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Motion of kinks in the ac-driven damped Frenkel-Kontorova chain

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It is demonstrated that a spatially homogeneous ac drive applied to a chain of interacting particles in a periodic substrate potential may support progressive motion of a dislocation, or of a periodic array of them, in the presence of friction (which may be either weak or strong). The progressive motion of the periodic array of dislocations implies motion of the whole chain with some mean velocity. Although these effects usually disappear in the continuum limit, we also give an example of "exotic" continuum damped models where the ac-driven propagation of a kink is possible. Physical implementations of the effects predicted here are briefly discussed.

In this paper we will analyze propagation of a dislocation (a kink) in an infinite chain of interacting particles placed in a potential when friction and an ac driving force are present. In a general case, the equations of motion for the displacements of the particles from their equilibrium positions x_n are

$$\ddot{x}_n+\alpha\dot{x}_n=-\partial U(x_n)/\partial x_n-\partial[V(x_{n+1}-x_n)+V(x_n-x_{n-1})]/\partial x_n+F\cos(\omega t)\,. \quad (1)$$

Here α is the friction coefficient (the common mass of the particles is set equal to one), $U(x)$ is the substrate periodic potential of period 2π (this value can always be achieved by an obvious renormalization), $V(x_n-x_{n-1})$ is the potential of the interparticle interaction, assumed to be nearest neighbor for simplicity, and F and ω are the drive's amplitude and frequency, respectively. In the simplest case, corresponding to the standard Frenkel-Kontorova (FK) model,[1,2] the substrate potential is harmonic and the interparticle interaction is linear:

$$\ddot{x}_n+\alpha\dot{x}_n=-\sin x_n+a^{-2}(x_{n+1}+x_{n-1}-2x_n)+F\cos(\omega t)\,. \quad (2)$$

Here a^{-2} is the stiffness of the linear interparticle interaction. The presence of a dislocation in the chain implies that all the wells of the substrate potential are filled except for one, or, alternatively, that one extra particle is added to the completely filled configuration.

It is well known that the motion of a dislocation can be supported by a dc drive applied to the chain: The dislocation may propagate along the chain when the dc drive does not exceed the threshold value for chain depinning. We aim to demonstrate that the progressive motion of the dislocation, or of a periodic array of dislocations, can be supported by the *ac* drive. Recently the same effect has been demonstrated by one of the present authors for the ac-driven underdamped ($\alpha \ll 1$) Toda lattice (TL).[3] The main difference between the FK and TL models is that in the latter one there is no substrate potential, but the nearest-neighbor interaction is strongly nonlinear. This gives rise to qualitative differences in the manifestations of the above-mentioned effect in the two models: (i) In the FK model, not only weak, but also strong damping can be treated analytically; (ii) in the FK model the ac drive may support propagation of a periodic array of the dislocations, which means, as a matter of fact, that the chain as a whole may move with some mean velocity.

The idea that underlies the analysis of the ac-driven propagation of the dislocation in the damped FK chain is the same as in Ref. 3, namely to consider the equation for the energy balance. Let the propagating dislocation have the form

$$x_n(t)=\xi(2\pi n-Vt)\,, \quad (3)$$

where V is its velocity. To investigate the possibility of compensating the dissipative losses by the work of the ac driving force, we insert Eq. (3) into the following equation:

$$\int_{-\infty}^{\infty}\alpha\dot{x}_n^2dt=\int_{-\infty}^{\infty}F\cos(\omega t)\dot{x}_n dt\,. \quad (4)$$

Equation (4) is exactly the energy balance for each particle in the chain, i.e., the condition that the dissipative losses are compensated by the energy input from the ac drive at each site. The crucial point is to guarantee that, after the insertion of Eq. (3), Eq. (4) is identically satisfied for all n. Evidently, this is only possible if the velocity V takes on one of the resonant values found in Ref. 3 for the TL model (with a slightly different notation):

$$V=V_N\equiv\omega/N,\ N=\pm1,\pm2,\dots\,. \quad (5)$$

The ac-driven propagation of the dislocation is possible if the drive's amplitude F exceeds a threshold value F_N,[3] i.e., a minimum value of F for which Eq. (4) is satisfied with $V=V_N$. Just as in the case of the TL model,[3] at $F=F_N$ there appear two different solutions with the same value V_N. These solutions correspond to different values of a constant phase shift between the ac drive and the periodic passage of particles in the chain through the moving dislocation. This is a typical saddle-node (tangent) bifurcation, so that we expect that one of the solutions is stable.[3]

To give an example where this general scheme is realized, let us consider in some detail the FK model (2) in the quasicontinuum approximation, in which the dislocation size ($\sim a^{-1}$) is assumed to be much larger than the chain spacing 2π. In this approximation, the unperturbed ($\alpha = F = 0$) dislocation is described by the well-known kink solution of the sine-Gordon equation:

$$x_n(t) = 4\tan^{-1}\{\exp[\sigma(an - \tilde{V}t)(1-\tilde{V}^2)^{-1/2}]\}\,, \qquad \tilde{V} \equiv aV/(2\pi)\,, \tag{6}$$

$\sigma = \pm 1$ being the kink's polarity. Straightforward calculations yield the following expression for the threshold amplitude:

$$F_n = 4\alpha\pi^{-1}\tilde{V}_N(1-\tilde{V}_N^2)^{-1/2}\cosh[\tfrac{1}{2}\pi\omega(1-\tilde{V}_N^2)^{1/2}/\tilde{V}_n]\,, \tag{7}$$

where $\tilde{V}_N \equiv a\omega/(2\pi N)$ [see Eq. (5)]. According to Eq. (7), the threshold amplitude is lowest for the lowest resonance ($|N| = 1$), and it monotonically increases with $|N|$. It also follows from Eqs. (5) and (7) that F_N becomes infinitely large as $a \to 0$ with α, ω, and N fixed. This implies that ac-driven propagation is not possible in the continuum limit.[3]

Consider now the ac-driven propagation of a periodic array of dislocations. This generalization is important as it makes possible to consider the propagation of one dislocation in a finite ring-like chain (the one with periodic boundary conditions).

We assume that, in the general case, the array has the form [cf. Eq. (3)]

$$x_n(t) = -(2\pi n - Vt)p + \xi[(2\pi n - Vt)p]\,, \tag{8}$$

where p is the density of the array and the function ξ is assumed to be 2π periodic. For the periodic array, we may repeat the previous analysis of the single dislocation with one modification: instead of integrating the energy equation (4) from $-\infty$ to $+\infty$, we integrate it over the drive's period $2\pi/\omega$. It is easy to see that the compensation of the dissipative losses by the ac drive is again possible if the velocity takes on the same resonant values (5), the array density takes the values

$$p = p_N \equiv N \tag{9}$$

[here N is the same as in Eq. (5)], and the drive amplitude exceeds some threshold value $F_N \sim \alpha$.

From Eqs. (5), (8), and (9), we may extract the important inference that the chain as a whole moves at a mean velocity equal to the drive's frequency:

$$\bar{v} \equiv pV = \omega\,. \tag{10}$$

It is interesting to note that, while the velocity of the dislocations depends on N [Eq. (5)], the mean velocity of the particles does not, according to Eq. (10).

Thus the ac drive may give rise to the mean transfer of mass in the FK model, which is not possible in models of the TL type.[3]

This general pattern can again be applied to the particular model (2) in the quasicontinuum approximation. The periodic array of kinks is described by the known solution of the sine-Gordon equation [cf. Eq. (6)],

$$x_n(t) = \pi - 2\,\mathrm{am}[(an - \tilde{V}t)(1-\tilde{V}^2)^{-1/2}/k]\,, \tag{11}$$

where am is the Jacobi elliptic amplitude. This solution contains one arbitrary parameter—the modulus k ($0 < k < 1$). For the wave form (11), the density of the array of dislocations is

$$p = a/[2k(1-\tilde{V}^2)^{1/2}K(k)]\,, \tag{12}$$

where $K(k)$ is the complete elliptic integral of the first kind. According to what was said above, we must set $\tilde{V} = \tilde{V}_N \equiv a\omega/(2\pi N)$ [cf. Eq. (7)] in Eqs. (11) and (12). Finally, the energy balance equation yields the following expression for the threshold amplitude of the driving force

$$F_N = 4\alpha\omega\pi^{-2}K(k)E(k)\cosh(\pi K'/K)\,, \tag{13}$$

where $K' \equiv K[(1-k^2)^{1/2}]$, and $E(k)$ is the complete elliptic integral of the second kind. The value of k is no longer arbitrary; instead it is determined by the transcendental equation

$$kK(k) = (a/2N)(1-\tilde{V}_N^2)^{-1/2}\,, \tag{14}$$

which follows from Eqs. (9) and (12). For fixed N, Eq. (14) has at most one solution. Note that, according to Eq. (14), $k \to 0$ in the continuum limit (corresponding to $a \to 0$), so that the threshold amplitude given by Eq. (13) diverges in this limit, as it also did in the case of one dislocation.

Along with the model (1), we can consider that described by the equation

$$\ddot{x}_n + \alpha\dot{x}_n = -\partial U(x_n)/\partial x_n - \partial[V(x_{n+1}-x_n) + V(x_n - x_{n-1})]/\partial x_n + (-1)^n F\cos(\omega t)\,. \tag{15}$$

This modification[3] corresponds, e.g., to the chain of alternating positively and negatively charged particles in an external ac electric field. The only substantial difference between the models (15) and (1) is that for (15) the spectrum of the resonant velocities has the form[3] [cf. Eq. (5)]

$$V = V_N \equiv \omega/(N+\tfrac{1}{2}),\, N = 0, \pm 1, \pm 2, \ldots\,. \tag{16a}$$

Accordingly, the expression (9) for the density of the ac-driven array of dislocations changes to

$$p_N \equiv (N + \tfrac{1}{2})\,. \tag{16b}$$

Finally, the chain moves as a whole with a mean velocity still given by Eq. (10), as a glance at Eqs. (16) shows.

The important feature of the results obtained in this work is that, unlike what happened in the TL model,[3] they are applicable to both the underdamped ($\alpha \ll 1$) and over-damped ($\alpha \sim 1$ or $\alpha \gg 1$) cases. Of course in the latter case, the "relativistic" radical $(1-\tilde{V}_N^2)^{1/2}$ should be dropped in all the formulas.

It is relevant to mention that in the underdamped case the dominating role will be played by the radiative losses[4] (emission of quasilinear waves by the moving dislocation), not by the direct dissipative ones, provided the underdamped model is far from being exactly integrable. The ac drive may be equally efficient in supporting the progressive motion of the dislocations in this situation. The kinematic relations (5), (9), and (10) remain valid while the calculation of the threshold amplitude is much more difficult in the case of radiation-dominated damping.

Let us emphasize again that, as in the case of the underdamped TL system, the ac-driven propagation of kinks is possible only in discrete models as those considered here, and that this effect disappears in the continuum limit. Nevertheless, one can invent continuum models where this effect is also possible, but they seem artificial from the standpoint of physical applications. As an example, let us consider the perturbed sine-Gordon model [cf. Eq. (2)],

$$u_{tt}-u_{xx}+\sin u=-\alpha u_t+Fg'(u)\cos(\omega t)\,,$$

where $g(u)$ is some function to be specified later. To analyze kink motion in this model, we can employ the momentum balance approach[5] in the form

$$dP/dt \equiv d/dt\int_{-\infty}^{\infty}u_xu_t dx=-\alpha\int_{-\infty}^{\infty}u_xu_t dx + F\cos(\omega t)\int_{-\infty}^{\infty}g'(u)u_x dx\,. \tag{17}$$

The first term of the right-hand side of Eq. (17) is equal to

$$(dP/dt)_- = -8\alpha V(1-V^2)^{-1/2}\,, \tag{18}$$

whereas the second one adopts the form

$$(dP/dt)_+ = F\cos(\omega t)\{g[u(+\infty)]-g[u(-\infty)]\}\,. \tag{19}$$

When the kink is present, the wave fields at $x=-\infty$ and at $x=+\infty$ are respectively

$$u(-\infty)=Fg'(0)[(1-\omega^2)^2+\alpha^2\omega^2]^{-1}\times[(1-\omega^2)\cos(\omega t)+\alpha\omega\sin(\omega t)]\,, \tag{20}$$

$$u(+\infty)=2\pi+Fg'(2\pi)[(1-\omega^2)^2+\alpha^2\omega^2]^{-1}\times[(1-\omega^2)\cos(\omega t)+\alpha\omega\sin(\omega t)] \tag{21}$$

[we have set $\sigma=+1$ in Eq. (6) for the sake of definiteness]. Inserting Eqs. (20) and (21) into (19), we find that the mean input rate of momentum from the ac drive is, to leading order in powers of F,

$$\langle (dP/dt)_+\rangle = \tfrac{1}{2}F^2(1-\omega^2)[(1-\omega^2)^2+\alpha^2\omega^2]^{-1}\times\{[g'(2\pi)]^2-[g'(0)]^2\}\,. \tag{22}$$

Thus it is possible to compensate the momentum lost by dissipation, Eq. (18), by the input from the ac drive if the quantity between curly brackets in the right-hand side of (22) is not zero. The simplest perturbation satisfying this condition is

$$g'(u)=\sin(u/3)\,,$$

which is not of paramount physical interest. Nevertheless, if the right-hand side of (22) is nonzero, one can find the equilibrium velocity $V_0 \sim F^2/\alpha$ for the ac-driven propagation of a kink in the infinite damped continuum system. In contrast with the resonant velocities of discrete systems [Eqs. (5) and (16)], this equilibrium velocity may take only one value, V_0 (the value $-V_0$ corresponds to the negative polarity of the kink).

Finally, let us discuss briefly feasible experimental realizations of the effect predicted. A natural choice is a chain of ions trapped by a metallic surface[2] with an external ac electric field playing the role of the drive.

A second possibility is a chain of fluxons in a long Josephson junction with a regular lattice of narrow inhomogeneities (the so-called microresistors) that provides the effective substrate potential.[6] In this system, the ac drive may be the uniformly distributed low-frequency ac-bias current. The progressive motion of the dislocation (or of the periodic array of dislocations) in the chain of fluxons will be manifest as the so-called reverse Josephson effect: The ac-bias current gives rise to a mean dc voltage across the Josephson junction.[3]

Last, an interesting object of study is a charge-density-wave system with a regular ionic superlattice.[7] In this case, the system may be driven by the external ac voltage. Here an effect akin to the reverse Josephson effect may be expected: the ac voltage may support a mean dc current in the system.[3]

In conclusion, it is worth mentioning that much more information about the dynamics of both the FK and TL ac-driven damped models can be elicited from numerical simulations. Work in this direction is under way now.

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[1] Ya. I. Frenkel and T. Kontorova, Zh. Eksp. Teor. Fiz. **8**, 1340 (1938).
[2] F. C. Frank and J. H. van der Merwe, Proc. R. Soc. London Ser. A **189**, 205 (1949).
[3] B. A. Malomed (unpublished); in Proceedings of the Workshop Macroscopic Aspects of Nonlinearity in Condensed Matter Physics, edited by A. R. Bishop, V. L. Pokrovsky, and V. Tognetti (unpublished).
[4] M. Peyrard and M. D. Kruskal, Physica D **14**, 88 (1984).
[5] P. L. Christiansen and O. H. Olsen, Wave Motion **4**, 163 (1982); D. J. Bergman, E. Ben-Jacob, Y. Imry, and K. Maki, Phys. Rev. A **27**, 3345 (1983); O. A. Levring, M. R. Samuelsen, and O. H. Olsen, Physica D **11**, 345 (1984).
[6] A. V. Ustinov, Phys. Lett. A **136**, 155 (1989); Boris A. Malomed and Alexey V. Ustinov, J. Appl. Phys. **67**, 3791 (1990); Phys. Rev. B **41**, 254 (1990); Boris A. Malomed, *ibid.* **41**, 2616 (1990).
[7] G. Grüner and A. Zettl, Phys. Rep. **119**, 117 (1985).