Crossover from a fractal lattice to a Euclidean lattice for the thermodynamic properties of a triplet-interaction Ising model

Yung Qin
Department of Physics, Beijing Normal University, Beijing, 100875, China

Z. R. Yang
CCAST (World Laboratory) P. O. Box 8730, Beijing, 100080, China
and Department of Physics, Beijing Normal University, Beijing, 100875, China

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We study how thermodynamic properties of the triplet-interaction Ising model on a family of Sierpinski-type gasket fractals cross over to that of the Ising model on a triangular lattice with three-spin interaction in half of the triangular faces. By using a spin-variable transformation, the exact free energy f_t for the triangular model and f_s for the fractals are obtained in closed form and found to be analytic in temperature. The free energy f_s varies smoothly with the parameter b (which labels each member of the fractal family), and for large b the difference between f_t and f_s is asymptotically directly proportional to $1/b$. For fractal lattices with $b=2^m$ ($m=1,2,3,\ldots$), the crossover behavior of the critical exponents is also discussed by using a renormalization-group approach. In the meantime we find that the correlation-length exponent $\nu=\ln 2/\ln 3$, which is independent of the parameter b and hence the fractal dimension d_f and is different from $\nu=\infty$ for the two-spin-interaction Ising model on the fractal lattice given by Gefen *et al.* It shows that the universality hypothesis is violated here.

I. INTRODUCTION

Recently there has been increasing interest in exact mathematical fractals.[1] The principal reason is that there are many objects in nature that can be modeled by these fractal lattices. Much of the current interest in the systems on the fractals concentrates on the influence of the geometrical structure on the physical properties.[2–11] As yet, however, how physical laws on fractal substrates cross over to the laws on translationally invariant Euclidean lattices is an open question, and far less is known about it. One way to answer this question would be to construct a sequence of fractal models which increasingly more closely approximate a translationally invariant lattice and to see how the physical properties vary as the fractal model becomes closer and closer to the translationally invariant lattice. A suitable candidate for fractals to this end is the fractal family of Sierpinski gaskets proposed by Hilfer and Blumen.[12] Each member of the fractal family can be labeled by a geometrical parameter b, where b is an integer that runs from 2 to infinity, in such a fashion that $b=2$ is the Sierpinski gasket and $b=\infty$ is a wedge of the triangular lattice. Its fractal dimension d_f depends on b as follows:

$$d_f=\ln[b(b+1)/2]/\ln b \tag{1}$$

and crosses over from its value for a fractal lattice to its value of 2 for a triangular wedge with the form

$$d_f=2-\ln 2/\ln b\ . \tag{2}$$

Some studies of systems on Sierpinski-type fractal in this direction have been carried out. Among these, the following are very significant: The difference between the exact value of spectral dimension d_s for the fractal lattice and the exact value $d_2=2$ for a triangular lattice is asymptotically a logarithmic function of parameter b,[13] and the ground-state entropy[14] σ of the Ising antiferromagnet on the fractal family varies smoothly with b and approaches for large b the exact value $\sigma_{\rm Baxter}$ calculated by Baxter and Tsang[15,16] for the hard-hexagon problem on the triangular lattice.

In this paper we study a triplet interaction Ising model situated on Sierpinski-type fractal lattices.[12] The definition of the model will be seen in Sec. II. In order to discuss the crossover behavior for thermodynamic properties of the model from fractal lattice to regular lattice, we also consider a triangular lattice Ising model which has a triplet interaction in every up-pointing triangle. Using a spin variable transformation the exact free energy $f_t(T)$ for the triangular Ising model with triplet interaction in alternate triangular faces (see Fig. 2) and $f_s(T,b)$ for the triplet interaction Ising model on the fractal lattices are obtained in a closed form and found to be analytic in temperature. The systems, thus, exhibit no phase transition in any positive temperature. When b is sufficiently large and the fractal lattice is very close the translationally invariant one, we find that the difference between the free energies $f_t(T)$ and $f_s(T,b)$ is asymptotically directly proportional to $1/b$.

To show the critical behavior near $T_c=0$, the decimation RG transformation is employed, and results in the exact recursion relations for the system on Sierpinski-type gasket fractals with $b=2^m$ ($m=1,2,3,\ldots$). Through calculating the critical exponents, the crossover behavior of critical exponents is also discussed. We find

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in the meantime that the correlation length exponent, $\nu = \ln 2/\ln 3$, is independent of the parameter b, in contrast with $\nu = \infty$ for the two-spin interaction Ising model on the fractal lattices given by Gefen *et al.*[4] It reveals that the critical behavior depends on the form of interaction.

The outline of this paper is as follows. In Sec. II, the triplet interaction Ising model is defined. In Sec. III, we calculate the free energy for both triangular lattice and fractal lattices, and analyze the crossover behavior for the free energy. In Sec. IV, the critical exponents are given by using a RG method and the crossover behavior of the critical exponents is exhibited. Section V is a summary.

II. DEFINITION OF MODEL

As we know, each member of Sierpinski-type fracal lattices may be built iteratively from a generator $G(b)$, here $b = 2, 3, \ldots,$ and $G(b)$ is an equilateral triangle (Fig. 1) that contains b^2 an identical smaller triangle of unit side length, of which only the upward oriented are physically present. One can repeatedly use an operation of enlarging each unit side length in the structure by a factor b in linear dimension, then filling the upward-pointing triangles with the generator $G(b)$ and leaving the downward triangles empty (Fig. 1). Until an infinite number of times, one obtains the fractal lattice.

Consider a system of N spins $s_i = \pm 1$ located at the vertices of the fractal lattice (Fig. 1). The three spins surrounding each up-pointing triangular face interact with a three-spin interaction of strength J so that the Hamiltonian reads

$$-\beta H = K \sum_{\Delta} s_i s_j s_k \tag{3}$$

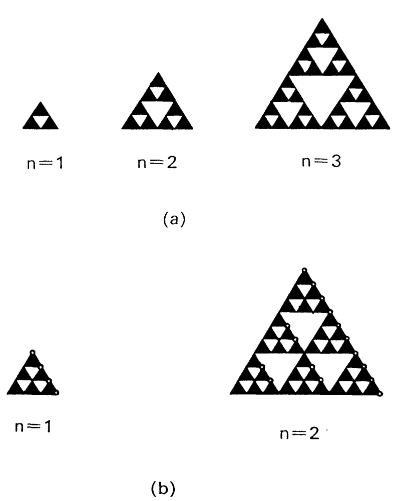

FIG. 1. Growth of the fracal lattice with (a) $b=2$ and (b) $b=3$. The first stage ($n=1$) is termed the generator. For $b>2$, there are two kinds of sites, some have four nearest neighbors and others have six.

with the summation extending over all up-pointing triangles of the lattice. Where $K = \beta J$ is the reduced interaction parameter and $\beta = 1/kT$. We wish to evaluate the partition function

$$Z_s(T,b) = \sum_{\{s\}} \prod_{\Delta} \exp(K s_i s_j s_k) , \tag{4}$$

where $\{s\}$ represents all site spins and the product is taken over all up-pointing triangles.

From expression (3) it is obvious that reversing all spins is equivalent to negating J, and hence K. Since such reversal leaves $Z_s(T,b)$ unchanged, we shall hereinafter take $J>0$ without loss of generality.

III. FREE ENERGIES

A. Two-dimensional triangular lattice

For obvious reasons, when $b \to \infty$ the generators and corresponding fractals are approaching the ordinary triangular lattice. Accordingly, it is necessary to consider an Ising model with the three-spin interaction $J>0$ among every three spins surrounding an up-pointing triangular face on a regular triangular lattice. The situation is shown in Fig. 2, where the shaded triangles indicate the Ising triplet interaction. The partition function can take the form as

$$Z_t(T) = \sum_{\{s\}} \prod_{\Delta} \exp(K s_{i,j} s_{i,j+1} s_{i+1,j}) , \tag{5}$$

where s_{ij} denotes an Ising spin on the lattice site (i,j). The sum is over all spins and the product is over all up-pointing triangles. It is fortunate for us that the partition function in (5) can be calculated explicitly by using a spin variable transformation. The calculation can be done as follows: Let the lattice be $L \times L$ (Fig. 2), where L is the number of horizontal bonds and the bonds which along the direction of a 60° angle to horizon. For convenience we call the vertical and horizontal lines columns and rows respectively. Then the lattice holds L^2 identical up-pointing triangles and $(L+1)^2$ lattice sites. We associate with each up-pointing triangular face a variable:

$$\sigma_{ij} = s_{i,j} s_{i,j+1} s_{i+1,j} , \tag{6}$$

where s_{ij}, $s_{i,j+1}$, and $s_{i+1,j}$ are the three-apex spin vari-

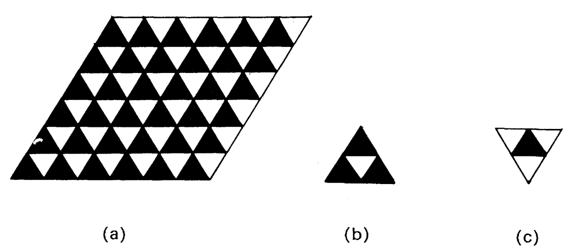

FIG. 2. (a) The Ising model on a $L \times L$ triangular lattice with three-spin interaction at each shaded triangle. (b) and (c) two simplest finite lattices present after the transformation is applied.

ables of an up-pointing triangular face. Using the variables $\{t_{i,L+1}\}$ and $\{t_{L+1,i}\}(i=1,2,\ldots,L+1)$ instead of $\{s_{i,L+1}\}$, spin variables on the $(L+1)$th row sites, and $\{s_{L+1,i}\}$, spin variables on the $(L+1)$th column sites, respectively. That is,

$$t_{i,L+1}=s_{i,L+1} \quad (i=1,2,\ldots,L+1) \quad t_{L+1,i}=s_{L+1,i} . \tag{7}$$

It can be found that the new variables $\{\sigma,t\}$ are all independent, that each of them takes only the values ± 1, and that the set $\{\sigma,t\}$ is a complete set of coordinates in the sense that to each configuration $\{s\}$ there corresponds a unique $\{\sigma,t\}$ and vice versa. We take a 3×3 lattice as an example to illustrate this situation in the Appendix. Thus the partition function $Z_t(T)$ may be described by the variable set $\{\sigma,t\}$ and the $\{\sigma,t\}$ summation can be performed to give

$$\begin{aligned} Z_t(T) &= \sum_{\{s\}} \prod_{\Delta} \exp(K s_{i,j} s_{i,j+1} s_{i+1,j}) \\ &= \sum_{\{t\}} \sum_{\{\sigma\}} \prod_{\Delta} \exp(K\sigma_{ij}) \\ &= 2^{2L+1}(e^{K}+e^{-K})^{L^2} . \end{aligned} \tag{8}$$

Here $2L+1$ is the number of the t variable. In the thermodynamic limit $L\to\infty$ the free energy per site is

$$\begin{aligned} f_t(T) &= -\lim_{L\to\infty} \frac{kT}{(L+1)^2} \ln Z_t(T) \\ &= -kT\ln(e^{K}+e^{-K}) . \end{aligned} \tag{9}$$

It is surprising that the free energy $f_t(T)$ in (9) was the same as that of the one-dimensional nearest-neighbor-interaction Ising model. Therefore $f_t(T)$ is analytic in temperature T and no long-range order exists in finite temperature. It will be interesting to compare the triangular Ising model with the triangular Potts model, both have three-spin interactions only in every up-pointing triangle. We know that the triangular Potts model which has anisotropic two-spin interactions K_1, K_2, K_3, and a three-spin interaction L in every up-pointing triangle has the exact critical condition:[17]

$$e^{L+K_1+K_2+K_3}-e^{K_1}-e^{K_2}-e^{K_3}+2=q , \tag{10}$$

which is valid for $K_i\geq 0$ and $L+K_1+K_2+K_3\geq 0$. When $K_1=K_2=K_3=0$, i.e., only the three-spin interaction L exists, the system exhibits a finite-temperature phase transition at $e^{L}=q+1$ even if $q=2$. However, for the Ising case our result shows that there is no phase transition at a finite temperature, and the physics underlying the property awaits further discussion.

B. Sierpinski-type gasket fractal lattices

We now turn to the model on the Sierpinski-type gasket fractal lattices defined in Sec. II. The spin-variable transformation used in Sec. III (A) can be straightforward to apply to this case. One introduces to each up-pointing triangular face a σ variable, and a t variable instead of the spin-variable s on each site indicated by an open circle (see Fig. 1). We find that for the structure of order n of the lattice the partition function in (4) is as follows:

$$\begin{aligned} Z_s^{(n)}(T,b) &= \sum_{\{s\}} \prod_{\Delta} \exp(K s_i s_j s_k) \\ &= \sum_{\{t\}} \sum_{\{\sigma\}} \prod_{\Delta} \exp(K\sigma_i) \\ &= 2^{M_t}(e^{K}+e^{-K})^{N_\sigma} , \end{aligned} \tag{11}$$

where M_t and N_σ are the number of t variables and σ variables, respectively, and

$$M_t=2+2[(b^{d_f})^n-1]/(b+2) , \tag{12}$$

$$N_\sigma=(b^{d_f})^n . \tag{13}$$

The free energy per site in the thermodynamic limit $n\to\infty$ is then

$$\begin{aligned} f_s(T,b) &= -kT \lim_{n\to\infty} \frac{1}{N_n} \ln Z_s^{(n)}(T,b) \\ &= -kT[2\ln 2+(b+2)\ln(e^{K}+e^{-K})]/(b+4) , \end{aligned} \tag{14}$$

where

$$N_n=3+(b+4)[(b^{d_f})^n-1]/(b+2) \tag{15}$$

denotes the number of lattice sites on the nth structure state of the fractal lattice with b.

The expression (14) states that the free energy $f_s(T,b)$ is simply associated with the parameter b, and is an analytic function of temperature T. To reveal the way that $f_s(T,b)$ converges to $f_t(T)$, we now investigate the behavior of $f_s(T,b)$ for very large b. From (14) we find that the asymptotic form of the free energy $f_s(T,b)$ is

$$f_s(T,b)=f_t(T)-\frac{B(T)}{(b+4)} \tag{16}$$

when b is sufficiently large, where $B(T)=-2kT\ln(\cosh K)$.

Consequently, we can come to the conclusion that all thermodynamic quantities, which are determined by the temperature derivative of the free energy, have the same asymptotic form as (16). We also note that (16) has the point of similarity with the asymptotic power law of the ground-state entropy $\sigma(b)$ for the antiferromagnet Ising model

$$\sigma(b)=\sigma_{\text{Baxter}}-\frac{P}{b^{\alpha}} \tag{17}$$

found by Stosic *et al.*,[14] where P and α are the fitting constants and σ_{Baxter} calculated by Baxter and Tsang[15,16] is the exact value of the ground-state entropy for the Ising antiferromagnet on the ordinary triangular lattice.

For the sake of comparison we now give the free energy $f_g(T,b)$ of the generators (Fig. 1, $n=1$)

$$f_g(T,b)=-kT[2\ln 2+b\ln(e^{K}+e^{-K})]/(b+2) . \tag{18}$$

It is evident that as b increases the difference between $f_s(T,b)$ in (14) and $f_g(T,b)$ in (18) decreases and both converge to $f_t(T)$ according to the same form (16). This is a consequence of that, both the fractals and the corresponding generators, in the limit $b \to \infty$, approach the triangular lattice.

IV. CRITICAL EXPONENTS

In this section we analyze the critical property of the Ising model placed on the lattices with $b = 2^m$ ($m = 1,2,3,\ldots$), which is an infinite subsequence of the sequence of the fractal lattices discussed above. In the absence of a magnetic field and given m, the RG equations are obtained by summing over the internal spins of all the triangles of linear size 2^m. The rescaling factor here is $b_r = 2^m$. The RG procedure is shown schematically in Fig. 3. The resulting recursion relation for K is

$$\tanh K' = (\tanh K)^{3^m} . \tag{19}$$

The only fixed points of Eq. (19) are $K^* = 0$ ($T = \infty$) and $K^* = \infty$ ($T = 0$). The point $K^* = 0$ is an infinite temperature stable fixed point and corresponds to the disordered phase in high temperature. $K^* = \infty$ is a zero-temperature unstable fixed point and denotes the critical point. Linearizing (19) near $T_c = 0$ yields the coefficient of the linear term

$$\lambda = 3^m . \tag{20}$$

The correlation length exponent is found to be

$$\nu = \ln b_r / \ln \lambda = \ln 2 / \ln 3 . \tag{21}$$

Equation (21) tells us that the value of ν is a constant independent of m, and hence fractal dimension d_f. It is noteworthy that our result ν differs from that of the two-spin interaction Ising model on the Sierpinski gasket. The latter has $\nu = \infty$ given by Gefen *et al.*[4] This distinction means that the form of interaction among spins influences the critical behavior of the spin system. It is well known that the two different exact solutions for the spin model on the two-dimension translationally invariant lattice had revealed that the notion of universality in critical phenomena is not so all embracing that all Ising models with short-range interactions and the same spatial dimension are in the same universal class. One of these is the nearest-neighbor spin-pair interaction Onsager solution,[18] which contains the correlation length exponent $\nu = 1$. The other is Baxter and Wu's[19] solution of the three-spin interaction Ising model, which exhibits a different exponent $\nu = \frac{2}{3}$. Our present result and the result of Gefen *et al.*[4] are worked out on the fracal lattice and also violate the universality hypothesis. It would seem to show that such violations have generality for regular or fractal lattice.

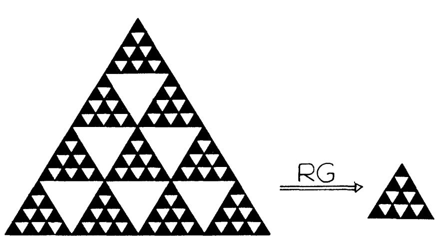

FIG. 3. The RG transformation from the second to the first construction stage for the $b = 4$ case.

Since the number of spins in a volume of the correlation length size ξ is ξ^{d_f}, we expect the singular free energy per spin to behave as

$$f_s \propto \xi^{-d_f} . \tag{22}$$

If we wrote this as $t^{2-\alpha(b)}$, where $t = e^{-2K}$ is chosen from (19) and $\alpha(b)$ is the specific heat exponent, we would conclude that

$$\alpha(b) = 2 - d_f \nu . \tag{23}$$

We now add a small magnetic field. As can be seen in Fig. 1 the first member of the family differs from all the others. Each site of the lattice with $m = 1$ ($b = 2$) has four nearest neighbors (except for the three apex site), whereas in the case of lattices with $m \geq 2$ ($b \geq 4$) some sites have four nearest neighbors and others have six. Since one iteration of the RG will generate different magnetic fields for the sites with different coordination numbers, we thus introduce two kinds of magnetic fields h_2 and h_3 for the $m \geq 2$ case, where h_2 and h_3 denote the magnetic fields at the sites with four and six nearest neighbors, respectively. To derive the recursion relations for h_i ($i = 2,3$), we take $\tanh K$, $\tanh K' \to 1$ (zero temperature) and find

$$h_2' = 3h_2 = b_r^{d_f} h_2 \quad (m = 1) \tag{24}$$

and

$$h_2' = h_2 + \frac{2}{3}\left[3(b-1)h_2 + h_3 \sum_{i=1}^{b-2} i\right] \quad (m > 1) , \tag{25}$$

$$h_3' = h_3 + 3(b-1)h_2 + h_3 \sum_{i=1}^{b-2} i .$$

From Eqs. (24) and (25) we always get one eigenvalue $b_r^{d_f}$, as expected by the general argument near a discontinuity fixed point.[20,21] Thus we expect the general scaling relation:

$$f_s(T,h) = \xi^{-d_f} f_s(\xi^{1/\nu} t, \xi^{d_f} h) , \tag{26}$$

and we find that $M \propto \xi^0$, $\chi \propto \xi^{d_f}$, etc. One can now identify the critical exponents, e.g.,

$$\beta = 0, \quad \gamma_{(b)} = \nu d_f, \quad \delta = \infty . \tag{27}$$

Next we evaluate the critical exponents for the two-dimension system described in Sec. III (A). An approximate RG scheme[22] used for the triplet interaction Ising model on triangular lattice by Imbro and Hemmer,[22] a finite-lattice method, is employed to produce the recursion relations. We use the smallest finite lattices shown in Figs. 2(b) and 2(c), corresponding to a rescaling factor

$b_r=2$. The yielding recursion relations for K and h are, respectively,

$$\tanh K' = (\tanh K)^3 , \quad (28)$$

$$h' = 4h = (b_r^2)h . \quad (29)$$

Note that the RG calculation corresponded to the Fig. 2(c) leads to a factor independent of spin variables and it has only a contribution to the regular part of the free energy. Equation (28) is obtained from Fig. 2(b).

Using Eqs. (28) and (29) we can immediately gain the critical exponents

$$\nu = \ln 2/\ln 3, \quad \beta = 0 , \quad (30)$$

and if all the scale relations were valid, we should have

$$\alpha = 2 - 2\nu, \quad \gamma = 2\nu . \quad (31)$$

From comparison of (23), (27), and (31) we will reach the following asymptotic forms of the exponents for b very large:

$$\alpha(b) = \alpha + Q/\ln b , \quad (32)$$

$$\gamma(b) = \gamma - Q/\ln b , \quad (33)$$

where $Q = \nu \ln 2$.

We may argue that, according to our results, some exponents, like correlation length and magnetic exponent, are independent of b and d_f, and have the same values for both fractal and regular lattices; some, such as, specific heat and susceptibility exponent, vary smoothly with parameter b and when b is a sufficiently large approach to that of the regular lattice with a logarithmic asymptotic law which is the same with that of fractal dimension d_f and spectral dimension d_s.[12,13]

V. SUMMARY

We have examined a triplet-interaction Ising model on the fractal-lattices family of the Sierpinski-type gasket and the model on the triangular lattice which was regarded as the case of limit $b \to \infty$ for the fractal family. The exact free energies are obtained in closed form and found to be analytic in temperature. Considered as a function of the parameter b, the free energy of fractals crosses over to that of a triangular lattice with a asymptotic power-law form as $b \to \infty$. The critical properties near $T_c = 0$ are studied. We find that the correlation-length exponent is a finite constant independent of the fractal dimension d_f and different from that of the two-spin-interaction Ising model on a Sierpinski gasket.

The crossover behavior from fractal to Euclidean lattices for the critical exponents which are relative to the parameter b is also discussed. A logarithmic asymptotic law of b for exponents is found. However, it should be noted that our analysis on the critical exponents is based on the scaling equations (22) and (26) of the free energy and the modified various scaling relations at the zero-temperature transition which had been argued to be valid for any self-similar lattice by Gefen *et al.*[3,4] our findings about the crossover behavior of the critical exponents, thus, needs to be verified in further work.

ACKNOWLEDGMENTS

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APPENDIX

To illustrate that $\{s\}$ configurations and $\{\sigma,t\}$ configurations are in a one-to-one correspondence, consider a 3×3 lattice shown in Fig. 4, in which there are nine up-pointing triangular faces and 16 lattice sites. The variables s_{ij} $(i,j=1,2,3,4)$ denote Ising spins placed on the lattice sites. Using (6) we introduce new variables σ_{ij} $(i,j=1,2,3)$ associated with the up-pointing triangular faces, and

$$t_{i4} = s_{i4} \quad (i=1,2,3,4) , \qquad t_{4j} = s_{4j} \quad (j=1,2,3,4) . \quad (A1)$$

If we give an $\{s\}$ configuration, for example,

$$(s_{11},s_{12},s_{13},s_{14},s_{21},s_{22},s_{23},s_{24},s_{31},s_{32},s_{33},s_{34},s_{41},s_{42},s_{43},s_{44}) = (-1,1,1,-1,1,-1,-1,1,1,-1,1,-1,-1,-1,1,-1) . \quad (A2)$$

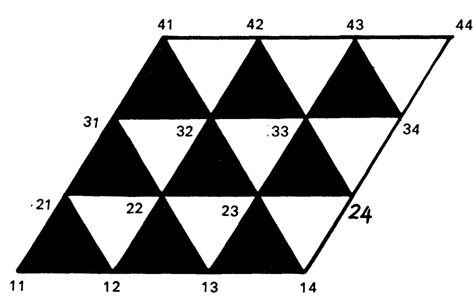

FIG. 4. A 3×3 triangular lattice, on which there are 9 up-pointing triangular faces (shaded faces) and 16 lattice sites.

From (A1), t variables are known as

$$(t_{14},t_{24},t_{34},t_{44},t_{41},t_{42},t_{43}) = (-1,1,-1,-1,-1,-1,1) . \quad (A3)$$

Using (6), (A2), and (A3), the values of σ variables can be readily determined as

$$(\sigma_{11},\sigma_{12},\sigma_{13},\sigma_{21},\sigma_{22},\sigma_{23},\sigma_{31},\sigma_{32},\sigma_{33}) = (-1,-1,1,-1,-1,-1,1,1,-1) . \quad (A4)$$

Otherwise, if $\{\sigma,t\}$ variables are given by (A3) and (A4), according to (A1), we have

$$(s_{14},s_{24},s_{34},s_{44},s_{41},s_{42},s_{43}) = (-1,1,-1,-1,-1,-1,1) \, . \quad \text{(A5)}$$

Other s variables can be calculated through using (6), (A4), and (A5). From (6)

$$\sigma_{33}=s_{33}s_{34}s_{43} \, . \quad \text{(A6)}$$

Substituting $\sigma_{33}=-1$, $s_{34}=-1$, and $s_{43}=1$ given by (A4) and (A5) to (A6), then $s_{33}=1$ can be obtained. Again because

$$\sigma_{32}=s_{32}s_{33}s_{42} \, , \quad \text{(A7)}$$

$$\sigma_{23}=s_{23}s_{24}s_{33} \, . \quad \text{(A8)}$$

Using known $\sigma_{32}=1$, $\sigma_{23}=-1$, $s_{33}=1$, $s_{24}=1$, and $s_{42}=-1$, one gets $s_{23}=-1$ and $s_{32}=-1$. Similarly, one can successively calculate remainder s variables, and have $s_{13}=1$, $s_{22}=-1$, $s_{31}=1$, $s_{12}=1$, $s_{21}=1$, and $s_{11}=-1$. It can be seen that from (A4) and (A5) one reaches (A2) uniquely. Therefore, we may say that the system described by the $\{s\}$ variable is completely equivalent to that described by the $\{\sigma,t\}$ variable.

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