Quantum Hall effect in quasi-one-dimensional conductors

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The integer topological invariant called the Chern number is calculated for a quasi-one-dimensional conductor in the magnetic-field-induced spin-density-wave state. Due to the nonzero value of the Chern number the Hall conductivity per layer has the quantized value $\sigma_{xy} = 2Le^2/h$ and in the effective action of the system there is a so-called Hopf term, which describes topologically nontrivial configurations of the spin-density-wave polarization vector. The dependence of the integer number L on magnetic field H is calculated in the parquette approximation. The theory is applied to the Bechgaard-salt family of organic conductors $(TMTSF)_2X$, where TMTSF is tetramethyltetraselenafulvalene.

I. INTRODUCTION

Hall plateaus of good quality with the ratios 1:2:3:4:5 are observed[1,2] in the quasi-one-dimensional conductor $(TMTSF)_2PF_6$ where TMTSF is tetramethyltetraselenafulvalene.[3] The ratio of the components of resistivity tensor ρ_{xy}/ρ_{xx} is large. It equals 75 – 22, depending on the number of the plateau, and strongly decreases in the narrow regions between plateaus.[1] Such behavior clearly shows that it is really a quantum Hall effect. The Hall plateaus are observed also in $(TMTSF)_2ClO_4$.[4,5]. In more detail the experiments will be discussed in Sec. V of the paper in comparison with the theory. Comprehensive reviews of the subject, both theoretical and experimental, can be found in Ref. 6.

$(TMTSF)_2X$ crystals consist of conducting chains. The interchain overlap of the electron wave functions is considerable in one direction. In another direction it is very small, so it will be neglected in the paper. Thus the compounds may be thought of as two-dimensional materials with strong quasi-one-dimensional anisotropy. The magnetic field is applied perpendicular to the planes of the chains.

The peculiar features of the $(TMTSF)_2X$ compounds are the appearance of magnetic-field-induced spin-density waves (FISDW's) and the cascade of phase transitions between them. As was shown in Refs. 7-11, the wave vector of a FISDW deviates from the value $2k_F$ (k_F is the Fermi momentum) by an integer number $-L$ multiplied by the magnetic wave vector

$$q = ebH/c, \tag{1}$$

where b is the distance between the chains, H is the magnetic field, e is the electron charge, c is the velocity of light, and the Planck constant divided by 2π, $\hbar$, is set to unity. It was shown in Ref. 12 that in the FISDW state there is the integer quantum Hall effect with the value $\sigma_{xy} = 2Le^2/h$ per plane. Unlike in semiconductors, the quantum Hall effect in the $(TMTSF)_2X$ compounds appears only as the result of the phase transition into the FISDW phase, which is the state of the Hall dielectric. The transitions between the plateaus, related to the change of L as a function of H, are also phase transitions.

The Streda formula was utilized in Ref. 12 to derive σ_{xy}.[13] In the present paper, following Ref. 14, the same result is obtained by calculating for the FISDW state a so-called Chern number which is the topological invariant of electron wave functions. The advantage of the method is that it manifests explicitly topological invariance of the Hall conductivity. It guarantees that σ_{xy} does not depend on any perturbation of the model parameters and is integer quantized. In this way it is easy to calculate σ_{xy} in the case of coexistence of several FISDW's with different values of L. The coexistence may have a relation to the explanation of the puzzling self-similar treelike phase diagram of $(TMTSF)_2ClO_4$.[15]

The conductivity tensor for FISDW's was calculated in Ref. 16 (see also Ref. 17). Unfortunately, the quantized contribution to σ_{xy} was completely lost in this paper. On the other hand, this theory describes very well the Hall effect in the FISDW phase with $L = 0$, which also exists in $(TMTSF)_2PF_6$.[1,2].

In Ref. 18 the Hall conductivity was discussed using a quasiclassical consideration of electron orbits in the presence of impurities. It was claimed there that there are plateaus but the value of σ_{xy} at the plateau is not integer quantized and depends on the impurities. In the present paper the effect of the impurities is not considered.

The paper is organized in the following way. In Sec. II the Chern number is calculated for the FISDW state.

To simplify the calculations a number of assumptions are made in this section. These assumptions are relaxed in Sec. III, to show that the result is quite general. In Sec. IV the dependence $L(H)$ is calculated using the parquette method. Obtained results are compared with experiment in Sec. V. In Sec. VI it is shown that so-called Hopf term is present in the FISDW state in the effective action of the $\mathbf{n}$ field, where $\mathbf{n}$ is the polarization vector of FISDW. Conclusions are given in Sec. VII.

II. CALCULATION OF THE CHERN NUMBER

For simplicity let us consider initially the case of spinless fermions with magnetic-field-induced charge-density-wave (FICDW's) instead of FISDW's. This permits us to study separately the orbital effect of magnetic field, which is the most important. The spin generalization will be done in Sec. III E.

The Hamiltonian of the fermions is as follows:

$$\hat{H} = \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} dx\, \psi^* \begin{pmatrix} -iv\partial_x + \varepsilon_\perp(k_y - qx) & \Delta(x) \\ \Delta^*(x) & iv\partial_x + \varepsilon_\perp(k_y + Q_y - qx) \end{pmatrix} \psi. \tag{2}$$

Here x is the coordinate along the chains, v is the Fermi velocity, k_y is the momentum in the direction perpendicular to the chains, and $\varepsilon_\perp(k_y)$ is the dispersion law of perpendicular motion which obeys the condition

$$\int_{-\pi}^{\pi} dk_y\, \varepsilon_\perp(k_y) = 0. \tag{3}$$

The amplitudes of tunneling between the nearest and the next-nearest neighboring chains t_1 and t_2 are assumed to be nonzero, so that

$$\varepsilon_\perp(k_y) = -2t_1 \cos k_y - 2t_2 \cos(2k_y). \tag{4}$$

In the gauge $A_y = Hx$, $A_x = A_z = 0$ the magnetic field appears in Hamiltonian (2) through the Peierls-Onsager substitution $k_y \to k_y - qx$. Note that the actual carriers in the $(TMTSF)_2X$ compounds are holes, so the charge e in (1) is positive. The vector $\psi = (\psi_+(x, k_y), \psi_-(x, k_y + Q_y))^T$ is the column vector of fermion wave functions. The index $\pm$ denotes two types of fermion with longitudinal momenta close to $\pm k_F$. The factors $\exp(\pm ik_F x)$ are extracted from the wave functions and the dispersion law of the longitudinal motion is linearized in (2) in the vicinity of $\pm k_F$. The complex function $\Delta(x)$ represents the order parameter of FICDW's. Q_y is the wave vector of the order parameter in the y direction. It will be seen below that $\Delta(x)$ may depend periodically on x.

Now let us make the transformation of the wave functions:[20,23]

$$\psi(x, k_y) = \begin{pmatrix} \psi'_+(x, k_y) \exp[-i \int^{qx} d\xi\, \varepsilon_\perp(k_y - \xi)/vq] \\ \psi'_-(x, k_y + Q_y) \exp[i \int^{qx} d\xi\, \varepsilon_\perp(k_y + Q_y - \xi)/vq] \end{pmatrix}. \tag{5}$$

After substitution of (5) in (2) the transformed Hamiltonian $\hat{H}'$, acting on the ψ', has the form:

$$\hat{H}' = \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} dx\, \psi'^* \begin{pmatrix} -iv\partial_x & \Delta(x) \exp\{\frac{i}{vq} \delta^{qx} d\xi [\varepsilon_\perp(k_y - \xi) + \varepsilon_\perp(k_y + Q_y - \xi)]\} \\ \text{c.c.} & iv\partial_x \end{pmatrix} \psi', \tag{6}$$

where c.c. stands for the complex conjugate of the other off-diagonal matrix element. Below the primes at ψ' are omitted. Due to condition (3) the phase factor in (6) is a periodic function of qx with the period 2π and can be expanded in the Fourier series with some coefficients $f_m(Q_y)$:

$$\hat{H}' = \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} dx\, \psi^* \begin{pmatrix} -iv\partial_x & \Delta(x) \sum_m f_m(Q_y) \exp[im(k_y - qx)] \\ \text{c.c.} & iv\partial_x \end{pmatrix} \psi, \tag{7}$$

where c.c. stands for the complex conjugate of the other off-diagonal matrix element.

Now, according to Refs. 7-11, let us assume that $\Delta(x)$ has the form

$$\Delta(x) = \Delta_0 \exp(-iLqx) \tag{8}$$

with some integer L. Physically it means the appearance of the FICDW with the longitudinal wave vector $Q_x = 2k_F - Lq$. After substitution of (8) in (7) we have

$$\hat{H}' = \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} dx\, \psi^* \begin{pmatrix} -iv\partial_x & \Delta_0 \exp(-iLk_y) \sum_m f_{L+m} \exp[im(k_y - qx)] \\ \text{c.c.} & iv\partial_x \end{pmatrix} \psi, \tag{9}$$

where c.c. stands for the complex conjugate of the other off-diagonal matrix element.

As was indicated in Refs. 19, the magnetic field induces the charge-density wave (CDW) because the one-loop staggering susceptibility of electrons becomes logarithmically divergent when temperature T is lower than the orbital magnetic energy

$$\Omega = vq = ebHv/c. \tag{10}$$

Taking also into account that the interaction constant between electrons is small, this means that the FICDW transition temperature T_c is always smaller than Ω. As the thermodynamics of the FICDW state is BCS-like,[9,20–23] so $\Delta_0 \sim T_c$ and we assume that

$$\Delta_0 \ll \Omega. \tag{11}$$

In this case the effect of each nondiagonal term in the sum in (9) can be studied separately. After Fourier transformation over x it becomes clear that the mth term in the sum (9) opens the gap $2|\Delta_0 f_{L+m}|$ in the spectrum at the wave vectors $k_x = \pm mq/2$.[23] Let us initially restrict the consideration to the simplest, so-called single-gap approximation[20] when all the terms in (9) are neglected except the term with $m = 0$. This term gives the gap at $k_x = 0$ that is at the Fermi level:

$$\hat{H}'' = \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \psi^*(k_x, k_y) \begin{pmatrix} vk_x & \Delta_0 f_L(Q_y) \exp(-iLk_y) \\ \text{c.c.} & -vk_x \end{pmatrix} \psi(k_x, k_y). \tag{12}$$

To calculate σ_{xy} at zero temperature the well-known formula (see Refs. 24 and 14) is utilized:

$$\begin{aligned} \sigma_{xy} &= -ie^2 \sum_a \int \frac{dk_x}{2\pi} \int \frac{dk_y}{2\pi} \left(\frac{\partial \langle \psi_a|}{\partial k_x} \frac{\partial |\psi_a\rangle}{\partial k_y} - \frac{\partial \langle \psi_a|}{\partial k_y} \frac{\partial |\psi_a\rangle}{\partial k_x} \right) \\ &= -ie^2 \sum_a \int \frac{dk_x}{2\pi} \int \frac{dk_y}{2\pi} \left(\frac{\partial}{\partial k_x} \langle \psi_a| \frac{\partial |\psi_a\rangle}{\partial k_y} - \frac{\partial}{\partial k_y} \langle \psi_a| \frac{\partial |\psi_a\rangle}{\partial k_x} \right). \end{aligned} \tag{13}$$

In formula (13) the integral is taken over the Brillouin zone and the summation is taken over all completely occupied fermion bands. It is assumed that there are no partially filled bands. The wave functions $|\psi_a(k_x, k_y)\rangle$ are the normalized eigenvectors of the Hamiltonian, whose number of components, generally speaking, is equal to the total number of bands in the energy spectrum.

Let us consider initially the case $Q_y = 0$. In this case the band structure of the model (12) is the following. The wave functions are defined on the Brillouin zone torus $|K_x| \le k_F$, $0 \le k_y < 2\pi$, where $k_x = K_x \mp k_F$ for $\pm$ fermions. There is a gap $\Delta = 2|\Delta_0 f_L|$ at the Brillouin zone boundaries $|K_x| = k_F$ and there is one completely filled fermion band under the Fermi energy. Let us start from some point (K_x^0, k_y), which is sufficiently far from the Brillouin zone boundaries $v|K_x^0 \pm k_F| \gg |\Delta|$, and change K_x along the closed line encircling the torus at fixed k_y. The fermion wave function $|\psi(K_x, k_y)\rangle$ changes as a function of K_x and, when we return to the starting point, it will coincide with the starting wave function $|\psi(K_x^0, k_y)\rangle = |\psi_0\rangle$ multiplied by the phase factor $\exp[i\phi(k_y)] = \exp(iLk_y)$. It appears due to the phase factor of the nondiagonal matrix element in Hamiltonian (12) when we cross the region $|K_x| \sim k_F$ ($k_x \sim 0$) near the gap (see similar calculations in Sec. VII B of Ref. 14). From this consideration follows that the first term in Eq. (13) equals

$$-ie^2 \int_0^{2\pi} \frac{dk_y}{(2\pi)^2} \langle \psi_0| \exp[-i\phi(k_y)] \partial_{k_y} \exp[i\phi(k_y)] |\psi_0\rangle - \langle \psi_0 | \partial_{k_y} | \psi_0 \rangle = [\phi(2\pi) - \phi(0)] e^2/(2\pi)^2 = Le^2/2\pi = Le^2/h, \tag{14}$$

where the dimensional Planck constant h is restored in the last equality.

The second term in (13) is equal to zero. In this term the expression under the integral can be rewritten as the difference $\langle \psi | \partial_{k_x} | \psi \rangle |_{k_y=0}^{k_y=2\pi}$. But the Hamiltonian (12) at $k_y = 2\pi$ is just the same as at $k_y = 0$, thus the wave functions are also the same, so the difference equals zero.

The conclusion is that for the model (12) the Hall con-

ductivity is integer quantized:

$$\sigma_{xy} = Le^2/h. \tag{15}$$

It will be shown in Sec. III E that the inclusion of spins doubles the result:

$$\sigma_{xy}^{(\text{spins})} = 2Le^2/h. \tag{16}$$

These results coincide with the ones found in Ref. 12.

III. SOME GENERALIZATIONS

In this section some of the assumptions made in the previous section to derive (15) are relaxed.

A. Multiple gaps

Let us relax the single-gap approximation (12) and take into account all nondiagonal terms in matrix (9). In this case the Brillouin zone has dimensions $|k_x| \le q/2$, $0 \le k_y < 2\pi$. There are many fermion bands under the Fermi level. When we change k_x along the closed loop encircling the Brillouin zone torus at the fixed value k_y, the wave function $|\psi_a\rangle$ gets the phase factor $\exp[i\phi_a^{(t)}(k_y) + i\phi_a^{(b)}(k_y)]$. The phase $\phi_a^{(t)}(k_y)$ appears while crossing the region near the top gap and the phase $\phi_a^{(b)}(k_y)$, the region near the bottom gap. It is easy to see from the behavior of the eigenvectors of (12) that the gap with the number m gives opposite phase contributions $\mp\phi_m$ for the upper and the lower neighboring bands.[14] So instead of (14) we now have

$$\sigma_{xy} = \frac{e^2}{(2\pi)^2}\sum_{a=1}^{\infty}[\phi_a^{(t)}(k_y) + \phi_a^{(b)}(k_y)]\,|_{k_y=0}^{k_y=2\pi} = \frac{e^2}{(2\pi)^2}\sum_{m=0}^{\infty}[\phi_m(k_y) - \phi_{m+1}(k_y)]\,|_{k_y=0}^{k_y=2\pi}$$
$$= [\phi_0(2\pi) - \phi_0(0)]e^2/(2\pi)^2 = Le^2/2\pi\ , \tag{17}$$

and result (14) is reproduced. In formula (17) the first sum is taken over the numbers of the occupied bands, beginning from the upper band, and the second sum is taken over the numbers of the gaps, starting from the gap at the Fermi level.

For Hamiltonian (9) we have

$$\phi_m(k_y) = (L - m)k_y, \tag{18}$$

so each band contributes one quantum to the Hall conductivity. Formula (18) becomes not valid far from the Fermi level, where the linearization of the dependence $\varepsilon(k_x)$ breaks. But this does not matter for result (17) which does not depend at all on the exact expressions for $\phi_m(k_y)$ at $m \neq 0$, because they cancel each other in sum (17).

In conclusion, result (15) is also valid in the multiple-gap approximation.

B. Transverse wave vector

Let us consider the case when the FICDW order parameter has the wave vector $Q_y = 2\pi/M$ in the y direction. The Hall conductivity may be calculated in the single-gap approximation (12), where now $\psi(k_x, k_y)$ is equal to the column vector $(\psi_+(k_x, k_y), \psi_-(k_x, k_y + Q_y))^T$. Let us repeat the reasoning following formula (13). After the change of K_x along the line encircling the Brillouin zone along the x direction, we do not get the initial wave function $|\psi(K_x^0, k_y)\rangle$ but obtain the function $\exp(iLk_y)|\psi(K_x^0, k_y + 2\pi/M)\rangle$. To get the initial wave function it is necessary to repeat this operation M times. The total phase, which the wave function $|\psi(K_x^0, k_y)\rangle$ acquires, is equal to $\phi(k_y) = MLk_y + \pi(M - 1)L$. We substitute the phase in formula (14) and take into account that the Brillouin zone for k_y is now reduced to $0 \le k_y < 2\pi/M$:

$$\sigma_{xy} = [\phi(2\pi/M) - \phi(0)]e^2/(2\pi)^2 = Le^2/2\pi. \tag{19}$$

Result (19) is the same as in (14). The generalization to the case $Q_y = 2\pi N/M$ is straightforward and gives the same result. An irrational $Q_y/2\pi$ can be represented as the limit of a sequence of rational values. So for the irrational $Q_y/2\pi$ result (15) also holds.

In conclusion, σ_{xy} does not depend on the transverse wave vector of the FICDW order parameter.

C. Three-dimensional case

Let us include in Hamiltonian (2) the tunneling along the third z direction perpendicular to the planes of the chains with the dispersion law

$$\varepsilon_z(k_z) = -2t_3 \cos k_z. \tag{20}$$

Due to this term the FICDW acquires the wave vector $Q_z = \pi$ in the z direction. Thus the vector of wave functions in (2) becomes equal to $\psi = (\psi_+(x, k_y, k_z), \psi_-(x, k_y + Q_y, k_z + \pi))^T$ and the diagonal matrix

$$\begin{pmatrix} -2t_3 \cos k_z & 0 \\ 0 & 2t_3 \cos k_z \end{pmatrix} \tag{21}$$

has to be added to matrix (2). The transformation

$$\psi = \psi' \exp(2ixt_3 \cos k_z/v) \tag{22}$$

eliminates term (21) from the Hamiltonian. We return to the initial problem (2) and all the results, obtained above, hold. In formula (13) the integration

$$\int_0^{2\pi} dk_z/2\pi \tag{23}$$

has to be added. As the function under the integral does not depend on k_z, integral (23) gives simply the factor of unity. This means that the Hall conductivity per layer even in the three-dimensional case is exactly given by expression (15). This result is in accordance with a general consideration[25] of the three-dimensional quantum Hall effect (see also Ref. 13). There are two important points here. Spectrum (20) has the property of the perfect nesting which permits the energy gap to cover the whole three-dimensional Fermi surface. There are no topological changes of the Fermi surface along the z direction.

D. Coexistence of several order parameters

It was assumed in Sec. II that the order parameter of the FICDW has the form (8). But in the general case several order parameters with different values of L_j may coexist:

$$\Delta(x) = \sum_j \Delta_j \exp(-iL_j qx). \tag{24}$$

To calculate σ_{xy} in this case let us follow the single-gap approximation. It is clear from Eq. (14) and the discussion before it that the integer L, which appears in (15), is, in fact, the phase acquired by the nondiagonal matrix element in (12) when k_y goes from 0 to 2π, divided by 2π and taken with the opposite sign. In the case when several order parameters coexist the nondiagonal matrix element is equal to

$$\tilde{\Delta}(k_y) = \sum_j \Delta_j f_j \exp(-iL_j k_y). \tag{25}$$

To find the Hall conductivity it is necessary to find the change of the phase of the complex number (25) when k_y goes from 0 to 2π. This phase, divided by 2π and taken with the opposite sign, is the integer number, which must be substituted instead of L in formula (15).

For given values of $\Delta_j f_j$ and L_j it is easy to perform this procedure. One particular case is extremely simple — when for some l the partial gap $|\Delta_l f_l|$ is larger than the sum of all other partial gaps:

$$|\Delta_l f_l| > \sum_{j\ (\neq l)} |\Delta_j f_j|. \tag{26}$$

In this case the Hall conductivity is determined by the largest term of (25) only:

$$L = L_l. \tag{27}$$

When two order parameters coexist the Hall conductivity is determined by the value L_j, whose partial gap $|\Delta_j f_j|$ is larger.

In conclusion, if several order parameters coexist the Hall conductivity is not the superposition of the partial Hall conductivities but is determined by the winding number of the complex function (25).

E. Spins

Let us now include in the consideration the spins of electrons. The corresponding Hamiltonian can be obtained from (2) if we assume that the elements of matrix (2) are themselves matrices 2×2 with respect to the spin indices. Zeeman terms $-\mu_B H \sigma_z$ should be added to the diagonal elements of matrix (2). They are eliminated by the transformation

$$\psi = \begin{pmatrix} \psi'_+ \exp(i\mu_B H x \sigma_z/v) \\ \psi'_- \exp(-i\mu_B H x \sigma_z/v) \end{pmatrix}. \tag{28}$$

After transformation (28) the nondiagonal element of (2) has the form

$$\hat{\Delta}(x) = \begin{pmatrix} \Delta_\uparrow(x) \exp(-2i\mu_B H x \sigma_z/v) & \Delta_x(x) - i\Delta_y(x) \\ \Delta_x(x) + i\Delta_y(x) & \Delta_\downarrow(x) \exp(2i\mu_B H x \sigma_z/v) \end{pmatrix}. \tag{29}$$

Let us consider firstly the case when $\Delta_x = \Delta_y = 0$. In this case spins up and down are decoupled. Choosing $\Delta_{\uparrow,\downarrow}$ in the form

$$\Delta_{\uparrow,\downarrow} = \Delta_0 \exp(\pm 2i\mu H x \sigma_z/v - iLqx), \tag{30}$$

we eliminate the oscillating factors from (29) and return to the problem of Sec. II separately for spins up and down. For each spin we have expression (15), and totally, we have (16) .

Let us consider now the case $\Delta_\uparrow = \Delta_\downarrow = 0$. By a phase transformation of wave functions it is possible to set $\Delta_y = 0$ in (29). Let us repeat the reasoning after formula (13). If we start from the wave function $|\psi_\uparrow(K_x^0, k_y)\rangle$ then after the trip around the Brillouin zone we receive the wave function $\exp(iLk_y)|\psi_\downarrow(K_x^0, k_y)\rangle$. To restore the initial wave function we have to repeat this procedure once more. So the total phase acquired is equal to $2Lk_y$. Thus the answer is given by formula (16).

IV. PARQUETTE APPROACH

In the preceding sections formula (15) for the Hall conductivity was derived. According to this formula the dependence $\sigma_{xy}(H)$ is determined by the dependence

$L(H)$. The dependence $L(H)$ should be, strictly speaking, determined at zero temperature T, where formula (15) is valid. But it is more simple to find the dependence $L(H)$ at the transition line $T_c(H)$ from the metallic to the FICDW phase. Both in experiment[1-6] and in theory[20-23,9] the values of the magnetic field, at which the transition from one FISDW phase to another appears, depend very slightly on the temperature. So the topology of the phase diagram in the $H-T$ plane is such that the sequence of the integer numbers L under the monotonous change of H is exactly the same at $T=0$ and at $T=T_c(H)$. Additional harmonics (24) of the order parameter, which may appear away from the metal-FICDW transition line, also does not change the value of L provided their amplitudes are sufficiently small (see Sec. III D). So, this section is devoted to the calculation of the dependence $L(H)$ at $T=T_c(H)$, which is argued to be the good approximation to the dependence $L(H)$ at $T=0$.

The general way to solve this problem is the following.[19] In the metallic phase at the given value of H it is necessary to determine as a function of T the electron-hole susceptibility with respect to the appearance of the infinitesimal order parameter (8). At some temperature $T_c(L,Q_y)$ the susceptibility becomes singular. The FICDW order parameter appears with the values L and Q_y, corresponding to the highest temperature $T_c(L,Q_y)$ below this temperature.

There are, basically, two approaches to the calculation of the susceptibility. In one approach the susceptibility is calculated as the sum of the ladder diagrams in the electron-hole channel.[7-11,19]. In the limit $T_c \ll \Omega$ in this approach the values L and Q_y, which maximize $|f_L(Q_y)|$ in (9), also maximize $T_c(L,Q_y)$.[9,10] The numerical calculation of the susceptibility was done for the spectrum (4) in Ref. 11. The result is that L acquires subsequent integer numbers $0,1,2,3,\ldots$ as $1/H$ increases from the zero value with the characteristic period $\sim ebv/ct_2$.

Another approach,[26-28] where the so-called parquette approximation is used, is more general. It takes into account not only the contribution to the susceptibility of the electron-hole loops but also the contribution of the electron-electron loops which are both logarithmically divergent as functions of temperature. This approach is briefly outlined below and the results for $L(H)$ are presented.

The starting Hamiltonian consists of two terms:

$$\hat{H}=\hat{H}_0+\hat{H}_{\rm int}. \tag{31}$$

The first term $\hat{H}_0$ is the same as (2) but with $\Delta=0$ because now we are in the metallic phase at $T>T_c$:

$$\hat{H}_0=\sum_{\alpha=\pm}\int_{-\pi}^{\pi}\frac{dk_y}{2\pi}\int_{-\infty}^{\infty}dx[-i\alpha v\hat{\psi}^\dagger_\alpha(x,k_y)\partial_x\hat{\psi}_\alpha(x,k_y)+\varepsilon_\perp(k_y-qx)\hat{\psi}^\dagger_\alpha(x,k_y)\hat{\psi}_\alpha(x,k_y)]. \tag{32}$$

The second term $\hat{H}_{\rm int}$ describes the interaction between the fermions:

$$\hat{H}_{\rm int}=G\int_{-\infty}^{\infty}dx\prod_{j=1}^{4}\left(\int_{-\pi}^{\pi}\frac{dk_y^{(j)}}{2\pi}\right)\hat{\psi}^\dagger_+(x,k_y^{(1)})\hat{\psi}^\dagger_-(x,k_y^{(2)})\hat{\psi}_-(x,k_y^{(3)})\hat{\psi}_+(x,k_y^{(4)})\delta(k_y^{(1)}+k_y^{(2)}-k_y^{(3)}-k_y^{(4)}), \tag{33}$$

where G is the interaction constant.

Now let us make transformation (5) and denote $\hat{\psi}'_\alpha(x,k_y)=\hat{a}_\alpha(x,k_y)$. The transformed Hamiltonians have the following form:

$$\hat{H}_0=\sum_{\alpha=\pm}\int_{-\pi}^{\pi}\frac{dk_y}{2\pi}\int_{-\infty}^{\infty}dx[-i\alpha v\hat{a}^\dagger_\alpha(x,k_y)\partial_x\hat{a}_\alpha(x,k_y)], \tag{34}$$

$$\hat{H}_{\rm int}=G\int_{-\infty}^{\infty}dx\prod_{j=1}^{4}\left(\int_{-\pi}^{\pi}\frac{dk_y^{(j)}}{2\pi}\right)g(x,k_y^{(1)},k_y^{(2)},k_y^{(3)},k_y^{(4)})\hat{a}^\dagger_+(x,k_y^{(1)})\hat{a}^\dagger_-(x,k_y^{(2)})$$
$$\times\hat{a}_-(x,k_y^{(3)})\hat{a}_+(x,k_y^{(4)})\delta(k_y^{(1)}+k_y^{(2)}-k_y^{(3)}-k_y^{(4)}), \tag{35}$$

where the matrix element of scattering $g(x,k_y^{(1)},k_y^{(2)},k_y^{(3)},k_y^{(4)})$ is equal to

$$g(x,k_y^{(1)},k_y^{(2)},k_y^{(3)},k_y^{(4)})=\exp\left[\frac{8it_1}{\Omega}\cos\frac{k_y^{(1)}-k_y^{(3)}}{2}\sin\frac{k_y^{(1)}-k_y^{(4)}}{2}\cos\left(\frac{k_y^{(1)}+k_y^{(2)}}{2}-qx\right)\right.$$
$$\left.+\frac{4it_2}{\Omega}\cos(k_y^{(1)}-k_y^{(3)})\sin(k_y^{(1)}-k_y^{(4)})\cos(k_y^{(1)}+k_y^{(2)}-2qx)\right]. \tag{36}$$

In formula (36) the explicit form of dispersion law (4) was used.

We see from (36) that the matrix element of scattering $g(x,k_y^{(1)},k_y^{(2)},k_y^{(3)},k_y^{(4)})$ periodically depends on qx. Let us

expand it in the Fourier series with respect to qx:

$$g(x,k_y^{(1)},k_y^{(2)},k_y^{(3)},k_y^{(4)})=\sum_m g(m,k_y^{(1)},k_y^{(2)},k_y^{(3)},k_y^{(4)})\exp(imqx). \tag{37}$$

The Fourier harmonics with $m\neq 0$ do not conserve the momentum k_x. In these scattering vertices the logarithmic singularity is cut off at the energy of order $m\Omega$, so they will be neglected.[27] What remains is the term of (37) with $m=0$. With this term Hamiltonian (35) can be rewritten as

$$\hat{H}_{\rm int}=G\int_{-\pi}^{\pi}\frac{dk_y}{2\pi}\int_{-\pi}^{\pi}\frac{dp_y}{2\pi}\int_{-\pi}^{\pi}\frac{dq_y}{2\pi}\int_{-\infty}^{\infty}dx f(q_y,p_y)\hat{a}_+^{\dagger}(x,k_y)\hat{a}_-^{\dagger}(x,k_y+q_y+p_y)\times\hat{a}_-(x,k_y+q_y)\hat{a}_+(x,k_y+p_y), \tag{38}$$

where q_y and p_y are the total and the relative momenta in the transverse direction in the electron-hole channel and

$$f(q_y,p_y)=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\exp\left[-\frac{8it_1}{\Omega}\cos\left(\frac{q_y}{2}\right)\sin\left(\frac{p_y}{2}\right)\cos\theta-\frac{4it_2}{\Omega}\cos q_y\sin p_y\cos(2\theta)\right]. \tag{39}$$

Now let us calculate the sum of so-called parquette diagrams which consist of electron-electron and electron-hole loops inserted into each other.[26,27]. The reason for the selection of such diagrams is that the loop consisting of two lines, one belonging to the + and another to the − fermion, is divergent as $\ln_{10}(1/T)$. Summing these diagrams we find the renormalized scattering vertex $\gamma(\xi,q_y,p_y)$ which now depends on temperature T through the variable

$$\xi=\frac{|G|}{2\pi v}\ln_{10}\frac{\Omega}{T}, \tag{40}$$

where the cutoff energy Ω is magnetic energy (10). The renormalization at the energies larger than Ω is not essential.[27] The dependence $\gamma(\xi,q_y,p_y)$ on ξ is determined from the parquette or the renormalization-group equations, derived for the model in Refs. 26 and 27:

$$\frac{d\gamma(\xi,q_y,p_y)}{d\xi}=\int_{-\pi}^{\pi}\frac{dk_y}{2\pi}[\gamma(\xi,q_y,k_y)\gamma(\xi,q_y,p_y-k_y)-\gamma(\xi,q_y+p_y-k_y,k_y)\gamma(\xi,q_y-k_y,p_y-k_y)], \tag{41}$$

$$\gamma(0,q_y,p_y)=\mathrm{sgn}(G)f(q_y,p_y). \tag{42}$$

It was shown in Ref. 29 and for the present model in Refs. 26 and 27 that the stable singular asymptotic solution of Eqs. (41) has the form of a so-called moving pole:

$$\gamma(\xi,q_y,p_y)=\exp(iLp_y)[\xi_c-\xi+\beta(q_y-Q_y)^2]^{-1},\qquad |q_y-Q_y|\ll 1, \tag{43}$$

where β is some coefficient. Singular vertex (43) is characterized by the parameters L, Q_y, and ξ_c. With vertex (43) the susceptibility can be easily calculated,[29] and it appears that the values L and Q_y of (43) are just the parameters of order parameter (8) which occurs at $T<T_c$.[26,27] The transition temperature T_c is determined by the value of ξ_c:

$$T_c=\Omega\exp(-2\pi v\xi_c/|G|). \tag{44}$$

Thus the procedure of determining of the phase diagram is as follows. For some value of H initial conditions (42) and (39) are calculated numerically. With these initial conditions Eqs. (41) are solved numerically by the fourth-order Runge-Kutta method. At some value of the "time" $\xi=\xi_c$ the solution becomes the singular one of the form (43), characterized by some values L and Q_y. Then the value of H is changed and the procedure is repeated. In such a way the dependence of L, Q_y, and ξ_c on H is found. The calculations are done for both signs of G and for different values of the ratio t_2/t_1.[30]

The results of the numerical solution are shown in Table I for $G<0$. This case is the most interesting because all compounds, where the cascade of FISDW transitions is observed, are superconductors at $H=0$. The dependence $Q_y(H)$ is not shown because σ_{xy} does not depend on Q_y. Generally, Q_y depends continuously on H.

In the second column of the table the dependence $\xi_c(H)$ is shown for $t_2/t_1=0.02$. Taking into account (44) it follows that T_c has a global maximum at $8t_1/\Omega\approx 5$. The transition temperature decreases both when H decreases toward zero and when H increases toward infinity.[26–28] There are also oscillations in the dependence $T_c(H)$ with the period in $8t_1/\Omega$ approximately equal to π, due to which the dependence is nonmonotonic. When the value t_2/t_1 is increased, the dependence $\xi_c(H)$ (not shown in the table) changes smoothly and the period of oscillations changes only slightly.

Concerning the values of L in the table we note that, as follows from (39)–(43), the sign of L is determined by the

TABLE I. Results of the numerical solution of Eqs. (41). In the first column the values of $8t_1/\Omega$ are indicated. In the first row the values of t_2/t_1 are indicated. The value of the integer L as a function of $8t_1/\Omega$ and t_2/t_1 is shown in the table, except in the second column where the value ξ_c as a function of $8t_1/\Omega$ is shown for $t_2/t_1 = 0.02$.

$8t_1/\Omega$ \ t_2/t_1	0.02		0.1	0.2	0.25	0.3	0.4	0.5	1
0.25	6.5	−1							
0.5	5.2	−1			−1			−1	−1
1.0	4.1	−1			−1			−1	−1
1.5	3.7	−1			−1			−1	−1
2.0	3.5	−1			−1			−1	−1
2.5	3.7	−1			−1			−1	−1
3.0	4.2	−1			−1			−1	−1
3.5	5.3	−1			−1			−1	−1
4.0	5.6	0			0			0	2
4.5	3.5	0			0			0	0
5.0	3.2	0			0			0	0
5.5	3.4	0			0			0	0
6.0	4.1	0	0	0	0	0	0	0	−1
6.5	5.4	0	0	0	0	0	0	0	−1
7.0	7.5	1	1	1	0	0	0	0	1
7.5	5.4	−1	−1	−1	−1	1	1	1	1
8.0	4.7	−1	−1	−1	−1	−1	1	1	1
8.5	4.6	1	1	1	−1	−1	−1	1	1
9.0	4.9	1	1	1	−1	−1	1	1	1
9.5	5.7	−1	−1	−1	−1	−1	−1	1	1
10.0	6.8	2	−1	−1	−1	−1	−1	1	1
10.5	6.3	0	0	0	0	0	0	−1	1
11.0	5.1	0	0	0	0	0	0	2	1
11.5	4.9	0	0	0	0	0	2	2	1
12.0	5.0	0	0	0	0	0	2	1	1
12.5	5.6	0	0	0	−2	−2	1	1	−3
13.0	6.6	0	−2	−2	−2	−2	−2	−1	−3
13.5	7.7	−2	−2	−2	−2	−2	−2	−2	−1
14.0	6.5	1	1	−1	−1	−1	−2	−2	3
14.5	6.0	1	1	−1	−1	−1	−2	−2	−1

sign of t_2: $L \to -L$ when $t_2 \to -t_2$. The results, shown in the table, correspond, in fact, to the case $t_2 < 0$. It follows from the table that $L = -1$ when $H \to \infty$ for any value of t_2/t_1. This result was obtained analytically in Refs. 26 and 27 in the case $t_2 = 0$. When H decreases, then L always acquires the value $L = 0$. Further behavior of L under the decrease of H strongly depends on the value of t_2/t_1. The dependence of L on both H and t_2/t_1 seems to be rather chaotic. The only observation is that the sign of L is oscillating and $|L|$ increases, in general, when H decreases. The table demonstrates that in the case $G < 0$ the solution of Eqs. (41) is strongly sensitive to initial conditions (42), which are determined by the parameters t_1/Ω and t_2/t_1. Strong sensitivity to the initial conditions is the characteristic property of the chaotic dynamical systems. So, it is not excluded that here we are faced with the intriguing possibility of the chaos in the renormalization-group equations.

The results of calculations in the case $G > 0$ are not shown in the table, because they are rather simple. In the limit $H \to \infty$ $L = 0$. When H decreases L acquires subsequent integer numbers. The sign of L is constant. The dependence $\xi_c(1/H)$ is nonmonotonic due to the oscillations with the period $\sim 1/t_2$. The dependence $L(H)$ is the same in the parquette and in the ladder approximations. The latter corresponds to the neglecting of the second term in Eqs. (41). The dependence $\xi_c(H)$ is also qualitatively the same except in the region $H \to \infty$, where $T_c \to 0$ in the parquette approximation and $T_c \to$ const in the ladder approximation.[26–28] So, except the phenomenon of the reentrance to the metallic phase at large H, the ladder approximation is the good one in the case $G > 0$.

V. COMPARISON WITH EXPERIMENT

The FISDW is found experimentally in the three compounds $(TMTSF)_2PF_6$,[1,2] $(TMTSF)_2ClO_4$,[4,5] and $(TMTSF)_2ReO_4$.[31] The Hall conductivity has not been measured yet in $(TMTSF)_2ReO_4$, so we discuss here the behavior of the first two compounds.

In $(TMTSF)_2PF_6$ the Hall conductivity per layer is found experimentally to be equal to $\sigma_{xy} = 2Le^2/h$,[1] in accordance with (16). For reasons unclear the value of

TABLE I. (*Continued*).

$8t_1/\Omega$ \ t_2/t_1		0.02	0.1	0.2	0.25	0.3	0.4	0.5	1
15.0	6.0	1	1	1	−1	−2	−2	−2	4
15.5	6.4	−3	−3	−3	−2	−2	−2	−2	−2
16.0	7.1	−3	−3	−3	−2	−2	−2	−2	−2
16.5	8.0	−3	−3	−2	−2	−2	−2	−2	−2
17.0	7.1	0	0	0	−2	−2	−2	−2	−4
17.5	6.1	0	0	0	−3	−2	−2	−2	−4
18.0	6.0	0	0	0	−3	−3	−2	−2	−4
18.5	6.2	0	0	−3	−3	−3	−2	−2	−4
19.0	6.9	0	0	−3	−3	−2	−2	−2	−5
19.5	8.2	−4	−4	−3	−3	−2	−2	3	−5
20.0	8.5	−5	1	−1	−1	−1	−2	3	−5
20.5	7.3	1	1	−1	−1	−1	−2	3	−1
21.0	7.0	1	1	−1	−1	−1	3	3	4
21.5	7.1	1	1	−1	−1	−4	2	3	4
22.0	7.6	5	1	−4	−4	−3	3	3	−6
22.5	8.3	−5	−5	−3	−3	−3	3	3	−6
23.0	9.2	−5	−4	−3	−3	−2	−4	3	−6
23.5	7.7	0	0	2	−2	−2	−4	−4	−6
24.0	7.0	0	0	2	−2	−2	3	−4	−6
24.5	6.9	0	0	2	−4	−4	3	−1	5
25.0	7.3	0	0	−4	−4	−4	3	−1	5
25.5	8.2	0	0	−4	−3	−3	−4	−1	6
26.0	9.5	−6	−4	−4	−4	−4	3	3	−7
26.5	8.9	1	−1	−1	−1	−4	3	3	−7
27.0	8.0	1	−1	−1	−1	3	3	−1	−3
27.5	7.8	1	−1	−1	−1	−5	3	−1	−3
28.0	8.1	1	−1	−1	−5	−5	3	−1	−3
28.5	8.7	1	1	−5	−4	−4	−1	4	6
29.0	9.6	7	−6	2	−4	−4	4	4	
29.5	9.5	0	0	−2	−2	−2	3	−6	
30.0	8.1	0	0	−2	−2	−2	3	−6	
30.5	7.7	0	0	−2	−2	3	−6	−6	
31.0	7.8	0	0	−2	−6	3	−1	5	
31.5	8.4	0		−6	−5	−4	−1	5	
32.0	9.4	0		−5	−5	−4	−1	−3	

σ_{xy}, found in Ref. 2, is two times larger. In a very strong magnetic field close to the experimental limit there is the FISDW phase where σ_{xy} and σ_{xx} are very small and thermally activated.[1] This is presumably the FISDW phase with $L = 0$. Under the decrease of H the sequence of the quantum Hall FISDW phases appears with L equal subsequently to 1, 2, 3, 4, 5.[1,2] For a lower magnetic field there are no well pronounced Hall plateaus. The sign of σ_{xy} is constant and negative in the quantum Hall regime. It is opposite to the sign in the low-H region where the carriers are holes. The described dependence $L(H)$ seems to be in excellent agreement with the theoretical results in the case $G > 0$ (see Sec. IV).[32] The sign of L can be easily fitted theoretically by taking the appropriate sign of t_2.

Nevertheless earlier experiments[35,36] reveal a more complex picture. In Ref. 35 an oscillatory behavior of the sign of ρ_{xy} was found. The reversals of the sign of ρ_{xy} were also found in Ref. 36 in the quantum Hall region later studied in Ref. 2. In Ref. 1 the reversals of the sign of ρ_{xy} were found in the transient regions between the plateaus.

It is impossible to explain the oscillatory sign of the Hall effect within the model with $G > 0$. On the other hand it is easy to do so in the case $G < 0$ (see Table I). Unfortunately, the absolute value of L in the table changes with H too chaotically. This may be a mathematical drawback of the parquette approximation, but the physical idea that in the case $G < 0$ the dependence $L(H)$ may be oscillatory in the sign, seems to be attractive. The negative sign of G is favorable for superconductivity, which is really found in all three compounds in a small magnetic field (see Refs. 26 and 27). The crucial experiment, which may distinguish which sign of G is

appropriate to $(TMTSF)_2PF_6$, is the following.[37] In the case $G > 0$ with the increase of H beyond the present experimental limit there must appear a reentrant phase transition to the metallic phase directly from the FISDW phase with $L = 0$. In the case $G < 0$ with the increase of H firstly there must be a transition from $L = 0$ to $|L| = 1$ and then the reentrant transition from the phase $|L| = 1$. According to the estimates of Ref. 2, the reentrant magnetic field for $(TMTSF)_2PF_6$ is of the order of 65–90 T.

The situation in $(TMTSF)_2ClO_4$ is different. Due to the ordering of the anions ClO_4 an additional crystalline superstructure doubles the unit cell in the direction perpendicular to the chains. Thus the results of the above calculations are not directly applicable in this case. But the qualitative statement about $|L| = 1$ in the last FISDW phase holds. In fact, the crucial experiment, described above, has been performed in $(TMTSF)_2ClO_4$:[5] the metallic reentrance directly from the phase $|L| = 1$ has been found.[38] The sign of the quantum Hall effect in $(TMTSF)_2ClO_4$ is strongly oscillatory,[39] also in qualitative agreement with the results of the calculations in the case $G < 0$. So, it seems that in $(TMTSF)_2ClO_4$ really the sign of G is negative, although further calculations of the dependence $L(H)$ in the presence of the crystalline superstructure are desirable in this case.

As was shown analytically in Ref. 27 in the last FISDW phase before reentrance $|L| = 1$ and $Q_y = \pi$. It follows from formulas (29), (59), and (60) of Ref. 27 that with the parameters of the FISDW state, indicated above, the modulation of the spin density is equal to zero or, at least, is very small if t_2 is taken into account. In this phase the order parameter is nonlocal, corresponding to the pairing between an electron at one chain and a hole at the neighboring chain.[27,40] So it does not change the density of spins on one chain. Usually, the appearance of FISDW's manifests itself by the disappearance of the nuclear magnetic resonance (NMR) signal due to the effect of the nonhomogeneous local magnetic field created by the modulated spin density.[41] According to the above consideration the NMR signal must be present in the last FISDW phase. One unconfirmed experimental result of such a type for the smaller values of H was reported in Ref. 42.

Concerning the dependence $T_c(H)$ we find that qualitatively the calculated dependence is in agreement with experiment. T_c really has a global maximum at some value of H (see Fig. 3 of Ref. 5).[43] The nonmonotonic behavior of $T_c(H)$ in the region of moderate fields was found experimentally as the partial reentrances of the metallic phase between the adjacent FISDW phases.[44] It should be stressed that these qualitative features of the curve $T_c(H)$ are valid theoretically in both cases of positive and negative G.[37]

A puzzling treelike phase diagram was recently found in $(TMTSF)_2ClO_4$.[15,45,46] Under the decrease of T the lines of the phase transitions between the FISDW phases split into several lines. Then these new lines split and so on. Surprisingly, these transitions are seen only in the thermodynamical measurements but do not manifest themselves in the transport measurements, in particular, in the value of σ_{xy}. I put forward the hypothesis that these transitions correspond to the appearance of additional order parameters (see Sec. III D). Let us discuss first what may happen in the vicinity of the metal-FISDW line. Depending on the coefficients of the Landau expansion of the free energy the transition between two FISDW phases may be of the first or second order. In the latter case, in fact, there are two lines of phase transitions so that in the intermediate phase the two FISDW order parameters coexist. I suggest that under the lowering of temperature more and more FISDW order parameters may coexist. Appearance of each new order parameter is the second order phase transition which can be seen thermodynamically. But, as shown in Sec. III D, provided the amplitudes of them are sufficiently small, the value of σ_{xy} remains constant. The value of σ_{xx} probably also does not change very much because the gap is always present at the Fermi level. To justify this hypothesis it is necessary to develop the thermodynamical description of the FISDW at $T < T_c$. Unfortunately, it is not clear how to continue the parquette technique for $T < T_c$. The thermodynamics of the FISDW at $T < T_c$ was studied in Refs. 20–23 and 9 in the mean-field approximation in the case $G > 0$. The above-described phenomenon has not been found. So it may be intrinsically related to the case $G < 0$ where the parquette method is necessary to obtain the phase transitions.

It was mentioned in Sec. III E that, with the spins taken into account, the two types of the FISDW order parameters are possible. Referring to the direction of H they can be called the longitudinal FISDW with $\Delta_{\uparrow,\downarrow} \neq 0$ and the transverse FISDW with $\Delta_{x,y} \neq 0$. It was argued in Sec. 7 of Ref. 27 that for the realistic values and signs of g_1 and g_2 the longitudinal FISDW with the two incommensurable potentials $\Delta_\uparrow$ and $\Delta_\downarrow$ is more favorable. But, generally speaking, the most preferable type of FISDW may depend on H. It is tempting to interpret the phase transition around 17 T, observed in the transport [47] and in the thermodynamical[48] measurements, as the phase transition between the longitudinal and transverse types of FISDW. If both types of FISDW have the same value of L ($|L| = 1$) then σ_{xy} does not change at the phase transition, while σ_{xx} may change if the gap at the Fermi level changes. Precisely such behavior has been found experimentally (see Fig. 2 of Ref. 4).

VI. TOPOLOGICAL HOPF TERM

A. Isotropic case

In the absence of a magnetic field the SDW order parameter has a vector nature:

$$\hat{\Delta}_{\zeta\eta} = \Delta_0 \mathbf{n} \boldsymbol{\sigma}_{\zeta\eta}. \tag{45}$$

Here Δ_0 is the absolute value of the order parameter,

the vector $\mathbf{n}$ determines the polarization of SDW, and the Pauli matrices $\sigma_{\zeta\eta}$ act on the spin indices of electrons. In the absence of relativistic effects the energy of the system does not depend on the direction of $\mathbf{n}$. The magnetic field breaks this spin invariance by distinguishing the direction along H. Formally it happens due to the appearance of the Zeeman term in the Hamiltonian. For simplicity let us initially consider the artificial case when there is no Zeeman term: $\mu_B = 0$. In this case the global spin rotational symmetry is preserved.

Let us integrate out fermions to find the effective action of the $\mathbf{n}$ field, which may slowly vary in space-time (x, y, t). The effective action can be found as a series in powers of gradients on the $\mathbf{n}$ field. Apart from the standard term $(\mathrm{grad}\mathbf{n})^2$ it may also contain the topologically nontrivial Hopf term.[49] This term can be written in the following form:

$$S_H = \frac{C\varepsilon_{\mu\nu\lambda}}{32\pi} \int dx\, dy\, dt\; A_\mu F_{\nu\lambda}, \tag{46}$$

$$F_{\mu\nu} = \mathbf{n}(\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}), \tag{47}$$

$$\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}, \quad \mu = t, x, y. \tag{48}$$

Here $\varepsilon_{\mu\nu\lambda}$ is the completely antisymmetric Levi-Cività tensor of rank 3. For a given configuration of the $\mathbf{n}(x, y, t)$ field it is necessary to determine first $F_{\mu\nu}$ from (47), then A_μ from (48), and then $S(H)$ from (46). The coefficient C in (46) determines the spin and statistics of the particlelike topological solitons of the $\mathbf{n}$ field, called skyrmions.[50] In the skyrmion $\mathbf{n}$ is up at infinity, down in the center of the skyrmion and there is a concentric domain wall in between, where $\mathbf{n}$ rotates between up and down directions.

In Refs. 51 and 52 it was shown that the value of the coefficient C in (46) and the value of the quantum Hall conductivity are determined by the same expression:

$$\sigma_{xy} = Ce^2/h, \tag{49}$$

$$C = \frac{1}{4\pi^2}\mathrm{Tr} \int d\omega \int dk_x \int dk_y\; G\frac{\partial G^{-1}}{\partial \omega} G \frac{\partial G^{-1}}{\partial k_x} G \frac{\partial G^{-1}}{\partial k_y}, \tag{50}$$

where $G(\omega, k_x, k_y)$ is the electron Green function. The equivalence of expressions (50) and (13) for σ_{xy} was shown in Ref. 53. From formulas (16) and (49) it follows that

$$C = 2L. \tag{51}$$

Thus in the FISDW phase the Hopf term (46) is present with coefficient (51). According to theory [50] it means that the skyrmions are bosons with the integer spin L in the FISDW state. The quantum numbers of skyrmions depend on H due to the dependence $L(H)$ manifested through the dependence $\sigma_{xy}(H)$.

B. Anisotropic case

Let us now take into account the Zeeman term. It will be shown that result (51) of Sec. VI A holds, but in addition an anisotropic term, proportional to n_z^2, has to be added to the effective action.

To perform calculations explicitly a mean-field theory is used. The starting Hamiltonian $\hat{H}$ has the form:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{int}}, \tag{52}$$

$$\hat{H}_0 = \sum_{\alpha=\pm} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} dx \hat{\psi}^{+}_{\alpha\zeta} \{[-i\alpha v \partial_x + \varepsilon_\perp(k_y - qx)]\delta_{\zeta\eta} - \mu_B H \sigma^z_{\zeta\eta}\} \hat{\psi}_{\alpha\eta}, \tag{53}$$

$$\hat{H}_{\mathrm{int}} = \int_{-\infty}^{\infty} dx \prod_{j=1}^{4} \left(\int_{-\pi}^{\pi} \frac{dk_y^{(j)}}{2\pi} \right) \hat{\psi}^{\dagger}_{+\zeta} \hat{\psi}^{\dagger}_{-\eta} \hat{\psi}_{-\rho} \hat{\psi}_{+\tau} (-g_1 \delta_{\zeta\rho}\delta_{\eta\tau} + g_2 \delta_{\zeta\tau}\delta_{\eta\rho}). \tag{54}$$

The second index of the fermion operators is the spin index.

Using the Hubbard-Stratonovich transformation, Hamiltonian (54) can be rewritten in the form

$$\hat{H}_{\mathrm{int}} = \hat{H}_\Delta + \hat{H}_{\Delta,\psi}, \tag{55}$$

$$\hat{H}_\Delta = \frac{|\Delta_+|^2 + |\Delta_-|^2}{g_2} + \frac{1}{g_2(g_2 - 2g_1)} \begin{pmatrix} \Delta^*_\uparrow \\ \Delta^*_\downarrow \end{pmatrix} \begin{pmatrix} g_2 - g_1 & g_1 \\ g_1 & g_2 - g_1 \end{pmatrix} \begin{pmatrix} \Delta_\uparrow \\ \Delta_\downarrow \end{pmatrix}, \tag{56}$$

$$\hat{H}_{\Delta,\psi} = \psi^{\dagger}_{+,\zeta} \hat{\Delta}_{\zeta\eta} \psi_{-,\eta} + \mathrm{H.c.}, \tag{57}$$

$$\hat{\Delta}(x) = \begin{pmatrix} \Delta_\uparrow & \Delta_+ \\ \Delta_- & \Delta_\downarrow \end{pmatrix}. \tag{58}$$

In these formulas the common integration [see (53) and (54)] is omitted to make the formulas more transparent. Let us now make the transformation of the fermions (28) which eliminates the Zeeman term from (53). Then let us make the transformation $\Delta_{\uparrow,\downarrow} \to \Delta_{\uparrow,\downarrow} \exp(\pm 2i\mu H x\sigma_z/v)$, which restores the form (58) after the first transformation [see (29) and (30)]. Due to this transformation, the nondiagonal elements of the matrix in formula (56) acquire oscillating factors $\exp(\mp 4i\mu H x\sigma_z/v)$. We assume that the components $\Delta_{\uparrow,\downarrow}$ of the transformed order parameter are practically constant in space, so the oscillating nondiagonal terms in (56) are integrated to zero. Finally, we have

$$\hat{H}_\Delta = \frac{|\Delta_+|^2 + |\Delta_-|^2}{g_2} + \frac{g_2 - g_1}{g_2(g_2 - 2g_1)}(|\Delta_\uparrow|^2 + |\Delta_\downarrow|^2). \tag{59}$$

Let us discuss what we have received. In Hamiltonian (57) and (58) the parameters $\Delta_{\uparrow,\downarrow}$ now physically describe the potentials with the wave vectors $2k_F \pm \mu_B H/v$. Each of these potentials is related to the modulation of charge and, in the general case, is pinned. This means that the global phases of $\Delta_\uparrow$ and $\Delta_\downarrow$, related to the rigid shifts of the superstructures, are fixed. By an appropriate phase transformation of $\hat{\psi}$ operators we may set $\Delta_\uparrow = -\Delta_\downarrow = \Delta_z$. Taking these considerations into account, we can rewrite (58) in the form (45), where the n_z component of the order parameter represents two potentials with the periods $2k_F \pm \mu_B H/v$ and the $n_{x,y}$ components represent the potential with the period $2k_F$. The Hamiltonian $\hat{H}_{\rm ferm} = \hat{H}_0 + \hat{H}_{\Delta,\psi}$ has now the form (2) where the diagonal elements are assumed to be unity matrices 2×2 with respect to the spin indices and the nondiagonal elements have the matrix form (45). The Hamiltonian $\hat{H}_{\rm ferm}$ is invariant with respect to SU(2) spin rotations of the fermions. Its eigenvalues do not depend on the direction of $\mathbf{n}$. If $\mathbf{n}$ slowly varies in space-time, then applying the method of Refs. 51 and 52, we find, after integrating out fermions from $\hat{H}_{\rm ferm}$, the Hopf term (46) with the coefficient C (51).

The remaining Hamiltonian (59) can be rewritten as

$$\hat{H}_\Delta = 2|\Delta_0|^2 \left(\frac{1}{g_2} + \frac{g_1}{g_2(g_2 - 2g_1)} n_z^2\right) \tag{60}$$

and contains anisotropic term n_z^2. We assume that g_1 is sufficiently small: $|g_1/g_2^2| \ll 1$. From (60) we see that the anisotropic term gives an $\mathbf{n}$-dependent correction to the coupling constant. The equilibrium value of the order parameter is

$$|\Delta| = \Omega \exp\left[-\left(\frac{1}{g_2} + \frac{g_1}{g_2(g_2 - 2g_1)} n_z^2\right)\right] \approx \Delta_0(1 - n_z^2 g_1/g_2^2). \tag{61}$$

It is well known that the condensation energy, which consists of (60) and the fermion part, is proportional to $|\Delta|^2$:

$$F_{\rm cond} = -A|\Delta|^2, \tag{62}$$

where A is some coefficient. Substituting (61) into (62), we find

$$F_{\rm cond} = \text{const} + 2A|\Delta_0|^2 \frac{g_1}{g_2^2} n_z^2. \tag{63}$$

So, we have found that, in comparison with the isotropic case, an additional term, proportional to n_z^2 with the coefficient as small as g_1/g_2^2, has to be added to the effective action. I believe that this result is also true in the case $g_2 < 0$, where the parquette method has to be applied. In the general case the coefficient before n_z^2 should be considered as a phemonenological parameter. The transition between the longitudinal and the transverse FISDW's, mentioned in the end of Sec. V, is related to the change of the sign of this coefficient.

Depending on the sign of this coefficient two situations are possible. In one case, $\mathbf{n} \parallel \mathbf{z}$ is favorable (longitudinal easy-axis FISDW). In this case the topological excitations of the $\mathbf{n}$ field are skyrmions, which are concentric domain walls between $\mathbf{n}$ up and down.

In another case, $\mathbf{n} \perp \mathbf{z}$ is favorable (transverse easy-plane FISDW). In this case the topological excitations are vertices with $\mathbf{n} \parallel \mathbf{z}$ in the vortex core. They are half-skyrmions, so they have spin $L/2$ and appropriate statistics. They interact logarithmically.

Further work is necessary to understand the physical consequences of the presence of term (46) and the possibilities of their experimental observation. One effect was suggested in Refs. 54 and 51. It is the spin quantum Hall effect: spins flow in the direction perpendicular to the direction of the gradient of H.

VII. CONCLUSIONS

The new results found in the paper are listed in this section.

The new (topological) method has been applied to the FISDW state to derive the formula $\sigma_{xy} = 2Le^2/h$ for the quantum Hall conductivity. The method is convenient, in particularly, in the case of coexistence of several order parameters. Obtained results may give an explanation of the experimental treelike phase diagram of $(TMTSF)_2ClO_4$.[15]

The dependence $L(H)$ has been calculated using the parquette method with the electron tunneling to the nearest and next-nearest neighboring chains taken into account. In the case of attraction between electrons (the sign of interaction favorable to superconductivity) the dependence $L(H)$ is oscillating in sign. In a strong magnetic field $|L|$ acquires the value 0 and then in a stronger field — the value 1. The FISDW transition temperature T_c as a function of magnetic field H has a global maximum at the value of $H \sim t_1 c/ebv$. There are also oscillations in the dependence $T_c(H)$ with the period $\Delta(1/H) \approx \pi ebv/8ct_1$, due to which the dependence $T_c(H)$ is nonmonotonic. These results are in qualitative agreement with the experimental behavior of

$(TMTSF)_2ClO_4$. The comparison with the behavior of $(TMTSF)_2PF_6$ is controversial.

With spins taken into account the two types of FISDW (easy-axis and easy-plane) are discussed. It is suggested that in $(TMTSF)_2ClO_4$ at $H = 17$ T the transition between these two types takes place.

In the effective action for the $\mathbf{n}$ field, where $\mathbf{n}$ is the polarization vector of the FISDW, a so-called Hopf term is found. It determines the spin and statistics of solitons of the $\mathbf{n}$ field. The interest to this term has grown recently in relation to high-T_c superconductivity.[49] But at present in high-T_c materials there are no serious indications on the presence of this term. So, quasi-one-dimensional organic conductors form the second class of materials after the ^{3}He-A films [51] where this term is *really* present.

Note added in proof. The coefficients of the Landau expansion, discussed in Sec. V, were calculated recently by Lebed'.[55] The coexistence of several FISDW's was actually found. After submission of this paper I learned of important papers[56–58] related to the subject considered.

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