Interaction of quasiparticles in the $\frac{1}{3}$ quantum Hall state and hierarchical estimates of the energy of daughter states

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The interaction of quasiparticles in the $\frac{1}{3}$ state is determined by exact calculations using systems with up to nine electrons. Within the hierarchical scheme proposed by Halperin, which is based on quasiparticles obeying fractional statistics, we use these results to estimate the energies and gaps of the polarized and unpolarized $\frac{2}{5}$ states and of the polarized and partially polarized $\frac{4}{11}$ states. Our results confirm the ability of the hierarchical scheme to provide accurate quantitative estimates for the fractional states issuing from the $\frac{1}{3}$ state.

INTRODUCTION

The fractional quantum Hall states at $\frac{1}{3}$ and $\frac{1}{5}$ are admitted to be well understood within the Laughlin theory as incompressible fluids with fractionally charged excitations.[1,2] Building on this idea, it has been proposed to describe other observed fractional states within a hierarchical scheme[3,4] (HS) in which higher-order fractional states result from the condensation of quasiparticles of the "parent" state into an incompressible fluid. Various studies have shown that the HS predicts reliably the quantum numbers of fractional states.[5–7] It is of great interest to find out if, beyond this, the HS is really fit to provide accurate quantitative predictions for energies and gaps, or if an alternative theory is needed (cf., e.g., Ref. 8).

At the first level of hierarchy, the energy of the "daughter" states of the $\nu_0=1/m$ state which appear at filling factor $\nu_1=1/(m-1/p_1)$ with m odd and p_1 even can be expressed as

$$\mathcal{E}(\nu_1)=(1+1/2mp_1)\mathcal{E}(\nu_0)+\tilde{\varepsilon}_1^{+}/p_1+\tilde{\mathcal{U}}_{p_1}/p_1 \ . \quad (1)$$

Here $\mathcal{E}(\nu)$ is the energy per electron, $\tilde{\varepsilon}_1^{+}$ is the proper energy of a single quasiparticle,[9] and $\tilde{\mathcal{U}}_{p_1}$ is the interaction energy per quasiparticle of the quasiparticle system condensed into an incompressible fluid state. Here and in the following, the subscripts 0 and 1 refer to the level of hierarchy. The energy $\mathcal{E}(\frac{1}{3})$ and the proper energy $\tilde{\varepsilon}_1^{+}$ of the quasiparticle of the $\frac{1}{3}$ state have been accurately determined.[5,7,10,11] So, the problem consists in the computation of the interaction energy $\tilde{\mathcal{U}}_{p_1}$ of the dense quasiparticle (anyon) system.

Halperin proposed a formulation of the HS (Ref. 4) based on quasiparticles with fractional statistics which allows us to estimate the interaction energy $\tilde{\mathcal{U}}_{p_1}$ of the quasiparticle system. Using this formulation, the energy of the $\frac{2}{5}$ state has been estimated[4,9,12] by treating quasiparticles as *point* charges leading to $\mathcal{E}(\frac{2}{5})=-0.424$.[13] Throughout this paper energy will be measured in units $e^2/\epsilon l_0$ where $l_0=(\hbar c/eB)^{1/2}$ is the magnetic length and ϵ is the dielectric constant. On an absolute scale, this estimate may seem to be in reasonable agreement with the extrapolation of diagonalizations for small systems[6] $\mathcal{E}(\frac{2}{5})=-0.433$. However, the relevant energy scale, as discussed in Ref. 4, is given by the difference of these results from the classical plasma value $\mathcal{E}_{\text{plasma}}(\frac{2}{5})\approx-0.4429$.[14] This difference comes from the confinement of electrons in the lowest Landau level. We now clearly see that the hierarchical result for the $\frac{2}{5}$ state overestimates these quantum effects by a factor of 2. A further worry is the fact that fractional states with well-defined gaps at all filling fractions $\nu=p/q$ at arbitrary large odd q are predicted, which is in conflict with experiment.

However, previous investigations[11,9] have shown that the charge density of the quasiparticle is not at all point-like, but actually ring shaped, suggesting an important internal structure which affects the interaction of quasiparticles at short distance.

In this paper, we use the fractional statistics representation of quasiparticles to derive estimates for ground-state energy and gaps of fractional states at first hierarchical level. An accurate treatment of the quasiparticle interaction leads for the first time to estimates which are consistent with extrapolations of diagonalization results and states at filling fractions not observed in experiment are found to be gapless.

SPECIFYING THE QUASIPARTICLE INTERACTION

The interaction between quasiparticles differs in two ways from the Coulomb repulsion of point charges: (i) Multipolar corrections originate from the finite size of quasiparticles. For large quasiparticle separation, these corrections can be calculated from the charge distribution of isolated quasiparticles. (ii) At short distance (i.e., comparable to the magnetic length l_0), complicated quantum-mechanical polarization effects occur. Fortunately, exact calculations using systems with only a few electrons allow us to investigate the system of two close quasiparticles. We have performed such calculations with up to nine electrons confined to the surface of a sphere.[5]

We similarly study the quasiparticle involving an electron with reversed spin, which becomes the lowest quasiparticle excitation of the $\frac{1}{3}$ state at low magnetic field.[15] Using this kind of excitation in the hierarchical construction leads to a spin-unpolarized $\frac{2}{5}$ state instead of a fully polarized $\frac{2}{5}$ state.

Following Halperin,[4] the quantum Hall state at filling factor $\nu=1/(m-1/p_1)$ consists of a $1/m$ state of N_0 electrons plus a system of $N_1=N_0/p_1$ quasiparticles in an incompressible fluid state described by a *pseudo-wave-function* depending on quasiparticle coordinates. Its square modulus defines the probability distribution of quasiparticles. In the disk geometry it is given by

$$\psi_{m,p_1}^{(N_1)}(z_1,\ldots,z_{N_1})=\left[\prod_{i<j}(z_i-z_j)^{p_1-1/m}\prod_{k=1}^{N_1}e^{-|z_k|^2/4m}\right] \tag{2}$$

where $z_k=x_k+iy_k$ is the complex coordinate of the kth quasiparticle measured in units of l_0 and where p_1 must be an even integer in order to lead to the correct fractional statistics.[4]

We assume that the interaction of quasiparticles can be approximated by a pair potential $\tilde{V}(|z_i-z_j|)$. This interaction $\tilde{V}(R)$ is defined such that the energy $\tilde{V}_L$ of two quasiparticles in a state with given relative angular momentum L described by *pseudo*-wave-function $\psi_{m,L}^{(2)}$

$$\tilde{V}_L=\langle\psi_{m,L}^{(2)}|\tilde{V}|\psi_{m,L}^{2}\rangle/\langle\psi_{m,L}^{(2)}|\psi_{m,L}^{(2)}\rangle \tag{3}$$

coincides with the interaction energy of the corresponding *microscopic* two quasiparticle state [see Eq. (8)]. Owing to the form (2) of the pseudo-wave-function, in the bulk limit $N_1\to\infty$, the two-point correlation function of quasiparticles $g(z_i,z_j)$ depends only on the relative separation $R=|z_i-z_j|$ and can be expanded (cf. Ref. 16) as

$$g(R)=\sum_{k=0}^{\infty}\frac{2(1+C_k)}{\Gamma(2k+1-1/m)}\left[\frac{R^2}{4m}\right]^{2k-1/m}e^{-R^2/4m}. \tag{4}$$

The coefficients C_k vanish rapidly for large k and are calculated by fitting Monte Carlo results[17] for $g(R)$. The interaction energy $\tilde{\mathcal{U}}_{p_1}$ of the quasiparticle system can then be directly expressed in terms of the matrix elements $\tilde{V}_L$ (cf. Refs. 3 and 5) by

$$\tilde{\mathcal{U}}_{p_1}=4\pi m n_1\sum_{k=0}^{\infty}\left[(1+C_k)\tilde{V}_{2k}-\frac{(k+\frac{3}{8})\Gamma(2k-\frac{1}{2})}{m^{5/2}(2k+1)!}\right], \tag{5}$$

where $n_1=[2\pi(mp_1-1)]^{-1}$ is the density of quasiparticles. The last term describes the Coulomb interaction of the quasiparticle system with its compensating background.

The matrix elements for small L are calculated directly from microscopic trial wave functions by extrapolating exact results for small systems of electrons on a sphere. In order to describe states containing a quasiparticle involving one electron with reversed spin, we use the microscopic trial wave function proposed initially in Ref. 9, which is given on the sphere by

$$\Phi_{+\downarrow}^{(N_0)}=\sum_i\frac{u_i^{N_0-2}}{\prod_{j\neq i}(u_iv_j-v_iu_j)}\Phi_m^{(N_0)}S_i^-|up\rangle, \tag{6}$$

where the indices i and j run over $1,\ldots,N_0$, $u_i=\cos(\theta_i/2)e^{i\phi_i/2}$ and $v_i=\sin(\theta_i/2)e^{-i\phi_i/2}$ are the spinor coordinates of the ith electron on the sphere,[3] $\Phi_m^{(N_0)}$ is the Laughlin-Jastrow wave function, $|up\rangle$ denotes the fully polarized spin state, and S_i^- is the lowering operator which sets the spin of the ith electron antiparallel to the field. To describe the system containing two quasiparticles with reversed spin and in a state with relative angular momentum L, we write

$$\Phi_{\downarrow\downarrow,L}^{(N_0)}=\sum_{i<n}\mathcal{Y}\left[\frac{(u_iv_n-v_iu_n)^L u_i^{N_0-2-L}u_n^{N_0-2-L}}{\prod_{j\neq i}(u_iv_j-v_iu_j)\prod_{j\neq n}(u_nv_j-v_nu_j)}\Phi_m^{(N_0)}\right]\times S_i^-S_n^-|up\rangle, \tag{7}$$

where i, j, and n run over $1,\ldots,N_0$ and where $\mathcal{Y}$ is the appropriate Young symmetrizer of symmetry type $[N_0-2,2]$,[18] and where L can only take even values for symmetry reasons. The wave functions $\Phi_{+\downarrow}^{(N_0)}$ and $\Phi_{\downarrow\downarrow,L}^{(N_0)}$ are eigenstates of the total spin $(\sum_i\mathbf{S}_i)^2$.[19]

Finally, to describe fully polarized systems containing, respectively, a single quasiparticle or two quasiparticles with relative angular momentum L, we use $\Phi_+^{(N_0)}=\mathcal{O}\Phi_{+\downarrow}^{(N_0)}$ or $\Phi_L^{(N_0)}=\mathcal{O}^2\Phi_{\downarrow\downarrow,L}^{(N_0)}$, where $\mathcal{O}=\sum_{i=1}^{N_0}S_i^+L_i^+$, and L_i^+ is the raising operator of the angular momentum of the ith electron. $\Phi_+^{(N_0)}$ is in fact identical to Halperin's trial wave function,[20] in which two electrons are paired. The wave function $\Phi_L^{(N_0)}$ describes an electron system with total angular momentum $l_{\rm tot}=N_0-L$. For $m=3$ and $L=2$, 4, and 6, $\Phi_L^{(N_0)}$ provides excellent estimates for the ground-state energy at corresponding quantum number $l_{\rm tot}$, which is obtained by diagonalization for systems with up to nine electrons. For the system with quantum number $l_{\rm tot}=N_0$, it turns out that the ground state is better described by a trial wave function describing a system with two quasiparticles plus a supplementary neutral excitation consisting of a quasihole-quasiparticle pair, whose creation costs less energy than bringing two quasiparticles together by imposing $L=0$.[17]

The proper energy $\tilde{\epsilon}_1^+$ of the single quasiparticle is defined as the bulk limit $N_0\to\infty$ of the difference in total Coulomb energy between states $\Phi_+^{(N_0)}$ (or $\Phi_{+\downarrow}^{(N_0)}$) and $\Phi_m^{(N_0)}$ (cf. Ref. 9). For the quasiparticle of the polarized system and of the one involving the spin reversal of an electron, we obtain, respectively, $\tilde{\epsilon}_1^+=0.076\,84\pm0.000\,10$ and $\tilde{\epsilon}_1^+=0.0501\pm0.0005$ (cf. also Ref. 15).

The interaction energy $\tilde{V}_L^{(\rm MWF)}$ of two quasiparticles in a state of relative angular momentum L is defined as

$$\tilde{V}_L^{(\rm MWF)}=\lim_{N_0\to\infty}\Delta E_L(N_0)-2\tilde{\epsilon}_1^+. \tag{8}$$

TABLE I. Interaction energy of two quasiparticles with relative angular momentum L. $\tilde{V}_L^{(\lambda)}$ denotes the expectation value (3) of the multipole expansion of order $r^{-\lambda}$ for $\tilde{V}$. $\tilde{V}_L^{(\mathrm{MWF})}$ are results obtained with microscopic wave functions. Numbers in parentheses denote the standard deviation in unit of the last digit.

		Polarized		Reversed spin	
L	$\tilde{V}_L^{(1)}$	$\tilde{V}_L^{(5)}$	$\tilde{V}_L^{(\mathrm{MWF})}$	$\tilde{V}_L^{(5)}$	$\tilde{V}_L^{(\mathrm{MWF})}$
0	0.1318		0.0939(11)		0.089(1)
2	0.0231	0.0186	0.0114(3)	0.0211	0.019(1)
4	0.0162	0.0146	0.0206(9)	0.0157	0.013(2)
6	0.0132	0.0123	0.0118(16)	0.0129	

Here $\Delta E_L(N_0)$ is the difference in total Coulomb energy between states $\Phi_L^{(N_0)}$ (or $\Phi_{\downarrow\downarrow,L}^{(N_0)}$) and $\Phi_m^{(N_0)}$. Problems associated with the extrapolation to the bulk limit $N_0 \to \infty$ are discussed in Refs. 5–7.

Exact calculations for the matrix elements $\tilde{V}_L$ with large L are not possible. However, their estimation only involves the interaction of distant quasiparticles, which is well approximated by the Coulomb repulsion of *rigid* quasiparticle charges. The charge density $\rho(r)$ of a single quasiparticle has been obtained from Monte Carlo results with wave functions $\Phi_{+\downarrow}^{(N_0)}$ and $\Phi_{+}^{(N_0)}$ (cf. also Refs. 11 and 9). However, this charge density $\rho(r)$ contains the effect of the cyclotron motion of the quasiparticle which has to be removed for the calculation of $\tilde{V}$. It results from the convolution of the "intrinsic" charge density $\rho^{(e)}(r)$ with the form factor of cyclotron motion $e^{-r^2/2m}$. The large-distance behavior of $\tilde{V}(R)$ is then calculated from the multipole moments of $\rho^{(e)}(r)$, which can be, in turn, related to the multipole moments of $\rho(r)$.

Expectation values $\tilde{V}_L^{(\lambda)}$ of Eq. (3) are listed in Table I for small L. Here, λ denotes the order at which the multipole expansion of $\tilde{V}(R)$ is truncated. In the spin-polarized system, the polarization effects of quasiparticles lead to an important reduction of $\tilde{V}_2$ (50% of the Coulomb value) and to a significant increase of $\tilde{V}_4$. For larger L values, pseudopotential coefficients are largely influenced by the important quadrupole correction originating from the ring-shaped density of the quasiparticle. The agreement of the estimates $\tilde{V}_6^{(\mathrm{MWF})}$ and $\tilde{V}_6^{(5)}$ confirms the validity of the multipole expansion of $\tilde{V}$ in the evaluation of matrix elements with $L \geq 6$. For quasiparticles involving electrons with reversed spin, a reasonable agreement is found already for $L=2$.

RESULTS

We first discuss our results for the polarized and unpolarized $\frac{2}{5}$ states. In Table II are listed the hierarchical estimates of the ground state and of the gap assuming pointlike quasiparticles (first row) and using a more accurate representation of the interaction at short distance (second row). These last estimates are in good agreement with the extrapolation of results of diagonalization, which are given in the third row. For the polarized state, taking properly into account the short-range interaction of quasiparticles improves the agreement for the ground-state energy by one order of magnitude.

TABLE II. Ground-state energy per electron and gap of the polarized ($\uparrow\uparrow$) and unpolarized ($\uparrow\downarrow$) $\frac{2}{5}$ states. In the first row are listed the hierarchical estimates assuming pointlike quasiparticles, which are obtained using matrix elements $\tilde{V}_L^{(1)}$. $\tilde{V}_L^{(\mathrm{MWF})}$ denotes the hierarchical estimates using $\tilde{V}_L^{(\mathrm{MWF})}$ for $L \leq 4$ and $\tilde{V}_L^{(5)}$ otherwise. At the bottom, we list the extrapolations of exact diagonalizations. The relevant quantity in the estimates for the ground-state energy is given by the difference from the classical plasma value $\mathcal{E}_{\mathrm{plasma}} = -0.4429$.

	$\frac{2}{5}\uparrow\uparrow$		$\frac{2}{5}\uparrow\downarrow$	
	$\mathcal{E}_{\mathrm{g.s.}}$	$\varepsilon_{\mathrm{gap}}$	$\mathcal{E}_{\mathrm{g.s.}}$	$\varepsilon_{\mathrm{gap}}$
$\tilde{V}_L^{(1)}$	−0.4220(2)	0.082(3)	−0.4355(3)	0.082(3)
$\tilde{V}_L^{(\mathrm{MWF})}$	−0.4313(8)	0.073(4)	−0.4411(15)	0.059(4)
Exact	−0.4335(9)	0.061(15)	−0.4393(6)	

The values for the gap in Table II are inferred from a neutral excitation consisting of a free pair of particlelike and holelike excitations[22] of the quasiparticle system,[23] which leaves the total spin unchanged.

Our results for the energies of the polarized and unpolarized $\frac{2}{5}$ states can also be used to estimate the value B_c of the magnetic field at which, due to the Zeeman term, the polarized state becomes the ground state. In the situation of the experiment of Ref. 24, for an electron density $n_e = 2.3 \times 10^{11}$ cm^{-2} and using the values $\epsilon = 12.8$ and $g = 0.4$, we obtain $B_c = 9.1 \pm 1.6$ T, whereas extrapolations of diagonalizations lead to a smaller value $B_c = 5.4 \pm 1$ T.[25] The experimental value is $B_c = 7$ T.

We also investigate the state at filling factor $\nu = \frac{4}{11}$, which results for the choice $p_1 = 4$. While injecting quasiparticles with up spin leads to the polarized state, adding quasiparticles involving electrons with reversed spin leads to a partially polarized state with total spin $S = N_0/4$. Our estimates for the energies of these states are consistent with results of diagonalizations (see Table III). Our calculations seem to rule out the existence of a gap in the polarized $\frac{4}{11}$ state, consistent with results of diagonalizations for four and eight electrons where the ground states occur at $l_{\mathrm{tot}} = 2$ and a large number of excited states are nearby.[17,26] On the other hand, the partially polarized $\frac{4}{11}$ state may have a small finite gap (see Table III).

TABLE III. Ground-state energy per electron and gap of the polarized ($\uparrow\uparrow$) and partially polarized ($\updownarrow$) $\frac{4}{11}$ states given as in Table II. The classical plasma value is $\mathcal{E}_{\mathrm{plasma}} = -0.4256$. The $1/N_0$ linear extrapolation of exact results for $N_0 = 4$ and 8 for the polarized state as well as the exact result for $N_0 = 6$ for the partially polarized state are given for the sake of illustration.

	$\frac{4}{11}\uparrow\uparrow$		$\frac{4}{11}\updownarrow$	
	$\mathcal{E}_{\mathrm{g.s.}}$	$\varepsilon_{\mathrm{gap}}$	$\mathcal{E}_{\mathrm{g.s.}}$	$\varepsilon_{\mathrm{gap}}$
$\tilde{V}_L^{(1)}$	−0.4139(4)	0.0041(2)	−0.4207(4)	0.0041(2)
$\tilde{V}_L^{(\mathrm{MWF})}$	−0.4134(3)	~0	−0.4211(4)	0.003(2)
Exact	−0.4158	~0	−0.4217	

To summarize our results, in the spin-polarized case, the interaction of quasiparticles in the $\frac{1}{3}$ state differs strongly from the Coulomb interaction at short separation. It is characterized by a particularly small pair-state energy $\tilde{V}_L$ for relative angular momentum $L=2$ while those for $L=0$ and 4 are large. As a result, the stability of the quasiparticle Laughlin state (2) with $p_1=2$ is enhanced, while those with $p_1 \geq 4$ are destabilized. This mechanism[3] explains why at the same time the gap of the spin polarized $\frac{2}{5}$ is large while the polarized $\frac{4}{11}$ state is likely to be eliminated. On the other hand, the interaction of quasiparticles involving electrons with reversed spin is not very different from the repulsion of point charges and allows the existence of both unpolarized $\frac{2}{5}$ and partially polarized $\frac{4}{11}$ states.

In view of our results, the experimental observation of particularly large gaps for the polarized systems at $\nu=\frac{2}{5},\frac{3}{7},\frac{4}{9},\ldots,(s+1)/(2s+3)$, etc. appears to have a simple explanation within the hierarchical picture. Indeed, this will result if the interaction between quasiparticles at level $s+1$ in the state with $p_s=2$ is again characterized by strong $\tilde{V}_0^{(s+1)}$ and weak $\tilde{V}_2^{(s+1)}$ pseudopotentials.[30] The ring shape of quasiparticles may be an indication for this to happen. Such a ring shape has been observed in numerical calculations[7,22] also at $\nu=\frac{2}{5}$ and $\frac{3}{7}$ and is probably a general feature for the sequence $(s+1)/(2s+3)$. This point can be studied by generalizing our trial wave functions of one- and two-quasiparticle states to higher hierarchical levels. While analytical evaluation of these becomes quite involved due to the fractional statistics term, Monte Carlo calculation poses no problems.

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