{+}]) \ , \quad (2.33)$$

and operating on $\tilde{a}^{+}|G\rangle$ yields, using (2.29) and (2.30),

$$H\tilde{a}^{+}|G\rangle=(E_G+N^{-1}m[\tilde{a},\tilde{a}^{+}])\tilde{a}^{+}|G\rangle \ ,$$

and hence

$$N^{-1}m[\tilde{a},\tilde{a}^{+}]=\omega_1 \ , \quad (2.34)$$

and similarly for $\tilde{b}$ and $\tilde{c}$. Thus

$$H_{\text{c.m.}}=(n_1+\tfrac{1}{2})\omega_1+(n_2+\tfrac{1}{2})\omega_2+(n_3+\tfrac{1}{2})\omega_3 \ , \quad (2.35)$$

with

$$n_1\equiv\tilde{a}^{+}\tilde{a}/[\tilde{a},\tilde{a}^{+}] \ ,$$

etc. $n_{1,2,3}$ have integer eigenvalues. Equations (2.35) and (2.30) thus show us that the eigenstates of the center-of-mass motion can be characterized by quantum numbers similar to those of three uncoupled harmonic oscillators, despite the presence of a magnetic field. (This is in accordance with correspondence principle and the fact that $\omega_{1,2,3}$ are also the classical frequencies of motion.) Special cases of this statement (for a single particle) has been made before.[9,10] Crucial in the above demonstration is the fact that $H_{\text{C.M.}}$ is quadratic in coordinates and momenta; the external magnetic field, being uniform, provides a vector potential which is *linear* in position.

Now we come to oscillator strength associated with the absorption into a particular collective mode $\omega_{1,2,3}$. We shall be interested in the case where the initial state is $|G\rangle$ (for the other cases we simply need the suitable multiplication factors involving the numbers of quanta that are already present, since we have already shown the equivalence to harmonic oscillators). Consider a relevant *normalized* excited state $|\Psi_a\rangle\equiv\mathcal{L}_a^{-1}\tilde{a}^{+}|G\rangle$. Equation (2.31) shows that $\mathcal{L}_a=[\tilde{a}\,\tilde{a}^{+}]^{1/2}$. The oscillator strength of light absorption is given by[11]

$$f_a=\frac{e}{m\omega}\langle\Psi_a|\boldsymbol{\Pi}|G\rangle\cdot\hat{A}_1 \ ,$$

where $\hat{A}_1$ is the unit vector along the perturbing vector potential $\mathbf{A}_1(t)\equiv\mathbf{A}_1e^{-i\omega t}$. Using the commutation relations obtained and (2.28),

$$f_a=\frac{e}{\omega_1}(\tilde{\xi}_a^{(1)}\hat{A}_{1x}+x\leftrightarrow y+x\leftrightarrow z)[\tilde{a},\tilde{a}^{+}]^{1/2} \ , \quad (2.36)$$

where the commutator has already been evaluated in (2.34). The coefficients $\tilde{\xi}\cdots\tilde{\lambda}\cdots$ have to satisfy

$$|\tilde{\xi}_x^{(1)}|^2+\cdots+|\tilde{\lambda}_x^{(1)}|^2+\cdots=1$$

so that U is unitary.

In actual calculation it is often more convenient to use operators of the form (2.17) with the coefficients "unnormalized," i.e.,

$$a=\xi_x^{(1)}\Pi_x+\cdots+\lambda_x^{(1)}NX+\cdots \ , \quad (2.37)$$

with the overall magnitude of the coefficients not fixed. The relations between these are simply

$$\tilde{\xi}_x^{(1)}=\frac{m\xi_x^{(1)}}{(|m\xi_x^{(1)}|^2+\cdots+|\lambda_x^{(1)}/\omega_x|^2+\cdots)^{1/2}} \ ,$$

$$\tilde{a}=(|m\xi_x^{(1)}|^2+\cdots+|\lambda_x^{(1)}/\omega_x|^2+\cdots)^{-1/2}a \ , \quad (2.38)$$

and hence (2.36) reads

$$f_a=\frac{e}{\omega_1}\left[\frac{m\xi_x^{(1)}\hat{A}}{(|m\xi_x^{(1)}|^2+\cdots+|\lambda_x^{(1)}/\omega_x|^2+\cdots)}+\cdots\right][a,a^{+}]^{1/2} \ . \quad (2.39)$$

The complicated denominator can be calculated directly or by calculating $[a,a^{+}]$:

$$(|m\xi_x^{(1)}|^2+\cdots+|\lambda_x^{(1)}/\omega_z|^2+\cdots) =[a,a^{+}]/[\tilde{a},\tilde{a}^{+}]=\frac{m}{N\omega_1}[a,a^{+}] \quad (2.40)$$

via (2.34). Thus alternatively

$$f_a=Ne\xi^{(1)}\cdot\hat{\mathbf{A}}_1[a,a^{+}]^{-1/2} \ . \quad (2.41)$$

The inversion formula (2.28) can also be rewritten, by (2.40), as

$$\frac{\Pi_x}{m}=\frac{N\omega_1\xi_x^{(1)*}}{[a,a^{+}]}a+\frac{N\omega_1\xi_x^{(1)}}{[a,a^{+}]}a^{+}+\frac{N\omega_2\xi_x^{(2)*}}{[b,b^{+}]}b+\cdots \quad (2.42)$$

from which one can also directly obtain (2.41). The intensity of light absorption is proportional to $\omega_1|f_a|^2$.

In the above it is crucial that ω_x, ω_y, and ω_z are all nonzero. If one or more of these is (are) zero, we have to modify the argument. We shall defer the discussion to Sec. III where explicit examples will be given.

III. APPLICATIONS

In this section we shall turn to some applications of the general theory we developed in the preceding section, as

well as giving the modifications necessary when zero eigenvalue exists.

A. Kohn's theorem

To orient ourselves we consider case (i) mentioned in the Introduction. In this case there is no confinement potential: $\omega_x=\omega_y=\omega_z=0$. With the field along $\hat{z}$, the equations of motion are

$$\begin{aligned} -i[\Pi_x, H]&=\frac{e}{mc}\Pi_y B \ , \\ -i[\Pi_y, H]&=-\frac{e}{mc}\Pi_x B \ , \\ -i[\Pi_z, H]&=0 \ , \end{aligned} \tag{3.1}$$

$$-i[N\mathbf{R}, H]=\mathbf{\Pi}/m \ . \tag{3.2}$$

The right-hand side of (3.1) does not involve $\mathbf{R}$. At this point the classical analogy is useful: (3.1) is simply $Nm(d/dt)\mathbf{V}=(Ne/c)\mathbf{V}\times\mathbf{B}$, and (3.2) is $d\mathbf{R}/dt=\mathbf{V}$. The second equation simply gives us $\mathbf{R}(t)$ once we solve $\mathbf{V}(t)$ using the first equation. Moreover, the emotion in the $\hat{z}$ direction separates out: V_z is a constant. This suggests that we simply consider the set of operators

$$\hat{O}=\xi_x\frac{\Pi_x}{m}+\xi_y\frac{\Pi_y}{m} \tag{3.3}$$

instead of (2.3). Equation (3.1) then tells us that the operation on $\hat{O}$ defined by $[\hat{O},H]$ is simply a linear mapping on the coefficients (ξ_x,ξ_y) with the matrix a truncated form of (2.19), i.e.,

$$M_K=\begin{bmatrix} 0 & -i\omega_c \\ i\omega_c & 0 \end{bmatrix} , \tag{3.4}$$

where $\omega_c\equiv(|e|B)/mc$ is the cyclotron frequency.

Our theorem in the preceding section can then be adopted here, though most of the results are obvious by inspection. The eigenvalues are $\omega=\pm\omega_c$, and the eigenvectors give us the raising and lowering operators

$$a^{\pm}=\Pi_x\pm i\Pi_y \tag{3.5}$$

which were first obtained by Kohn.[5]

Classically one has $-i(d/dt)(V_x\pm iV_y)=\pm\omega_c(V_x\pm iV_y)$. One then integrates $d\mathbf{R}/dt=\mathbf{V}$ to obtain $\mathbf{R}(t)$. The analogous quantum-mechanical procedure gives us two invariants

$$\begin{aligned} K_1&=\Pi_x/m+\omega_c NY \ , \\ K_2&=\Pi_y/m-\omega_c NX \ , \end{aligned} \tag{3.6}$$

satisfying

$$[K_{1,2}, H]=0 \ .$$

$K_1\pm iK_2$ are in fact the "level" operators connecting degenerate states discussed in Ref. 12. $K_{1,2}$ are related to the position of the center of the cyclotron motion (of the center of mass).

B. Parabolic well

In this case $\omega_z\neq0$ but $\omega_x=\omega_y=0$. This has been treated before by BJH, however, our general theorem (with suitable modifications: see below) gives us some useful information and moreover puts the work of BJH within the same general framework as all the other cases. Thus we shall treat this case afresh.

We choose $\mathbf{B}=B(\sin\theta,0,\cos\theta)$ as in BJH. The quantum-mechanical equations of motion are, with $\omega_z\equiv\omega_0$,

$$\begin{aligned} -i[\Pi_x, H]&=-\omega_c\Pi_y\cos\theta \ , \\ -i[\Pi_y, H]&=\omega_c\Pi_x\cos\theta-\omega_c\Pi_z\sin\theta \ , \\ -i[\Pi_z, H]&=\omega_c\Pi_y\sin\theta-m\omega_0^2 NZ \ , \\ -i[NZ, H]&=\Pi_z/m \ , \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} -i[NX, H]&=\Pi_x/m \ , \\ -i[NY, H]&=\Pi_y/m \ . \end{aligned} \tag{3.8}$$

It is useful to recall (as in Sec. III A) again the classical analogy. Anyway the first four equations form a complete set of equations of motion involving Π_x, Π_y, Π_z, and Z only. If

$$\hat{O}\equiv\tilde{\xi}_x\frac{\Pi_x}{m}+\tilde{\xi}_y\frac{\Pi_y}{m}+\tilde{\xi}_z\frac{\Pi_z}{m}+\tilde{\lambda}_z(\omega_0 NZ) \ , \tag{3.9}$$

the operation $-[H,\hat{O}]$ defines a linear map on the subspace spanned by $\hat{O}$ into itself. The map is described by the Hermitian matrix

$$M_{\text{well}}=\begin{bmatrix} 0 & -i\omega_c\cos\theta & 0 & 0 \\ i\omega_c\cos\theta & 0 & -i\omega_c\sin\theta & 0 \\ 0 & i\omega_c\sin\theta & 0 & -i\omega_z \\ 0 & 0 & i\omega_z & 0 \end{bmatrix} , \tag{3.10}$$

multiplying the column matrix formed by the coefficients $(\tilde{\xi}_x,\tilde{\xi}_y,\tilde{\xi}_z,\tilde{\lambda}_z)$. The eigenvalues are $\omega=\pm\omega_{1,2}$ with

$$\omega_{1,2}^2=\frac{\omega_0^2+\omega_c^2\pm[\omega_0^4+\omega_c^4-2\omega_0^2\omega_c^2\cos(2\theta)]^{1/2}}{2} \tag{3.11}$$

and are nonzero. Our general theorem can thus be carried over except the trivial changes in the dimension of the linear map. We introduce the angle α (for a more intuitive understanding of α, see the Appendix), which satisfies

$$\begin{bmatrix} \omega_0^2\cos^2\theta & -\omega_c^2\sin\theta\cos\theta \\ -\omega_c^2\sin\theta\cos\theta & \omega_0^2+\omega_c^2\sin^2\theta \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$=\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} . \tag{3.12}$$

This α must exist since the first matrix in (3.12) is real and symmetric and must be diagonalizable by a rotation. The eigenvalues of this matrix are ω_1^2 and ω_2^2. (This is of course not an accident: see the Appendix.) The ("unrenormalized") raising and lowering operators can be obtained easily by finding the eigenvectors of M_{well},

$$\begin{aligned} a^{\pm} &= \mp i(\Pi_x \cos\alpha - \Pi_z \sin\alpha) + \frac{\omega_c}{\omega_1}\cos(\alpha-\theta)\Pi_y \\ &\quad - \frac{Nm\omega_0^2}{\omega_1}\sin\alpha\, Z \ , \\ b^{\pm} &= \mp i(\Pi_z \cos\alpha + \Pi_x \sin\alpha) + \frac{\omega_c}{\omega_2}\sin(\alpha-\theta)\Pi_y \\ &\quad + \frac{Nm\omega_0^2}{\omega_2}\cos\alpha\, Z \ . \end{aligned} \tag{3.13}$$

The form of these operators are chosen to allow an easy comparison with the work of BJH, the details of which will be relegated to the Appendix. Our general theorem (2.30) implies that $[a^{\pm}, b^{\pm}]=0$, which can be also directly verified. On the other hand, we can directly calculate

$$\begin{aligned} [a, a^+] &= 2Nm\omega_1 \ , \\ [b, b^+] &= 2Nm\omega_2 \ . \end{aligned} \tag{3.14}$$

Moreover, our general theorem (2.28) then guarantees that $\Pi_{x,y,z}$ and NZ are linear combinations of $a^{\pm}$ and $b^{\pm}$. Equation (2.42) provides us with the inversion formulas. With the help of (3.12) they are

$$\begin{aligned} \Pi_x &= \frac{i}{2}[\cos\alpha(a^+ - a^-) + \sin\alpha(b^+ - b^-)] \ , \\ \Pi_y &= \frac{1}{2\omega_c\cos\theta}[\omega_1\cos\alpha(a^+ + a^-) + \omega_2\sin\alpha(b^+ + b^-)] \ , \\ \Pi_z &= \frac{i}{2}[\cos\alpha(b^+ - b^-) - \sin\alpha(a^+ - a^-)] \ , \\ NZ &= \frac{1}{\omega_0^2\cos\theta}\left[-\omega_1\sin(\alpha-\theta)\frac{a^+ + a^-}{2} + \omega_2\cos(\alpha-\theta)\frac{b^+ + b^-}{2}\right] , \end{aligned} \tag{3.15}$$

which can also be obtained by simple algebra. The line intensities of absorption under long-wavelength light can be obtained by (3.13) and (2.41), or using (3.15) and (3.14). For example, if the incident light is polarized along $\hat{x}$, absorption is at ω_2 and ω_1 (and only these), with an intensity ratio of $\tan^2\alpha$. If the incident light is polarized along $\hat{y}$, the intensity ratio is [with the help of (3.12)] $(\omega_2^2/\omega_1^2)\tan^2\alpha$.

Finally, we can "integrate" (3.8) (recalling the classical analogy). We obtain two invariants of motion

$$\begin{aligned} K_1 &= \frac{\Pi_x}{m} + N\omega_c Y\cos\theta \ , \\ K_2 &= \frac{\Pi_y}{m} - N\omega_c X\cos\theta + NZ\omega_c\sin\theta \ , \end{aligned} \tag{3.16}$$

which, by the classical analogy, can be seen to be related to the translational symmetry along x and y. It can be easily verified that

$$[a^{\pm}, K_{1,2}] = [b^{\pm}, K_{1,2}] = 0 \ , \tag{3.17}$$

which can also be proven in a similar way as (2.30).

C. Asymmetric "coin"

In this case the z direction is frozen out, and we shall simply drop the degree of freedom associated with the motion in this direction. All the discussions in Sec. II carry over except the change in dimensionality.

If $\omega_x = \omega_y$, i.e., a symmetric coin, due to the cylindrical symmetry, it is useful to introduce the operators

$$X_{\pm} \equiv X \pm iY \ , \quad \Pi_{\pm} = \Pi_x \pm i\Pi_y \ . \tag{3.18}$$

In this case it can be easily seen that the equations of motion decouple into two sets, involving Π_+ and X_+ on the one hand, and Π_- and X_- on the other.

Generally from the linear map we get the raising and lowering operators

$$\begin{aligned} a^+ &= \frac{\Pi_+}{m} - i(\omega_c - \omega_+)NX_+ + g_a\left[\frac{\Pi_-}{m} + i(\omega_c + \omega_+)NX_-\right] , \\ b^+ &= \frac{\Pi_-}{m} + i(\omega_c + \omega_-)NX_- + g_b\left[\frac{\Pi_+}{m} - i(\omega_c - \omega_-)NX_+\right] , \\ a^- &= (a^+)^{\dagger}, b^- = (b^+)^{\dagger} \ , \end{aligned} \tag{3.19}$$

where

$$g_a \equiv \frac{\omega_x^2-\omega_y^2}{2\left[\omega_+^2+\omega_c\omega_+-\frac{\omega_x^2+\omega_y^2}{2}\right]}, \qquad g_b \equiv \frac{\omega_x^2-\omega_y^2}{2\left[\omega_-^2-\omega_c\omega_--\frac{\omega_x^2+\omega_y^2}{2}\right]}, \tag{3.20}$$

and

$$\omega_{1,2}^2 \equiv \omega_\pm^2 = \frac{\omega_x^2+\omega_y^2+\omega_c^2 \pm [\omega_c^4+2\omega_c^2(\omega_x^2+\omega_y^2)+(\omega_x^2-\omega_y^2)^2]^{1/2}}{2}, \tag{3.21}$$

satisfying

$$\omega^4-\omega^2(\omega_x^2+\omega_y^2+\omega_c^2)+\omega_x^2\omega_y^2=0, \tag{3.21a}$$

which are the eigenfrequencies. The finite difference between ω_x^2 and ω_y^2 couples the otherwise decoupled set. $g_{a,b}\to 0$ as $\omega_x^2\to\omega_y^2$. In this limit one can check that our results reduce to that of Li *et al.*,[7] and $\omega_\pm$ reduce to $[(\omega_c/2)^2+\omega_0^2]^{1/2}\pm\omega_c/2$. In general, at $B=0$ there are resonances at ω_x and ω_y. The difference between the frequencies increases with increasing field, and as $\omega_c \gg \max(\omega_{x,y})$ one mode approaches ω_c while the other branch approaches to zero as $1/B$.

It can be checked easily that $[a^\pm, b^\pm]=0$ in accordance with our general theorem. A straightforward direct calculation yields

$$[a, a^+]=\frac{2N}{m}(2\omega_+-\omega_c), \qquad [b, b^+]=\frac{2N}{m}(2\omega_-+\omega_c). \tag{3.22}$$

Our general theorem again guarantees that Π_x, and Π_y, and hence $\Pi_\pm$, can be expressed entirely in terms of $a^\pm$ and $b^\pm$, and hence the external long-wavelength perturbation can only cause transitions at $\pm\omega_{1,2}$. Equation (2.42) allows us to express $\Pi_{x,y}$ in terms of $a^\pm$ and $b^\pm$ easily. However, we shall not show them here. We directly use (2.41) to obtain the relevant ratios of line intensities.

First we recall that, in (2.41), $\xi_x^{(1)}$ is the coefficient of a^- (not a^+) in terms of Π_x. From (3.19)

$$\xi_x^{(1)}=\frac{1}{m}(1+g_a), \quad \xi_y^{(1)}=\frac{-i}{m}(1-g_a), \quad \xi_x^{(2)}=\frac{1}{m}(1+g_b), \quad \xi_y^{(2)}=\frac{i}{m}(1-g_b). \tag{3.23}$$

Thus for linearly polarized light along $\hat{x}(\hat{y})$ the intensity ratios are given by

$$\frac{\omega_-}{\omega_+}\frac{(1+g_b)^2}{(1+g_a)^2}\frac{(2\omega_+-\omega_c)}{(2\omega_-+\omega_c)} \left[\frac{\omega_-}{\omega_+}\frac{(1-g_b)^2}{(1-g_a)^2}\frac{(2\omega_+-\omega_c)}{(2\omega_-+\omega_c)}\right].$$

For a circular symmetric coin these ratios are simply ω_-/ω_+ irrespective of the direction of polarization. Now consider incident circular polarized light. For $E_y=+iE_x$, $(H_1\propto\Pi_+)$,

$$f_a = Ne(\xi_x^{(1)}+i\xi_y^{(1)})[a, a^+]^{-1/2} = N^{1/2}2e/m\left[\frac{2}{m}(2\omega_+-\omega_c)\right]^{-1/2}, \qquad f_b = Ne(\xi_x^{(2)}+i\xi_y^{(2)})[b, b^+]^{-1/2} = N^{1/2}2e/mg_b\left[\frac{2}{m}(2\omega_-+\omega_c)\right]^{-1/2}. \tag{3.24}$$

There is only absorption into ω_+ if the dot is symmetric. Generally the ratio of intensity of absorption into $\omega_2\equiv\omega_-$ to that of $\omega_1\equiv\omega_+$ is given by

$$g_b^2\frac{\omega_-}{\omega_+}\frac{2\omega_+-\omega_c}{2\omega_-+\omega_c},$$

which, to lowest order in $\omega_x^2-\omega_y^2$, is given by

$$\frac{(\omega_x^2-\omega_y^2)^2}{4\omega_+\omega_c}.$$

Similarly, for circularly polarized light where $E_y=-iE_x$ $(H_1\propto\Pi_-)$, if the dot is symmetric, absorption is entirely at ω_-. Taking asymmetry into account produces absorption into ω_+ also with an intensity ratio to that at ω_- given by $g_a^2(\omega_+/\omega_-)(2\omega_-+\omega_c)/(2\omega_+-\omega_c)$, which to lowest order in the difference $\omega_x^2-\omega_y^2$ is $(\omega_x^2-\omega_y^2)^2/4\omega_-\omega_c$.

D. Cylindrical wire

In the last two cases we can carry over our general discussion (of Sec. II B) provided we restrict ourselves to an

appropriate subspace. This, however, is not true when there are zero eigenvalues. The original Kohn's theorem in the absence of any confinement potentials, if we include Π_z in (3.2), actually consititutes an example. However, that case is so simple that one may as well work out any desired relations by simple algebra. As a nontrivial example we consider a cylindrical wire, i.e., $\omega_x=\omega_y=\omega_0$ and $\omega_z=0$. Without loss of generality we take $\mathbf{B}=B(\sin\theta,0,\cos\theta)$.

The equations of motion are as in (2.11). Our previous experience with the parabolic well suggests that we consider the set of operators.

$$\hat{O}=\tilde{\xi}_x\frac{\Pi_x}{m}+\tilde{\xi}_y\frac{\Pi_y}{m}+\tilde{\xi}_z\frac{\Pi_z}{m}+\tilde{\lambda}_x(\omega_0 NX)+\tilde{\lambda}_y(\omega_0 NY)\ . \tag{3.25}$$

Then the operation $-[H,\hat{O}]$ is a linear map acting on $(\tilde{\xi}_x,\tilde{\xi}_y,\tilde{\xi}_z,\tilde{\lambda}_x,\tilde{\lambda}_y)$. The map is given by the Hermitian matrix

$$M_{\text{wire}}=\begin{pmatrix} 0 & -i\omega_c\cos\theta & 0 & -i\omega_x & 0 \\ i\omega_c\cos\theta & 0 & -i\omega_c\sin\theta & 0 & -i\omega_y \\ 0 & i\omega_c\sin\theta & 0 & 0 & 0 \\ i\omega_x & 0 & 0 & 0 & 0 \\ 0 & i\omega_y & 0 & 0 & 0 \end{pmatrix}\ . \tag{3.26}$$

The eigenvalues are $\pm\omega_{1,2}$ and zero, where

$$\omega_{1,2}^2\equiv\omega_\pm^2=\omega_0^2+\frac{\omega_c^2}{2}\pm\left[\frac{\omega_c^4}{4}+\omega_0^2\omega_c^2\cos^2\theta\right]^{1/2}\ . \tag{3.27}$$

The eigenvector with zero eigenvalue corresponds to the operator

$$I=\frac{\Pi_z}{m}-NY\omega_c\sin\theta\ , \tag{3.28}$$

with

$$[I,H]=0\ , \tag{3.29}$$

and is related to the translational symmetry along the wire. It is simplest to proceed by eliminating the variables Π_z in (3.25) by making use of the invariant (3.28) (c.f. Appendix), reducing the dimensionality of the linear map by 1. After some algebra, it can be seen that the problem reduces to the asymmetric dot of Sec. III C provided we make the following identifications:

$$\begin{aligned} &\omega_x^2\to\omega_0^2\ , \\ &\omega_y^2\to\omega_0^2+\omega_c^2\sin^2\theta\ , \\ &\omega_c\to\omega_c\cos\theta\ , \\ &NY\to NY'=NY+I'\ , \end{aligned} \tag{3.30}$$

where

$$I'\equiv\frac{I\omega_c\sin\theta}{\omega_0^2+\omega_c^2\sin^2\theta}\ .$$

One can check that one indeed obtains Eq. (3.27) for the frequencies. The eigenvectors with finite eigenvalues corresponding to the raising and lowering operators (unrenormalized) are analogous to (3.19),

$$\begin{aligned} a^+=&\frac{\Pi_+}{m}-i(\omega_c\cos\theta-\omega_+)(NX_++iI') \\ &+g_a\left[\frac{\Pi_-}{m}+i(\omega_c\cos\theta+\omega_+)(NX_--iI')\right]\ , \\ b^+=&\frac{\Pi_-}{m}+i(\omega_c\cos\theta+\omega_-)(NX_--iI') \\ &+g_b\left[\frac{\Pi_+}{m}-i(\omega_c\cos\theta-\omega_-)(NX_++iI')\right]\ . \end{aligned} \tag{3.31}$$

a^- and b^- can be obtained by taking the Hermitian conjugate. Here, analogous to (3.20),

$$\begin{aligned} g_a=&-\frac{\omega_c^2\sin^2\theta}{2\left[\omega_+^2+\omega_c\cos\theta\omega_+-\left[\omega_0^2+\frac{\omega_c^2\sin^2\theta}{2}\right]\right]}\ , \\ g_a=&-\frac{\omega_c^2\sin^2\theta}{2\left[\omega_-^2-\omega_c\cos\theta\omega_--\left[\omega_0^2+\frac{\omega_c^2\sin^2\theta}{2}\right]\right]}\ . \end{aligned} \tag{3.32}$$

Our general theorem on commutators is still applicable for $[a^\pm,b^\pm]$, $[a^\pm,I]$, and $[b^\pm,I]$. To verify the latter, it is useful to notice

$$[\Pi_\pm,I]=0$$

and thus

$$[a_\pm,I]=[b_\pm,I]=0\ . \tag{3.33}$$

Analogous to (3.22),

$$\begin{aligned} &[a,a^+]=\frac{2}{m}(2\omega_+-\omega_c\cos\theta)\ , \\ &[b,b^+]=\frac{2}{m}(2\omega_-+\omega_c\cos\theta)\ . \end{aligned} \tag{3.34}$$

It can also be checked easily that indeed $[a^\pm,b^\pm]=0$.

With these commutators, the intensities of light absorption can be obtained via formulas as in Sec. II. For example, if the incident light is circularly polarized with electric vector in the $x-y$ plane, then the answer is thus completely analogous to Sec. III C except the proper replacements (3.30). The collective modes can also be excited by light polarized along the z direction. Applying (2.41) shows that the intensity of ω_- to that of ω_+ is given by

$$\frac{\omega_-}{\omega_+}\left[\frac{(\omega_c\cos\theta+\omega_-)+g_b(\omega_c\cos\theta-\omega_-)}{(\omega_c\cos\theta-\omega_+)+g_a(\omega_c\cos\theta+\omega_+)}\right]^2\frac{2\omega_+-\omega_c\cos\theta}{2\omega_-+\omega_c\cos\theta}.$$

It can be easily checked that, if $\theta=0$, then the results reduce to those of a symmetric dot (with extra free motion along the $\hat{z}$ axis). The intensity of absorption of polarized light along $\hat{z}$ vanishes as $\sin^2\theta$ for both modes [see (3.31) and (3.30)] as $\theta\to 0$.

E. Quantum "droplet"

In this case we have confinement potentials in all three directions. The general frequencies of the modes have already been given in Eq. (2.21). Finding the explicit expressions for $\omega_{1,2,3}$ and their optical oscillator strengths are algebraically extremely tedious and the results are not too illuminating, and will not be attempted here. We shall merely content ourselves with some comments. (i) If the magnetic field is along one of the principal axes x, y, or z, then the motion along the field is decoupled from the others. One simply obtains an oscillator along $\hat{B}$ and a quantum coin, in general asymmetric, where the results of Sec. III C applies. (ii) In general there are three modes. At $B\to 0$ they are ω_x, ω_y, and ω_z. At high fields $[\omega_c \gg \max(\omega_x,\omega_y,\omega_z)]$ an examination of Eq. (2.21) shows that one mode approaches ω_c, the second approaches a constant, whereas the third approaches zero with $\omega_3 \propto 1/B$. The first one is intuitively obvious. The last one semiclassically corresponds to a slow "$E\times B$" drift near the edge of the droplet. (iii) The line intensities can be obtained numerically with the help of (2.36) and (2.34).

IV. CONCLUSION

In this conclusion section we compare our results with existing theoretical and experimental works.

Experimentally far-infrared absorption measurements have confirmed the Kohn's theorem predictions in the case of parabolic quantum wells [case (ii)], as have already been discussed in the literature.[2,6] For circular symmetric quantum "coins" [case (iii)], Refs. 3(a) and 3(c) have confirmed Kohn's theorem and have shown spectral weights consistent with those predicted in Sec. III. The other experiments in Ref. 3 seem to have nonparabolic confinements, which exhibit themselves as having more than two resonances and/or absorption frequencies dependent on the number of electrons present. (See Ref. 7 for a more detailed discussion.)

These are various theoretical efforts in studying plasmons or magnetoplasmons in the presence or absence of parabolic confinement potentials. Approximations invariably have to be made at some stage in a many-body calculation. However, it is clear that (from the present investigation) a result consistent with Kohn's theorem will be guaranteed so long as the center-of-mass equation of motion is preserved. Classical hydrodynamics has been used, though only for some special cases of parabolic confinement potentials (or, equivalently, with proper interpretations, a uniform positively charged background) to study magnetoplasmons. It is gratifying that, at least for the case of a spherical three-dimensional droplet ($\omega_x^2=\omega_y^2=\omega_z^2=\omega_0^2$) and thus also of a symmetric coin, the classical hydrodynamics[13] does give results in agreement with the present investigation.

Quantum mechanically, there are two major types of efforts in calculating the relevant frequencies of optical absorption microscopically. Brey *et al.*[14] recently have demonstrated explicitly in the local-density approximation that Kohn's theorem is obeyed, at least in their case of a parabolic quantum well in the absence of the magnetic field. There are also various efforts using the standard (diagrammatic) techniques of many-body theory. In order that the equations of motion of the center of mass be preserved in the approximate calculation, it is obvious from a generalization of the work by Kadanoff and Baym[15] that one must use a "conserving approximation." In fact, for the case of a two-dimensional electron gas and in the absence of confinement potentials along the x-y plane [case (i)], this point has been demonstrated explicitly (in the strong-field limit) by Kallin and Halperin[16] in their (conserving) self-consistent Hartree and Hartree-Fock calculations.[17] In the presence of confinement potentials, the situation is less clear in the literature. Reference 18, for example, has calculated (in the absence of a magnetic field and effectively for the weak interaction or large subband separation limit) the intersubband collective excitation in the case of a "flat wire" [case (iv) of the Introduction], where states of subbands higher than the first are unoccupied. Though explicitly for a square-well confinement potential, this work would give a result violating the generalized Kohn theorem in the parabolic case. This work uses the response function as in Fig. 1, but with the full Green's function replaced by the bare ones. However, the response function corresponding to the self-consistent Hartree approximation should involve the full Green's function with the self-energy as in Fig. 2. (This self-energy vanishes in the uniform case.) In the parabolic case it is not difficult to show (in the weak-interaction limit) that the presence of this self-energy cancels the "depolarization shift" from the summation of

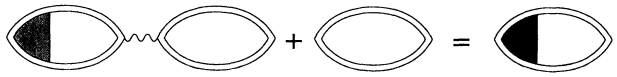

FIG. 1. Response function in the self-consistent Hartree approximation. The double lines represent electron Green's function and the wavy line represents electron-electron interaction.

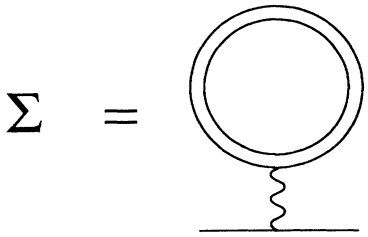

FIG. 2. Self-energy in the same approximation as in Fig. 1.

the series of bubbles ("the vertex correction"), leading to, at $q=0$, a resonance at the bare one-particle energy separation as required by the generalized Kohn theorem.[19]

Notes added. After this paper has been submitted, two papers [P.A. Maksym and T. Chakraborty Phys. Rev. Lett. **65**, 108 (1990); and F. M. Peeters, Phys. Rev. B **42**, 1486 (1990)] have appeared in which some of the results obtained in the present paper are reported.

ACKNOWLEDGMENTS

The author would like to thank Professor Das Sarma and his group, and Professor H. D. Drew for interesting him in this problem. He is particularly grateful to Qiang Li for an initial collaboration and to Dr. Herb Fertig for many discussions which have much clarified the physical picture for him. This research is supported by the National Science Foundation, the U.S. Army Research Office, and the U.S. Office of Naval Research.

APPENDIX

In this appendix we solve the problem of a parabolic well in a more intuitive way, and compare our results with BJH. It also allows us to understand how and why the rotation introduced in Ref. 9 decouples the problem to two independent harmonic oscillators. The equations of motion are as in (3.7) and (3.8). It is again advantageous to solve the problem in a way analogous to which one would do classically. One first obtains the invariants $K_{1,2}$ of (3.16) by inspection. Using K_2 to eliminate Π_y in (3.7) yields

$$\begin{aligned}
-i[\Pi_x, H] &= -m\omega_c^2\cos^2\cos^2\theta NX + m\omega_c^2\sin\theta\cos\theta NZ \\
&\quad - m\omega_c K_2\cos\theta \ , \\
-i[\Pi_z, H] &= m\omega_c^2\sin\theta\cos\theta NX - (\omega_0^2+\omega_c^2\sin^2\theta)NZ \\
&\quad + m\omega_2 K_2\sin\theta \ .
\end{aligned} \tag{A1}$$

These equations have to be solved together with $-i[NX, H]=\Pi_x/m$ and $-i[NZ, H]=\Pi_z/m$ [and then the $\hat{y}$ motion can be solved by (3.16)]. These are then two coupled harmonic oscillators. Introducing the rotation α about $\hat{y}$ as in (3.12) one obtains

$$\begin{aligned}
-i\left[\frac{\Pi_x'}{m}, H\right] &= -\omega_1^2 NX' - \omega_c K_2\cos(\alpha-\theta) \ , \\
-i\left[\frac{\Pi_z'}{m}, H\right] &= -\omega_2^2 NZ' - \omega_c K_2\sin(\alpha-\theta) \ , \\
-i[NX', H] &= \Pi_x'/m \ , \\
-i[NZ', H] &= \Pi_z'/m \ .
\end{aligned} \tag{A2}$$

Here

$$\begin{bmatrix} X' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} \ , \tag{A3}$$

and similarly for Π_x and Π_z, and $\omega_{1,2}$ are as in (3.11), i.e., we obtain two independent harmonic oscillators (oscillating about positions shifted from the origin). It can be shown that α defined by (3.12) is the same as in BJH. The raising and lowering operators are thus a simple generalization of those known for simple harmonic oscillators and can be written down as in BJH:

$$\begin{aligned}
a^{\pm} &= m\omega_1 NX' \mp i\Pi_x' + \frac{m\omega_c}{\omega_1}K_2\cos(\alpha-\theta) \ , \\
b^{\pm} &= m\omega_2 NZ' \mp i\Pi_z' + \frac{m\omega_c}{\omega_2}K_2\sin(\alpha-\theta) \ .
\end{aligned} \tag{A4}$$

Using (A3), (3.12), and (3.16), one can show that these are just (3.13). Notice that (3.13) does not involve X, as it must be [though this may not be obvious from (A4)].

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[1]For a review, see, for example, T. Ando, A. B. Fowler, and F. Stern, Rev. Mod. Phys. **54**, 437 (1982).

[2]See K. Karraï *et al.*, Phys. Rev. B **39**, 1426 (1988), **40**, 12020 (1989), and references therein.

[3]See, for example, (a) S. J. Allen, H. L. Störmer, and J. C. M. Hwang, Phys. Rev. B **28**, 4875 (1983); (b) D. B. Mast, A. J. Bahm, and A. L. Fetter, Phys. Rev. Lett. **54**, 1706 (1985); (c) Ch. Sikorski and U. Merki, *ibid.* **62**, 2164 (1989); (d) T. Demel, D. Heitman, P. Grambon, and K. Ploog, *ibid.* **64**, 788 (1990); (e) A. Lorke, J. P. Kotthaus, and K. Ploog, *ibid.* **64**, 2559 (1990); and other references cited in Ref. 7 below.

[4]See, for example, T. Demel, D. Heitmann, P. Grambow, and K. Ploog, Phys. Rev. B **38**, 12732 (1988). For a more complete list, see the references cited in Ref. 18 below.

[5]W. Kohn, Phys. Rev. **123**, 1242 (1961).

[6]L. Brey, N. F. Johnson, and B. I. Halperin, Phys. Rev. B **40**, 10647 (1989).

[7]Q. P. Li, K. Karraï, S. K. Yip, S. Das Sarma, and H. D. Drew (unpublished).

[8]It is straightforward to deduce the ground-state wave function (and thus also of the excited states) using this equation and (2.27). The author would like to thank Herb Fertig for a discussion on this point.

[9]For a single electron in a parabolic quantum well, see (a) J. C.

Maan, in *Two-Dimensional Systems, Heterostructures, and Superlattices,* edited by G. Bauer, F. Kuchar, and H. Heinrich, Vol. 53 of *Solid State Sciences* (Springer-Verlag, Berlin, 1984); (b) R. Merlin, Solid State Commun. **64**, 99 (1987).

[10]For a single electron in a circular symmetric quantum "coin" with parabolic confinement potential, see C. G. Darwin, Proc. Cambridge Philos. Soc. **27**, 86 (1930).

[11]We are using the definition (up to a factor of i) of W. A. Harrison, Solid State Theory (Dover, New York, 1980), Sec. 3.5.

[12]S. K. Yip, Phys. Rev. B **40**, 3682 (1989).

[13]See, e.g. A. L. Fetter, Phys. Rev. B **32**, 7676 (1985), and references therein; as well as other references cited in Ref. 3.

[14]L. Brey, J. Dempsey, N. Johnson, and B. I. Halperin (unpublished).

[15]L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, Reading, MA, 1962), see in particular Chaps. 6 and 7.

[16]C. Kallin and B. I. Halperin, Phys. Rev. B **30**, 5655 (1984).

[17]More precisely, the authors in Ref. 16 concentrated on the weak interaction or strong-field limit, keeping terms of the lowest order in $\langle U \rangle/\omega_c$ only. Here $\langle U \rangle$ is a matrix element for the electron-electron interaction. Also they have effectively concentrated on the two-particle Green's function, i.e., *all* collective excitations in the system (not only those relevant for the current-current response). They showed that the collective modes are at integral multiples of ω_c at $q=0$ (ignoring terms of order $\langle U \rangle^2/\omega_c$). Kohn's theorem, however, further asserts that, for the current-current response, there are resonances *only* at $\pm\omega_c$ (but not its multiples). This has not been shown explicitly in Ref. 16, but it can be easily verified that the other resonances are at least of order $\langle U \rangle^2/\omega_c^2$ weaker than main resonance at ω_c, and thus can be consistently ignored within the weak-field approximation in which they are interested.

[18]Q. Li and S. Das Sarma, Phys. Rev. B **40**, 5860 (1989).

[19]Caveats similar to those in Ref. 17 apply, i.e., the present author can show this statement explicitly only in the weak interaction or strong confinement limit ($\langle U \rangle/\omega_o \ll 1$). Restricting ourselves to the case where the subbands higher than the first are empty, the author has shown (unpublished) that, at $q=0$, the current-current response function has a resonance, corresponding to transition into the second subband, at $\omega=\pm\omega_0$ (ignoring terms of order $\langle U \rangle^2/\omega_0$ or higher). There seem to be other resonance corresponding to transitions into even higher subbands, but they are at least $\langle U \rangle^2/\omega_0^2$ weaker, and can be consistently ignored. The situation is unclear if the weak interaction condition is relaxed, for in order to evaluate the two-particle Green's functions, intermediate states involving transition into arbitrary high subbands have to be included, and one has to deal with an infinite-order matrix without the help of the small parameter $\langle U \rangle/\omega_0$.