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Wigner solid from finite-cluster studies

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We perform finite-cluster exact diagonalization studies for electrons in the lowest Landau level for filling factors from $\frac{1}{3}$ to $\frac{1}{11}$. In contrast to previous studies, we focus on the ground-state eigenfunction. This wave function changes continuously for filling factors from $\frac{1}{3}$ to $\frac{1}{9}$. At a filling factor of $\frac{1}{11}$, there is a dramatic qualitative change in the ground-state wave function. Associated with this change is a peak in the structure factor at a finite wave vector as well as excitation energies that are consistent with gapless excitations at $\frac{1}{11}$ filling. These results are consistent with recent experimental findings.

There has been much interest in the fractional quantized Hall effect (FQHE) and its transition to the solid phase recently. Two solid-fluid transitions were found.[1] Electrons in the lowest Landau level can exist in a particularly stable fractional quantized Hall state at odd-denominator filling factors or some other states with no long-range order at other filling factors. Either one of these can become a solid with long-range order as the density is changed and thus there can be two transitions. The transition between a nonfractionally quantized Hall state and the solid has been discussed recently.[2,3] Here we focus on the transition between the solid and the fractional quantized Hall state. The relative stability between these two states were discussed by Lam and Girvin[4] who calculated the quartic anharmonic correction to the energy of the solid with the magnetophonon wave function and compare this energy to that of the Laughlin wave function. They found that the fluid energy is higher than that of the solid at a filling factor ν of 1/6.5. Esfarjani and Chui[5] recently improved upon this estimate by including the cubic anharmonic correction to the solid energy and found that the transition occurs at $\nu=1/5.6$. These estimates are in apparent contradiction of recent experimental findings that discover a fractional quantized Hall state at $\nu=\frac{1}{7}$,[6] and at $\nu=\frac{1}{9}$.[7] There can be different reasons why the theoretical estimates are in disagreement with the experimental results. The nature of electron solids in a magnetic field have received considerable attention recently.[8] Kivelson *et al.*,[9] Baskaran,[10] and Tešanović, Axel, and Halperin[11] have discussed the idea of a solid as well as ring exchanges in the context of the FQHE. Our recent systematic studies of anharmonic corrections[12] suggest that the solid energy is extremely accurate. It is possible that the energy of the fractional quantized Hall state can be lowered if a more accurate wave function than that of Laughlin is used. It is also possible that one needs to include the next Landau level or the finite extent of the wave function to obtain agreement. It is with the purpose of clarifying these issues that we carried out the following finite-cluster exact diagonalization studies for particles in the lowest Landau level interacting with a $1/r$ potential.

There have been numerous finite-cluster exact diagonalization studies done previously for filling factors greater than $\frac{1}{11}$.[13-15] However, none of these discussed in detail the ground-state wave function. Nor is there any evidence of a solid function with true long-range order, which only comes in at $\frac{1}{11}$ filling, in previous finite-cluster studies. In principle, the FQHE state possesses a gap in its excitation spectrum, whereas the solid exhibits no gap. At $\frac{1}{11}$ filling, the number of basis states for four particles is 3091 and it is not practical to perform calculations with more particles. Because of the small number of particles that can be incorporated in a calculation, it is difficult to separate the existence of a gap in the excitation spectrum from finite-size effects. Fortunately, the ground-state wave function is very easy to examine and provides for much clearer evidence of an additional phase. We find that much of the spectral weight of the ground-state wave function is concentrated in a very small fraction of the basis states. This wave function changes continuously for filling factors from $\frac{1}{3}$ to $\frac{1}{9}$. At a filling factor of $\frac{1}{11}$ there is a dramatic qualitative change in the ground-state wave function. Associated with this change is a peak in the structure factor at a finite wave vector as well as excitation energies that are consistent with gapless excitations at $\frac{1}{11}$ filling. These results are consistent with recent experimental findings[6,7] and suggest that the $\frac{1}{11}$ state is a solid state. It also suggests that the discrepancy between theoretical estimates and experimental results lie neither in the exclusion of other Landau levels nor in the finite extent of the wave functions. We now describe our results in detail.

The basis set can be written as product wave functions of Landau orbitals given by[13]

$$\phi_j(\mathbf{r})=\exp[ix_jy-(x-x_j)^2/2]/(\pi^{1/2}L_y)^{1/2}, \quad (1)$$

$$x_j=(2\pi/L_y)j. \quad (2)$$

L_y is the width in the y direction. We set the magnetic length $l=(\hbar c/eB)^{1/2}=1$. The Hamiltonian in second quantized form can be written as

$$H=\sum_{|j|}A(j_1,j_2,j_3,j_4)C_{j_1}^{+}C_{j_2}^{+}C_{j_3}C_{j_4}+\sum_i e_M C_i^{+}C_i. \quad (3)$$

The A's are integrals of the Coulomb potential and the

Landau orbitals ϕ_j's:

$$A(j_a,j_b',j_b,j_a') = \tfrac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\phi_{j_a}^*(\mathbf{r})\phi_{j_b'}^*(\mathbf{r}')\phi_{j_b}(\mathbf{r})\phi_{j_a'}(\mathbf{r}')/|\mathbf{r}-\mathbf{r}'| . \tag{4}$$

e_M is the Madelung energy. We have assumed the particles are confined in a rectangle under periodic boundary conditions. An aspect ratio of 0.866 was used for the four-particle case. This is chosen so as to be commensurate with a hexagonal lattice. For a square e_M is equal to $-1.95e^2/L$. For a rectangular lattice with aspect ratio 0.866, it is equal to $-1.943458e^2/(L_xL_y)^{0.5}$. Here $L_xL_y = 2\pi N_s$ is the area of the rectangle.

We have performed calculations for filling factors from $\frac{1}{3}$ to $\frac{1}{11}$. We focus on the ground-state wave function by selecting only those amplitudes larger than 0.1. The number of such states n_s is typically 2 orders of magnitude less than the total number of basis states. Furthermore, most of the spectral weight of the total wave function is concentrated in them. This is shown in Table I.

TABLE I. The spectral weight and the number of states n_s with amplitude larger than 0.1. Also shown is the total number N of basis states.

Filling factor	Spectral weight	n_s	N
$\frac{1}{11}$	0.698	31	3091
$\frac{1}{9}$	0.8198	18	1641
$\frac{1}{7}$	0.862	19	735
$\frac{1}{5}$	0.937	21	245
$\frac{1}{3}$	0.97	11	43

Because of this simplicity of the ground-state wave function, it is easy to see how it changes as the filling factor is changed. We denote the states as $|\Psi\rangle = |j_1,j_2,j_3,j_4\rangle$, where $j_i = 1,\ldots,4/\nu$, with $i = 1$, 2, 3, and 4. In the fractional quantized Hall state, we find that the five states with the largest amplitude are given by

$$\begin{aligned}
|A\rangle &= |i,i+1,2/\nu+i,2/\nu+i+1\rangle - |i+1/\nu,i+1+1/\nu,3/\nu+i,3/\nu+i+1\rangle ,\\
|B\rangle &= -(|i,i+1,2/\nu+i-1,2/\nu+i+2\rangle + |i+1/\nu,i+1+1/\nu,3/\nu+i-1,3/\nu+i+2\rangle\\
&\quad + |i-1,i+2,2/\nu+i,2/\nu+i+1\rangle - |i-1+1/\nu,i+2+1/\nu,3/\nu+i,3/\nu+i+1\rangle) ,\\
|C\rangle &= (|i-1,i+2,2/\nu+i-1,2/\nu+i+2\rangle - |i-1+1/\nu,i+2+1/\nu,3/\nu+i-1,3/\nu+i+2\rangle) ,\\
|D\rangle &= (|i,i+1,2/\nu+i-2,2/\nu+i+3\rangle + |i+1/\nu,i+1+1/\nu,3/\nu+i-2,3/\nu+i+3\rangle\\
&\quad + |i-2,i+3,2/\nu+i,2/\nu+i+1\rangle - |i-2+1/\nu,i+3+1/\nu,3/\nu+i,3/\nu+i+1\rangle) ,\\
|E\rangle &= -(|i-1,i+2,2/\nu+i-2,2/\nu+i+3\rangle + |i-1+1/\nu,i+2+1/\nu,3/\nu+i-2,3/\nu+i+3\rangle\\
&\quad + |i-2,i+3,2/\nu+i-1,2/\nu+i+2\rangle - |i-2+1/\nu,i+3+1/\nu,3/\nu+i-1,3/\nu+i+2\rangle) .
\end{aligned} \tag{5}$$

This can also be illustrated schematically as follows, where the crosses (circles) indicate occupied (unoccupied) sites.

$$\begin{aligned}
|A\rangle &= |\text{--}\circ\circ\times\times\circ\circ\text{---}\circ\circ\times\times\circ\circ\text{--}(1/\nu\text{-})\rangle - |(1/\nu\text{-})\text{--}\circ\circ\times\times\circ\circ\text{---}\circ\circ\times\times\circ\circ\text{--}\rangle ,\\
|B\rangle &= -(|\text{--}\circ\times\circ\circ\times\circ\text{---}\circ\circ\times\times\circ\circ\text{--}(1/\nu\text{-})\rangle + |(1/\nu\text{-})\text{--}\circ\times\circ\circ\times\circ\text{---}\circ\circ\times\times\circ\circ\text{--}\rangle\\
&\quad + |\text{--}\circ\circ\times\times\circ\circ\text{---}\circ\times\circ\circ\times\circ\text{--}(1/\nu\text{-})\rangle - |(1/\nu\text{-})\text{--}\circ\circ\times\times\circ\circ\text{---}\circ\times\circ\circ\times\circ\text{--}\rangle) ,\\
|C\rangle &= |\text{--}\circ\times\circ\circ\times\circ\text{---}\circ\times\circ\circ\times\circ\text{--}(1/\nu\text{-})\rangle - |(1/\nu\text{-})\text{--}\circ\times\circ\circ\times\circ\text{---}\circ\times\circ\circ\times\circ\text{--}\rangle ,\\
|D\rangle &= |\text{--}\times\circ\circ\circ\circ\times\circ\text{---}\circ\circ\times\times\circ\circ\text{--}(1/\nu\text{-})\rangle + |(1/\nu\text{-})\text{--}\times\circ\circ\circ\circ\times\circ\text{---}\circ\circ\times\times\circ\circ\text{--}\rangle\\
&\quad + |\text{--}\circ\circ\times\times\circ\circ\text{---}\times\circ\circ\circ\circ\times\circ\text{--}(1/\nu\text{-})\rangle - |(1/\nu\text{-})\text{--}\circ\circ\times\times\circ\circ\text{---}\times\circ\circ\circ\circ\times\circ\text{--}\rangle ,\\
|E\rangle &= -(|\text{--}\times\circ\circ\circ\circ\times\circ\text{---}\circ\times\circ\circ\times\circ\text{--}(1/\nu\text{-})\rangle + |(1/\nu\text{-})\text{--}\times\circ\circ\circ\circ\times\circ\text{---}\circ\times\circ\circ\times\circ\text{--}\rangle\\
&\quad + |\text{--}\circ\times\circ\circ\times\circ\text{---}\times\circ\circ\circ\circ\times\circ\text{--}(1/\nu\text{-})\rangle - |(1/\nu\text{-})\text{--}\circ\times\circ\circ\times\circ\text{---}\times\circ\circ\circ\circ\times\circ\text{--}\rangle) .
\end{aligned} \tag{6}$$

Thus, the state with the largest amplitude consists of two clusters each with two electrons. The other states consist of an increasing number of hops away from this cluster configuration. The amplitude of these basis states in the ground state as a function of the filling factor is shown in Fig. 1. These amplitudes change continuously from $\nu = \frac{1}{3}$ to $\nu = \frac{1}{9}$. At $\nu = \frac{1}{11}$ it suddenly decreases by one order of magnitude. Indeed, the dominant states at $\nu = \frac{1}{11}$ are completely different. The state $|A'\rangle$ with largest amplitude consists of four occupied sites each, separated from each other by ten empty sites. The amplitude of other states decreases as the number of hops away from this equally spaced configuration increases. Specifically, the ground-state wave function is approximately given by

$$|\Psi'\rangle = 0.153|A'\rangle - 0.142|B'\rangle - 0.141|C'\rangle + 0.140|D'\rangle + 0.138|E'\rangle + 0.113|F'\rangle + 0.105|G'\rangle - 0.102|H'\rangle , \tag{7}$$

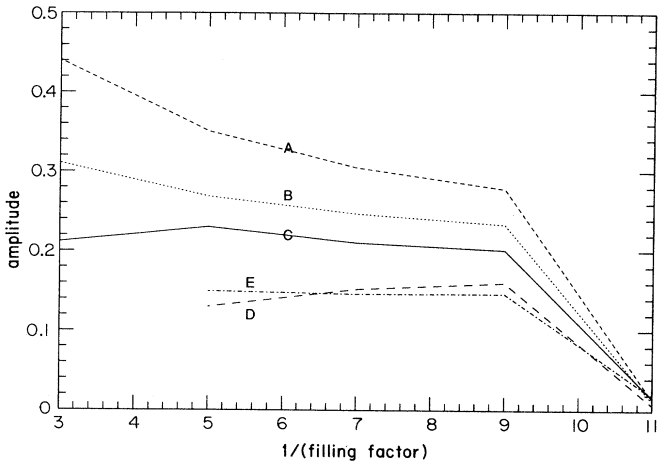

FIG. 1. The amplitude of the five states $|A\rangle$, $|B\rangle$, $|C\rangle$, $|D\rangle$, $|E\rangle$ in the ground-state wave function as a function of the inverse filling factor, $1/\nu$.

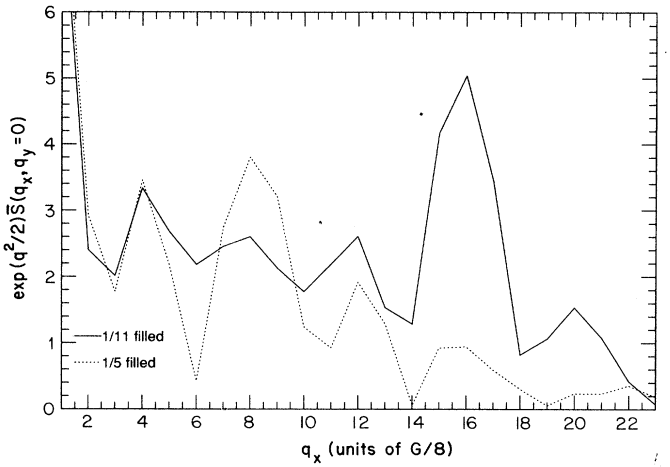

FIG. 2. The projected structure factor $\exp(q^2/2)\bar{S}(q_x,q_y=0)$ as a function of the momentum q_x.

where

$$
\begin{aligned}
|A'\rangle &= |7,18,29,40\rangle ,\\
|B'\rangle &= |6,18,29,41\rangle + |7,18,30,39\rangle + |8,17,29,40\rangle + |7,19,28,40\rangle ,\\
|C'\rangle &= |7,17,30,40\rangle + |7,18,28,41\rangle + |6,19,29,40\rangle + |8,18,29,39\rangle ,\\
|D'\rangle &= |8,17,30,39\rangle + |6,19,28,41\rangle ,\\
|E'\rangle &= |7,17,29,41\rangle + |6,18,30,40\rangle + |7,19,29,39\rangle + |8,18,28,40\rangle ,\\
|F'\rangle &= |7,17,31,39\rangle + |8,16,30,40\rangle + |6,18,28,42\rangle + |5,19,29,41\rangle \\
&\quad + |8,18,30,38'\rangle + |9,17,29,39\rangle + |7,19,27,41\rangle + |6,20,28,40\rangle ,\\
|G'\rangle &= |6,17,30,41\rangle + |6,19,30,39\rangle + |8,17,28,41\rangle + |8,19,28,39\rangle ,\\
|H'\rangle &= |8,16,31,39\rangle + |9,17,30,38\rangle + |6,19,27,42\rangle + |5,20,28,41\rangle .
\end{aligned}
\tag{8}
$$

The excitation energy as a function of the filling factor is shown in Table II. This energy changes continuously until $\nu = \frac{1}{11}$ when it decreases by 1 order of magnitude. The excitation energy at $\frac{1}{11}$ filling corresponds to a state with different total y momentum. This small energy lies within the accuracy of the calculation and probably indicates increased degeneracy in the ground states, as we expect from a solid phase with broken symmetry. The experimental result from Ref. 9 is also shown in Table II. The agreement between the present calculation and experiment at low filling factors is remarkably well. Because we have not performed systematic size-dependence studies, this agreement could be fortuitous. We do expect, however, that the size dependence will be less for the larger-size, smaller-filling-factor systems. The discrepancy at $\frac{1}{3}$ filling is most likely due to the finite extent of the wave function and the small number of sites in the numerical calculation.

The ground-state energy per particle E is shown in Table III together with the solid energy E_s and the energy of the Laughlin fluid E_L. E is close to but slightly lower than E_L and E_s, as expected. To display the trend more clearly, we have also shown $E/\sqrt{\nu}$. This energy first decreases as ν changes from $\frac{1}{3}$ to $\frac{1}{5}$. It then flattens out between $\nu = \frac{1}{7} - \frac{1}{9}$. After $\nu = \frac{1}{9}$, it shows a change in slope.

To gain more understanding of the nature of the ground state, we have also looked at the projected static structure factor $\bar{S}(k)$, obtained from the ordinary structure factor by requiring that all intermediate states lie in the lowest Landau level. This is defined to be

$$\bar{S}(k) = \langle \bar{\rho}_{-k} \bar{\rho}_k \rangle , \tag{9}$$

where $\bar{\rho}_k$ is the projected density operator defined by

$$\bar{\rho}_k = \sum_j \exp(ik\partial_{z_j}) \exp(ik^* z_j/2) , \tag{10}$$

where $z_{(i,j)} = x_{(i,j)} + iy_{(i,j)}$. For large k, $S(k)$ approaches

TABLE II. The excitation energy as a function of the filling factor, in units of $e^2/\epsilon l$.

Filling factor	$\frac{1}{11}$	$\frac{1}{9}$	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{3}$
Excitation energy	0.00008	0.0007	0.003	0.008	0.05
Expt. results	0.0	0.0009	0.004	0.007	0.02

TABLE III. The ground-state energy per particle from the diagonalization, $E_g/4$, the Madelung energy e_M, the total ground-state energy per particle E, the Laughlin fluid energy E_L, the crystal energy, and the energy divided by the squared root of the filling factor, in units of $e^2/\epsilon l$.

ν	$E_g/4$	e_M	E	E_L	E_C	$E/\sqrt{\nu}$
$\frac{1}{11}$	−0.11515	−0.11689	−0.23204	−0.2274	−0.2292	−0.7696
$\frac{1}{9}$	−0.12353	−0.12922	−0.25275	−0.25	−0.2516	−0.75825
$\frac{1}{7}$	−0.13778	−0.146523	−0.2843	−0.281	−0.2822	−0.75218
$\frac{1}{5}$	−0.15827	−0.17337	−0.33164	−0.3276	−0.327	−0.7416
$\frac{1}{3}$	−0.1909	−0.22382	−0.41472	−0.41	−0.399	−0.7183

1, whereas $\bar{S}(k)$ approaches 0. Our result is shown in Fig. 2 for $\exp(q^2/2)\bar{S}(q_x,q_y=0)$ as a function of q_x in units of the reciprocal-lattice vector G of the triangular lattice. At $\frac{1}{5}$ filling and $\frac{1}{11}$ filling, the magnitude of the peak at the $q=G$ is comparable. At $\frac{1}{11}$ filling, the peak at $q=2G$ is much larger. This is consistent with greater long-range order for $\frac{1}{11}$ filling than that at $\frac{1}{5}$ filling. Thus, this result is consistent with our expectation that the state at $\frac{1}{11}$ filling corresponds to a solid with long-range order.

In summary, we have performed finite-cluster exact diagonalization studies for electrons in the lowest Landau level interacting with a $1/r$ potential for filling factors from $\frac{1}{3}$ to $\frac{1}{11}$. We find that the ground-state wave function changes continuously for filling factors from $\frac{1}{3}$ to $\frac{1}{9}$. At a filling factor of $\frac{1}{11}$ there is a dramatic qualitative change in the ground-state wave function. Associated with this change is a peak in the structure factor at a finite wave vector, as well as excitation energies that are consistent with gapless excitations at $\frac{1}{11}$ filling. These results are consistent with recent experimental findings that observe the FQHE at $\frac{1}{7}$ filling and $\frac{1}{9}$ filling. It suggests that recent results by Lam and Girvin[4] and by Esfarjani and Chui[5] can be improved if a better estimate of the fluid energy can be obtained.

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