Hyperuniversality in quantum critical phenomena

Kihong Kim and Peter B. Weichman
Condensed Matter Physics 114-36, California Institute of Technology, Pasadena, California 91125
(Received 13 February 1991)

We apply generalizations of two-scale-factor universality, or hyperuniversality, to quantum phase transitions at zero temperature. We find new universal amplitude combinations involving the superfluid density in two- and three-dimensional Bose systems, as well as confirm and extend previous proposals for universal transport coefficients in two-dimensional superconducting films and magnetic-field-induced metal-insulator transitions. We also give a general proof for the existence of universal jumps in two-dimensional superfluids at finite temperature using finite-size hyperuniversality.

It has long been recognized[1,2] that, along with any critical-point scaling relation between exponents, there should be a corresponding universal relation between critical amplitudes. For example, from the equalities $\alpha=\alpha'$ and $\gamma=\gamma'$ for the specific-heat and susceptibility exponents, above and below the critical temperature T_c, follows the universality of the corresponding ratio of amplitudes A_+/A_- and Γ_+/Γ_-.[2] *Two-scale-factor universality*, or *hyperuniversality*, is the name given to the generalization of these ideas to include the classical hyperscaling relation $d\nu=2-\alpha$, where d is the dimensionality and ν the correlation-length exponent: $\xi\approx\xi_0^{\pm}|t|^{-\nu}$ for $t=(T-T_c)/T_c \gtrless 0$. When hyperscaling is valid the singular parts of the free-energy density integrated over a correlation volume (measured in units of $\beta^{-1}=k_BT$), $\beta\xi^d f_{\text{sing}}$, are dimensionless constants when $t\to 0^{\pm}$. Hyperuniversality is the statement that these constants are universal.[1,2] Equivalently, the amplitude combinations $R_\xi^{\pm}=A_{\pm}(\xi_0^{\pm})^d$, where $C_{\text{sing}}/k_B\approx(A_{\pm}/\alpha)|t|^{-\alpha}$, are universal.[3] It follows also that that in systems with a continuous symmetry (order parameter dimensionality $n\geq 2$), for which $\xi_0^-\equiv\infty\equiv\Gamma_0^-$,[4] one can extract an alternative diverging length $\xi_\Upsilon=(\Upsilon/k_BT)^{1/(2-d)}\approx\xi_0^\Upsilon|t|^{-\nu}$ below $T_c(d>2)$, with the corresponding universal ratio ξ_0^Υ/ξ_0^+. Here $\Upsilon\approx\Upsilon_0|t|^{v}$ is the helicity modulus, related to the superfluid density via $\rho_s=(m/\hbar)^2\Upsilon$, where m is the particle mass (or Cooper pair mass in superconductors), while the corresponding exponent relation is the Josephson hyperscaling relation $v=(d-2)\nu$.

Hyperuniversality may also be applied to finite-size systems.[5] If, for simplicity, one considers a classical cubically shaped system of volume L^d, finite-size hyperuniversality states that at the bulk critical temperature, $T=T_c$, $\lim_{L\to\infty}\beta_c L^d f_{\text{sing}}$ is a universal constant, depending in general only on the sample shape (here assumed cubic) and boundary conditions. In particular, for applications to the superfluid density, if one chooses periodic boundary conditions in the first $(d-1)$ dimensions, and imposes an order-parameter phase-angle twist of θ across the final dimension[4] one should find

$$\lim_{L\to\infty}\beta_c L^d f_{\text{sing}}(\theta)=F(\theta)\ , \tag{1}$$

where $F(\theta)$ is a universal function. Since we may assume that all boundary condition dependence of the free energy is contained in its singular part,[5] the helicity modulus at T_c is then given by

$$\beta_c\Upsilon(T_c)=\beta_c\frac{2}{\theta^2}\lim_{L\to\infty}L^2[f_{\text{sing}}(\theta)-f_{\text{sing}}(0)]$$
$$=\lim_{L\to\infty}\frac{2L^{2-d}}{\theta^2}[F(\theta)-F(0)]\ . \tag{2}$$

Clearly, for $d>2$, this correctly predicts $\Upsilon_c\equiv\Upsilon(T_c)=0$, while in $d=2$ one finds $F(\theta)=F(0)+F_2\theta^2$ for $|\theta|\leq\pi$, and $\beta_c\Upsilon_c=2F_2$ is a *universal number*.[6] This yields automatically the Nelson-Kosterlitz[7] universal jump for the superfluid density in two dimensions: Since one knows from detailed calculations in this case that $\beta_c\Upsilon_c=2/\pi$, we predict $F_2=1/\pi$. A caution is necessary here: The number F_2 is associated with a given fixed point, while in two dimensions one tends to have lines, or even higher dimensional surfaces of fixed points with (perhaps several) associated marginal variables. The value of F_2, and, in general of any other universal quantity, will vary along these fixed surfaces and will be specified uniquely only if all marginal variables are specified. For the Kosterlitz-Thouless transition the unique fixed point describing $T=T_c$ is specified by the marginality of the vortex degrees of freedom at $\beta_c\Upsilon_c=2/\pi$, or exponent $\eta=1/4$.[8]

In dimensions $d>2$ the function $F(\theta)$ is no longer quadratic in θ. If one assumes, as is very likely, that θ-boundary conditions in a finite system of size L are essentially equivalent to a uniform order-parameter twist with wave vector $k_0=\theta/L$ in an infinite system, appropriate scaling of k_0 with ξ [see, e.g., Eq. (5) below] predicts that $f_{\text{sing}}(k_0)-f_{\text{sing}}(0)\propto k_0^d$ at T_c with a universal coefficient. This yields $F(\theta)=F(0)+F_d|\theta|^d$, $|\theta|\leq\pi$.

In the rest of this note we generalize hyperuniversality to the case of quantum critical phenomena at zero temperature. The generalization itself is very straightforward, but the applications and consequences are very deep. We will recover the universal transport coefficients of Ref. 9 in a concise and unified way. We make new predictions for universal ratios involving ρ_s in two- and three-dimensional Bose systems. Applying the same ideas to metal-insulator transitions yields new predictions for universal transport coefficients which are consistent with previous work.

Consider then a quantum system at temperature $T=0$.

We assume the existence of an alternative field controlling the phase transition (for example, magnetic field H, particle mass density ρ, or chemical potential μ) which we denote generically by the dimensionless quantity δ. We assume that $\delta=0$ defines the critical point, while $\delta>0$ denotes the disordered phase and $\delta<0$ denotes the ordered phase. At $T=0$ and small $|\delta|$ one defines two correlation lengths, for definiteness, via the rate of the exponential decay of the Matsubara Green function: (a) the usual spatial correlation length $\xi \approx \xi_0^{\pm}|\delta|^{-\nu}$ and (b) the temporal correlation length $\xi_\tau \approx \xi_{\tau,0}^{\pm}|\delta|^{-\nu_\tau}$. We take ξ_τ to have the same units as β, i.e., inverse energy. The combination $\hbar\xi_\tau$ then has units of time. When $n\geq 2$ we have $\xi_0^- \equiv \infty \equiv \xi_{\tau,0}^-$ and Υ is again required: see below. The dynamical exponent z is defined as the ratio $z=\nu_\tau/\nu$. At positive temperatures the imaginary temporal extent, $0\leq\tau\leq\beta$, of the system is finite. Thus ξ_τ can never diverge and the quantum dynamics cannot affect the static critical behavior: this is the usual statement of irrelevancy of quantum mechanics at finite temperatures. Only at $T=0$ may ξ_τ diverge. When it does, the usual classical hyperscaling argument must be modified. Since the free-energy density is $f=-(\beta V)^{-1}\ln(Z)$, Z being the partition function and V the volume, where now *both* β and V diverge in the thermodynamic limit, the natural hyperscaling ansatz is that $f_{\text{sing}} \sim \xi^{-d}\xi_\tau^{-1} \sim |\delta|^{2-\alpha}$. This yields the generalized hyperscaling relation $2-\alpha=(d+z)\nu$. The corresponding amplitude relation is that $\xi^d\xi_\tau f_{\text{sing}}$ should be universal when $|\delta|\to 0$. Equivalently

$$R_\tau^{\pm}=\xi_{\tau,0}^{\pm}(\xi_0^{\pm})^d A_{\pm} \tag{3}$$

are universal amplitude combinations. Equation (3) is the basic result, which may be derived more formally within the renormalization-group framework by a straightforward generalization of the Appendix of Ref. 2(a). The Josephson hyperscaling relation is generalized similarly: one finds[10] $v=(d-2+z)\nu$. The related ordered phase diverging length is now

$$\xi_\Upsilon(\delta)=[\xi_\tau(-\delta)\Upsilon(\delta)]^{1/(2-d)} \approx \xi_0^{\Upsilon}|\delta|^{-\nu}, \quad \delta\to 0^- \tag{4}$$

(note that ξ_τ replaces β in the finite temperature result), and $R_\Upsilon \equiv \xi_0^{\Upsilon}/\xi_0^+$ is universal. In two dimensions Eq. (4) is problematical, and a better approach is to use Υ to define a divergent temporal scale: $\xi_\tau^{\Upsilon}(\delta) \equiv \xi(-\delta)^{2-d}\Upsilon(\delta)^{-1} \approx \xi_{\tau,0}^{\Upsilon}|\delta|^{-z\nu}$, $\delta\to 0^-$, and $R_\Upsilon^\tau \equiv \xi_{\tau,0}^{\Upsilon}/\xi_{\tau,0}^+$ is universal. This definition does not run into any problems in two dimensions. If the ordered phase has a propagating mode, such as second or high-order sound in ^{4}He, spatial and temporal scales may be related to one another through the speed of sound, and often the exponent z may be determined explicitly.[10] We do not address this issue here, however.

All of the results to follow can be based on various universal scaling forms for the superfluid density, and quantities derived from them. We begin with the scaling of the singular part of the free-energy density in the presence of an imposed order-parameter twist with wave vector k_0:[4,10]

$$f_{\text{sing}} \approx A|\delta|^{2-\alpha}\Phi_{\pm}(Bk_0|\delta|^{-\nu}) , \tag{5}$$

from which one derives

$$\Upsilon(\delta)=\lim_{k_0\to 0}\frac{\partial^2 f_{\text{sing}}}{\partial k_0^2}=AB^2|\delta|^{2-\alpha-2\nu}\Phi_{\pm}''(0) . \tag{6}$$

Here Φ'' denotes a second derivative with respect to the argument. A convenient normalization is to choose $\Phi_-(0)=\Phi_-''(0)=1$, making $\Phi_{\pm}(x)$ universal. Clearly $\Phi_+''(0)=0$, and, with the standard definition[2,3] $A_{\pm}=-A\alpha(1-\alpha)(2-\alpha)\Phi_{\pm}(0)$, one has $A_+/A_- =\Phi_+(0)/\Phi_-(0)$. Note that at $T=0$ we define α and $A_{\pm}$ via $-\partial^2 f_{\text{sing}}/\partial\delta^2 \approx (A_{\pm}/\alpha)|\delta|^{-\alpha}$. Universality requires that $B|\delta|^{-\nu}$ be universally related to ξ, in the present case finite only for $\delta>0$. Thus $R_B=B/\xi_0^+$ is universal. Hyperuniversality implies that $R_\tau \equiv A(\xi_0^+)^d\xi_{\tau,0}^+$ is universal. One then has $R_\Upsilon=[R_\tau R_B^2\Phi_-''(0)]^{1/(2-d)}$ and $R_\Upsilon^\tau=R_\Upsilon^{d-2}$ which are indeed universal.

Equation (6) can be extended in various ways. Of interest here are the extensions to small but finite temperature and frequency. We define the frequency-dependent superfluid density in terms of the temporal Fourier transform of the usual momentum-momentum (or current-current) correlation function.[11] The general scaling form we expect is

$$\Upsilon_{\text{sing}}(\delta,T,\omega)=AB^2|\delta|^{2-\alpha-2\nu} \times Y_{\pm}(C\hbar\omega|\delta|^{-z\nu},D\beta^{-1}|\delta|^{-z\nu}) , \tag{7}$$

where we also allow for a *regular* contribution to Υ, which, however, must *vanish* at $\omega=0$ (see below). Universality requires that $R_C\equiv C/\xi_{\tau,0}^+$ and $R_D\equiv D/\xi_{\tau,0}^+$ be universal, and clearly $Y_{\pm}(0,0)=\Phi_{\pm}''(0)$.

As a first application of Eq. (7) we consider the bosonic models of amorphous and granular superconductors[9,12] in which Cooper pairs are treated as conserved particles obeying Bose statistics, and unpaired electrons are either ignored or included as an effective-harmonic-oscillator heat bath.[12,13] See Ref. 9 for some discussion of the validity of these simplified models near the critical point. The frequency-dependent conductivity of these models is simply given by $\sigma(\delta,T,\omega)=(4e^2/\hbar)\Upsilon(\delta,T,-i\omega)/(-i\hbar\omega)$, where $2e$ is the Cooper pair charge. Consider now approaching the critical point at $\delta=0$, $\omega=0$, $T=0$ along some path in the (δ,ω,T) space in such a way that $x=C\hbar\omega|\delta|^{-z\nu}$, and $y=D\beta^{-1}|\delta|^{-z\nu}$ approach some fixed values x_0,y_0 ($x_0=0$ or ∞ and $y_0=\infty$ are probably the most useful experimentally). One finds then

$$(\hbar/4e^2)\xi(|\delta|)^{d-2}\sigma_{\text{sing}}\to R_\sigma(x_0,y_0) \equiv R_\tau R_B^2 R_C Y_{\pm}(-ix_0,y_0)/(-ix_0) \tag{8}$$

so that, in particular, in $d=2$ the limiting value of $(\hbar/4e^2)\sigma_{\text{sing}}$ is itself universal. A tacit assumption here is that no logarithmic factors appear: these are expected at the critical dimensions for the transition. For most applications of (8), the lower critical dimension is $d_<=1$, while the upper critical dimension is at least $d_>=4$.

Hence no problems are expected in $d=2$.

When the $\delta>0$ phase is an insulator, as when the model does not include a heat bath (i.e., is purely bosonic), any analytic nonuniversal background conductivity must vanish when $\omega=0$, independent of δ. In this case one may drop the subscript on $\sigma_{\rm sing}$ in (8). When the model includes a heat bath, which probably corresponds more closely to experimental reality at least in the case of granular films, the $\delta>0$ phase may be a metal, and may possess an analytic background conductivity $\sigma_0(\delta,T,\omega)=\sigma_{0,0}+\sigma_{0,1}\delta+\sigma_{0,2}\omega+\cdots$, in which $\sigma_{0,i}(T)$ are nonuniversal. In $d=2$ one will find $(\hbar/4e^2)\sigma\rightarrow R_\sigma(x_0,y_0)+(\hbar/4e^2)\sigma_{0,0}$, in place of (8). Often $\sigma_{0,0}$ is found to be very small, and hence, since $R_\sigma(x_0,y_0)$ is expected to be of order unity,[9] one may simply ignore its existence. In general one must take the difference between limits for two different values of $x_0=x_1,x_2$ and $y_0=y_1,y_2$ to obtain the universal result $R_\sigma(x_1,y_1)-R_\sigma(x_2,y_2)$.

As a second application of (7) we consider the recent scaling theory of the superfluid to Bose glass transition in disordered boson systems.[10] The results are equally applicable to the previous nondissipative models of amorphous and granular superconductors, though Υ is much harder to measure experimentally in these cases. We consider (7) with $\omega\equiv0$ but $T>0$. We now assume, as is often the case, that there is a line of finite temperature transitions, $T_c(\delta)$, ending at the special point $T=0$, $\delta=0$. The scaling form (7) then requires that

$$k_BT_c(\delta)=(y_c/D)|\delta|^{z\nu}\ , \tag{9}$$

where y_c is the universal value of the scaling function argument at which $Y_\pm(0,y)$ displays the finite temperature singularity. Thus

$$\Upsilon(T=0,\delta)/k_BT_c(\delta)\approx[R_\tau R_B^2R_DY_\pm(0,0)/y_c]\xi(T=0,|\delta|)^{2-d}\ , \tag{10}$$

so that in $d=2$, $\beta_c(\delta)\Upsilon(0,\delta)$ is a universal number in the limit $\delta\rightarrow0^-$ and $T_c(\delta)\rightarrow0$. It follows also that $\Upsilon(0,\delta)\propto T_c(\delta)^{(d+z-2)/z}$, with a nonuniversal coefficient of proportionality when $d\neq2$. The exponent was predicted in Ref. 10, but the possibility of universal ratios was not examined.

It has again been assumed that $d=2$ is not a critical dimension for the $T=0$ transition. For a *clean* interacting Bose gas, $d=2$ is the *upper* critical dimension,[10] and for the continuum problem the transition takes place at zero density, ρ. For this case, in the limit where $\ln\ln(m/\rho a^2)>>1$ (probably an experimentally inaccessible limit) one finds[14]

$$\Upsilon(T=0,\rho)/k_BT_c(\rho)\approx(1/2\pi)\ln\ln(m/\rho a^2)\ , \tag{11}$$

where a is the atomic hard-core diameter. Thus in order to obtain a universal ratio (in this case $1/2\pi$) the double logarithm should be divided out as well.

As a final point, it is also possible to construct more complicated universal amplitude combinations in three-dimensional Bose systems, an example of which is

$$\lim_{\delta\rightarrow0-}\hbar C_s(T=0,\delta)[\Upsilon(T=0,\delta)]^2/[k_BT_c(\delta)]^2\ , \tag{12}$$

where C_s is the fourth sound speed.[10] All of the input quantities are in principle experimentally measurable.

As our final application we speculate briefly about applying hyperscaling theory to the metal-insulator transition. Since there is no superfluid density in this case, we study the behavior of the current-current correlation function directly. We assume, without justification, that hyperscaling is indeed valid, and hence that the current-current correlation function scales in the same way the superfluid density would. Thus (7,8) are still valid, with the appropriate generalizations of the notations of correlation length and time. Thus in $d=2$ one again expects a universal limiting conductance at the critical point. Since σ is finite on both sides of the transition, presumably zero on the localized side, this may be rephrased as a prediction for a universal jump, $[\sigma(\delta=0^-)-\sigma(\delta=0^+)]$, of the static conductivity. We again emphasize that $d=2$ should not be critical, so the results do not apply to the standard Anderson transition. Models with strong spin-orbit scattering,[15] however, may show the predicted behavior. Similar arguments apply to the diagonal and Hall conductivities at the transition between plateaus in the quantum Hall effects.[9,16] On the insulating side of the transition one may look at the dielectric constant $\epsilon(\omega)=1+4\pi i\sigma(\omega)/\omega$, which, when combined with the correlation lengths, yields the hyperuniversal combination $\lim_{\delta\rightarrow0^+}(1/e^2)\xi^{d-2}\xi_\tau^{-1}\epsilon_{\rm sing}(\omega=0)$. Thus $\epsilon_{\rm sing}$ diverges as $|\delta|^{-\lambda}$ with $\lambda=(2+z-d)\nu$.

In many cases the magnetic field H is a thermodynamically relevant perturbation at the $H=0$ metal-insulator transition since it breaks the symmetry between positive and negative winding numbers in the coherent backscattering picture of localization. The relevant length scale is set by $\sqrt{\Phi_0/H}$, where $\Phi_0=\hbar c/e$ is the flux quantum. This quantity should then appear scaled by ξ as a third argument, $z=G|\delta|^{-\nu}\sqrt{H/\Phi_0}$, in (7), with $R_G\equiv G/\xi_0^+$ universal. Various further universal ratios may now be defined. A simple example is to consider

$$\sigma(\delta=0,T=0,H)\approx(e^2/\hbar)R_\tau R_B^2R_CR_G^{d-2}Y_\infty(H/\Phi_0)^{(d-2)/2}\ , \tag{13}$$

where $Y_\infty=\lim_{z\rightarrow\infty}z^{2-d}Y_\pm(0,0,z)$ which gives $\sigma\propto H^{(d-2)/2}$ with a *universal coefficient*.[17] Finally, the transition at small H must take place at a universal value, z_c, of the argument of $Y_\pm(0,0,z)$. This yields $\delta_c(H)\propto H^{1/2\nu}$. The constant of proportionality is nonuniversal, but this relation gives an experimental means of extracting the exponent ν.[17]

The authors are indebted to M. P. A. Fisher, P. C. Hohenberg, and M. P. Gelfand for several useful conversations, and to M. C. Cross for ongoing encouragement. Financial support from the NSF through Grant No. DMR-8715474 is gratefully acknowledged.

[1]D. Stauffer, M. Ferer, and M. Wortis, Phys. Rev. Lett. **29**, 345 (1972).

[2]P. C. Hohenberg, A. Aharony, B. I. Halperin, and E. D. Siggia, Phys. Rev. B **13**, 2986 (1976); A. Aharony and P. C. Hohenberg, *ibid.* **13**, 3081 (1976).

[3]For a renormalization-group proof see Ref. 2. See also the recent review by V. Privman, P. C. Hohenberg, and A. Aharony, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, New York, in press).

[4]These are the famous spin-wave divergences. See, e.g., M. E. Fisher, M. N. Barber, and D. Jasnow, Phys. Rev. A **8**, 1111 (1973) for a detailed discussion. See also P. B. Weichman and K. Kim, Phys. Rev. B **42**, 10 505 (1990) for explicit calculations within the ϵ-expansion formalism.

[5]V. Privman and M. E. Fisher, Phys. Rev. B **30**, 322 (1984).

[6]We are indebted to M. P. A. Fisher for proposing that F_2 be universal, and for pointing out its connection to the universal jump in two dimensions.

[7]D. R. Nelson and J. M. Kosterlitz, Phys. Rev. Lett. **39**, 1201 (1977).

[8]J. V. Jose, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977).

[9]M. P. A. Fisher, G. Grinstein, and S. M. Girvin, Phys. Rev. Lett. **64**, 587 (1990); M. P. A. Fisher, *ibid.* **65**, 923 (1990). See also A. F. Hebard and M. A. Paalanen, *ibid.* **65**, 927 (1990) for some recent experiments.

[10]D. S. Fisher and M. P. A. Fisher, Phys. Rev. Lett. **61**, 1847 (1988); M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989); P. B. Weichman and K. Kim, *ibid.* **40**, 813 (1989).

[11]See Fisher, Grinstein, and Girvin (Ref. 9), Eq. (4). Note, however, the misprint: the second equal sign should be a minus sign.

[12]S. Doniach, Phys. Rev. B **24**, 5063 (1981); see also M. P. A. Fisher, *ibid.* **36**, 1917 (1987) and references therein. There is still controversy in the experimental evidence with regard to the universal conductances discussed in this note: see S. J. Lee and J. B. Ketterson, Phys. Rev. Lett. **64**, 3078 (1990); D. B. Haviland, Y. Liu, and A. M. Goldman, *ibid.* **62**, 2180 (1989) and references therein.

[13]A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983).

[14]See D. S. Fisher and P. C. Hohenberg, Phys. Rev. B **37**, 4936 (1988), who clarify and extend earlier resuts of V. N. Popov, *Functional Integrals in Quantum Field Theory* (Reidel, Dordrecht, 1983), Chap. 6.

[15]T. Ando, Phys. Rev. B **40**, 5325 (1989) and references therein. The metallic conductivity here is believed to be infinite so rather than a universal jump, one should observe a universal critical conductivity, as in the superconducting case.

[16]H. P. Wei, D. C. Chui, M. A. Paalanen, and A. M. M. Pruisken, Phys. Rev. Lett. **61**, 1294 (1988).

[17]These results have been derived within a much less general framework by D. E. Khmel'nitskii and A. I. Larkin, Solid State Commun. **39**, 1069 (1981). We note that the result $|\delta_c(H)| \propto H^{1/2\nu}$ should be useful for superconductor-insulator transitions, too. Some data on magnetic-field dependence in the thin-film experiments are available: see H. M. Jaeger, D. B. Haviland, B. G. Orr, and A. M. Goldman, Phys. Rev. B **40**, 182 (1989).