Orthogonality exponent and the friction coefficient of an electron gas

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The Anderson orthogonality exponent K for the overlap of two ground states of a free-electron gas with a local potential at different positions is discussed. In the small-distance limit, it is shown that for arbitrary local potentials K can be expressed in terms of the friction tensor. In the one-dimensional case, K depends on the potential only via the friction coefficient for *arbitrary distance*, and the exponent is *bounded* for arbitrary shape of the potential in contrast to the two- or higher-dimensional case. The exact result for K is presented for a short-range potential on a lattice.

I. INTRODUCTION

In his paper[1] on the infrared catastrophe in Fermi gases with local scattering potentials, Anderson showed that the overlap of the many-body ground states with and without a local potential vanishes in the infinite-volume limit. In the language of quantum field theory, this is a manifestation of the problem of unitary inequivalent representations of the anticommutation relations for the field operators in an infinite system.[2] For a large but finite system, Anderson showed that the overlap goes to zero as a power law in the number of electrons in the system, and the exponent can be expressed in terms of the squared scattering phase shifts for a spherical potential.[1] From Anderson's expression of the exponent for a spherical potential, it was very suggestive to assume that the exponent for arbitrary local potentials can be expressed in terms of the trace of the square of the *phase-shift operator*. The proof of this fact was by no means straightforward.[3] This general result has been used to present a formal expression also for the overlap of two many-body ground states with the *same* local potential at *different positions* in the electron gas. This overlap plays an important role, e.g., in the theoretical description of muon diffusion in metals.[4]

A general calculation of the exponent as a function of the distance is rather complicated,[3,5–8] and not even for spherical potentials has an explicit expression in terms of the scattering phase shifts been derived. In the following, exact results are presented in various special cases. In the small-distance limit, it is shown that the overlap exponent can be exactly expressed in terms of the *friction coefficient* of a heavy particle giving rise to the local potential. Such a relation is known to hold for a bosonic heat bath[9] and was conjectured for the short-distance limit of an electronic bath by Sols and Guinea.[10] For spherical potentials, the friction coefficient is determined by the transport cross section,[11–14] for which a well-known expression in terms of the scattering phase shifts exists. For arbitrary nonspherical symmetric local potentials, the overlap exponent is expressed in terms of the friction tensor. For the one-dimensional case, it is shown that the overlap exponent depends on the potential only via the friction coefficient for *arbitrary* distances. This is in contrast to the two- and three-dimensional case. The exact result for the exponent K for electrons on a lattice is presented for a short-range potential.

In Sec. II two definitions of a "phase-shift operator" and their relation to the general result of Yamada and Yosida[3] are discussed. The general relation between the exponent and the friction coefficient is proven in Sec. III. After presenting the general result for one-dimensional potentials in Sec. IV, the model of electrons on a lattice is discussed in Sec. V. In the final section, Sec. VI, a comparison of the various exact results is given. A simple derivation of the relation between the exponent K and the friction coefficient without using the formal expression[3] for K is presented in the Appendix, and its extension to interacting electrons is discussed.

II. THE PHASE-SHIFT OPERATOR

As there are two different definitions for the phase-shift operator in the literature, it is useful to clarify their relation in connection with the Yamada-Yosida (YY) formula[3] for the overlap exponent.

The scattering by a spherical potential is best described in terms of angular-momentum states $|\varepsilon,l,m\rangle$, where ε is the (kinetic) energy of the particle and (l,m) are the angular-momentum quantum numbers. More generally, states $|\varepsilon,\alpha\rangle$ can be introduced, where α labels the "additional quantum numbers." The normalization of these states is chosen to be

$$\langle\varepsilon,\alpha|\varepsilon',\alpha'\rangle=\delta(\varepsilon-\varepsilon')\delta(\alpha,\alpha')\ , \tag{1}$$

where $\delta(\alpha,\alpha')$ is a Kronecker (Dirac) δ for discrete (continuous) quantum numbers α. The scattering matrix S defined in terms of the Møller operators can be expressed by the T matrix in the standard way:[15]

$$\langle\varepsilon,\alpha|S|\varepsilon',\alpha'\rangle=\delta(\varepsilon-\varepsilon')[\,\delta(\alpha,\alpha')-2\pi i\langle\varepsilon,\alpha|T(\varepsilon+i0)|\varepsilon,\alpha'\rangle\,]\ , \tag{2}$$

$$T(z)=V[1-G_0(z)V]^{-1}\ , \tag{3}$$

where $G_0(z)\equiv(z-H_0)^{-1}$ is the unperturbed resolvent, and V is the scattering potential. The expression multiplying the energy δ function on the right-hand side of Eq. (2) for *fixed energy* ε is a unitary matrix in the additional quantum numbers α,α',

$$[\underline{S}(\varepsilon)]_{\alpha\alpha'}\equiv\delta(\alpha,\alpha')-2\pi i\langle\varepsilon,\alpha|T(\varepsilon+i0)|\varepsilon,\alpha'\rangle\ . \tag{4}$$

The phase-shift matrix $\underline{\delta}(\varepsilon)$ is then defined as

$$\underline{\delta}(\varepsilon)\equiv\frac{1}{2i}\ln\underline{S}(\varepsilon)\ . \tag{5}$$

For a spherical potential, the phase-shift matrix is diagonal in the angular-momentum quantum numbers,

$$[\underline{\delta}(\varepsilon)]_{lm,l'm'}=\delta_{ll'}\delta_{mm'}\delta_l(\varepsilon)\ , \tag{6}$$

where the $\delta_l(\varepsilon)$ are the usual scattering phase shifts.

Alternatively, one can define an operator $\hat{S}(\varepsilon)$ in the full Hilbert space,[3]

$$\begin{aligned}\hat{S}(\varepsilon)&\equiv[1-G_0(\varepsilon-i0)V][1-G_0(\varepsilon+i0)V]^{-1}\\&=1-2\pi i\delta(\varepsilon-H_0)T(\varepsilon+i0)\ .\end{aligned} \tag{7}$$

This is *not* a unitary operator, but its *partial on-shell matrix elements* are related to the unitary scattering matrix S [see Eq. (2)] and the unitary matrix $\underline{S}(\varepsilon)$ defined in Eq. (4):

$$\langle\varepsilon'\alpha'|\hat{S}(\varepsilon)|\varepsilon,\alpha\rangle=\delta(\varepsilon-\varepsilon')[\underline{S}(\varepsilon)]_{\alpha'\alpha}\ . \tag{8}$$

The phase-shift operator is defined as

$$\hat{\delta}(\varepsilon)\equiv\frac{1}{2i}\ln\hat{S}(\varepsilon)\ . \tag{9}$$

The YY formula for the overlap of the many-body ground state $|i\rangle$ without the potential V and $|f\rangle$ with the potential V reads[3]

$$|\langle f|i\rangle|\sim(1/N)^{K_+}\ , \tag{10}$$

$$K_+=\tfrac{1}{2}\mathrm{Tr}[\hat{\delta}(\varepsilon_F)/\pi]^2\ , \tag{11}$$

where Tr denotes the trace in the (full) Hilbert space. If one defines a trace operation for matrices $\underline{A}$ in the additional quantum numbers (lower-case tr),

$$\mathrm{tr}\underline{A}\equiv\sum_\alpha(\underline{A})_{\alpha\alpha}\ , \tag{12}$$

a straightforward power expansion of the logarithm in the formula for the exponent K_+ shows that it can also be expressed in terms of the phase-shift matrix $\underline{\delta}(\varepsilon)$ and the lower-case tr,

$$K_+=\tfrac{1}{2}\mathrm{tr}[\underline{\delta}(\varepsilon_F)/\pi]^2\ . \tag{13}$$

It is a matter of convenience as to which of the two expressions for K_+ to use.

When a local potential V_i is included in the unperturbed Hamiltonian, which is *shifted* by $\mathbf{a}$ to present the potential V_f, straightforward algebraic manipulations using (7) yield[3]

$$K=-\frac{1}{8\pi^2}\mathrm{Tr}\{\ln[\hat{S}_f(\varepsilon_F)\hat{S}_i^{-1}(\varepsilon_F)]\}^2\ . \tag{14}$$

It is a simple exercise to show that K can alternatively be expressed in terms of the scattering matrices $\underline{S}(\varepsilon)$:

$$K=-\frac{1}{8\pi^2}\mathrm{tr}\{\ln[\underline{S}_f(\varepsilon_F)\underline{S}_i^{-1}(\varepsilon_F)]\}^2\ . \tag{15}$$

In Sec. III, Eq. (14) is the starting point for the discussion of the small-distance behavior of K, while Eq. (15) is used for the one-dimensional case in Sec. IV, as this is closer to the scattering description on the elementary textbook level.

III. RELATION WITH THE FRICTION COEFFICIENT

The scattering operator in the final state can be related to the scattering operator in the initial state with the help of the translation operator $U(\mathbf{a})$:[8]

$$\hat{S}_f(\varepsilon_F)=\hat{U}(\mathbf{a})\hat{S}_i(\varepsilon_F)\hat{U}^\dagger(\mathbf{a})\ , \tag{16}$$

$$\hat{U}(\mathbf{a})=\exp(i\hat{\mathbf{p}}\cdot\mathbf{a}/\hbar)\ . \tag{17}$$

In the following the arguments (as well as the carets) in $\hat{U}$ and $\hat{S}_i$ will be suppressed. To study the small-distance behavior, it is useful to commute S_i with $U^\dagger$,

$$US_iU^\dagger S_i^{-1}=U[S_i,U^\dagger]S_i^{-1}+UU^\dagger S_iS_i^{-1}\ . \tag{18}$$

This yields

$$K=-\frac{1}{8\pi^2}\mathrm{Tr}\{\ln(1+U[S_i,U^\dagger]S_i^{-1})\}^2\ , \tag{19}$$

which is a convenient starting point for the further discussion. To obtain the result for K quadratic in the distance, one can approximate $U^\dagger\approx1-i\hat{\mathbf{p}}\cdot\mathbf{a}/\hbar$, and $U\approx1$. The relation

$$[S_i,\hat{\mathbf{p}}\cdot\mathbf{a}]S_i^{-1}=-S_i[S_i^{-1},\hat{\mathbf{p}}\cdot\mathbf{a}]\ , \tag{20}$$

and the expansion of the logarithmic leads to

$$K=-\frac{1}{8\pi^2}\mathrm{Tr}([S_i,\hat{\mathbf{p}}\cdot\mathbf{a}][S_i^{-1},\hat{\mathbf{p}}\cdot\mathbf{a}])+O((k_Fa)^3)\ . \tag{21}$$

The trace is now performed using momentum eigenstates $|\mathbf{p}\rangle$, with the usual normalization $\langle\mathbf{p}|\mathbf{p}'\rangle=\delta(\mathbf{p}-\mathbf{p}')$. The matrix elements of the commutators in (21) can be expressed in terms of the T matrix using (7):

$$\langle\mathbf{p}|[\hat{S}_i,\hat{\mathbf{p}}\cdot\mathbf{a}]|\mathbf{p}'\rangle=2\pi i\delta(\varepsilon_F-\mathbf{p}^2/2m)(\mathbf{p}-\mathbf{p}')\cdot\mathbf{a}\langle\mathbf{p}|T_i(\varepsilon_F+i0)|\mathbf{p}'\rangle\ , \tag{22}$$

$$\langle\mathbf{p}'|[\hat{S}_i^{-1},\hat{\mathbf{p}}\cdot\mathbf{a}]|\mathbf{p}\rangle=2\pi i\delta(\varepsilon_F-\mathbf{p}'^2/2m)(\mathbf{p}-\mathbf{p}')\cdot\mathbf{a}\langle\mathbf{p}'|T_i(\varepsilon_F-i0)|\mathbf{p}'\rangle\ . \tag{23}$$

Therefore, one obtains

$$K=\frac{1}{2\hbar^2}\int d\mathbf{p}\int d\mathbf{p}'[(\mathbf{p}-\mathbf{p}')\cdot\mathbf{a}]^2|\langle\mathbf{p}|T(\varepsilon_F+i0)|\mathbf{p}'\rangle|^2\delta(\varepsilon_F-\mathbf{p}'^2/2m)\delta(\varepsilon_F-\mathbf{p}^2/2m)\ . \tag{24}$$

Because of the energy δ functions, only on-shell T-matrix elements occur which can be related to the differential scattering cross section $\sigma_{p_F}(\mathbf{n}\leftarrow\mathbf{n}')$, where $\mathbf{p}=p_F\mathbf{n}$, and $\mathbf{p}'=p_F\mathbf{n}'$. In three dimensions, one has

$$\sigma_{p_F}(\mathbf{n}'\leftarrow\mathbf{n})=(4\pi^2 m\hbar)^2|\langle p_F\mathbf{n}'|T(\varepsilon_F+i0)|p_F\mathbf{n}\rangle|^2 . \quad (25)$$

Performing the momentum integrations in (24) in spherical coordinates yields

$$K=\frac{p_F^4}{2(2\pi\hbar)^4}\int d\mathbf{n}\int d\mathbf{n}'[(\mathbf{n}-\mathbf{n}')\cdot\mathbf{a}]^2\sigma_{p_F}(\mathbf{n}\leftarrow\mathbf{n}') . \quad (26)$$

For a spherical potential, the differential cross section $\sigma_{p_F}(\mathbf{n}\leftarrow\mathbf{n}')$ depends on $\mathbf{n}\cdot\mathbf{n}'$ only, and the angular integrations can be expressed in terms of the "transport cross section" $\sigma_{\rm tr}(p_F)$:

$$K=\frac{4\pi}{3}\frac{p_F^4}{(2\pi\hbar)^4}\sigma_{\rm tr}(p_F)a^2 , \quad (27)$$

$$\sigma_{\rm tr}(p_F)\equiv\int(1-\cos\theta)\sigma_{p_F}(\cos\theta)d\Omega . \quad (28)$$

This should be compared with the result for the friction coefficient η for a heavy particle in a free-electron gas,[11–14,16] valid for velocities small compared with $v_F=p_F/m$,[13,14]

$$\eta=\frac{4\pi}{3}\frac{p_F^4}{(2\pi\hbar)^3}\sigma_{\rm tr}(p_F) . \quad (29)$$

Comparison with Eq. (27) leads to the relation

$$K=\frac{\eta a^2}{2\pi\hbar}+O((k_Fa)^3) , \quad (30)$$

which was conjectured by Sols and Guinea[10] from their linear-response calculation. The proof presented here is valid for any arbitrary strength of the potential. For the case of a spherical potential discussed in Eqs. (27)–(30), the transport cross section can be expressed in terms of the scattering phase shifts,[11–14]

$$\sigma_{\rm tr}(p_F)=\frac{4\pi}{(p_F/\hbar)^2}\sum_{l=0}^{\infty}(l+1)\sin^2[\delta_{l+1}(\varepsilon_F)-\delta_l(\varepsilon_F)] . \quad (31)$$

The short-distance behavior of the single-phase-shift model of Yamada and co-workers[5,6] is consistent with the exact relation Eq. (30).

For the case of a nonspherical potential, one has to go back to Eq. (26), and one obtains (using the repeated-index summation convention)

$$K=\frac{a_ia_j}{2\pi\hbar}\eta_{ij} , \quad (32)$$

where η_{ij} is the friction tensor:[11,14]

$$\eta_{ij}=\tfrac{1}{2}\frac{p_F^4}{(2\pi\hbar)^3}\int d\mathbf{n}\int d\mathbf{n}'(n_i-n_i')(n_j-n_j')\sigma_{p_F}(\mathbf{n}\leftarrow\mathbf{n}') \quad (33)$$

for velocities $v\ll v_F$.

It is tempting to use Eq. (30) or (32) to discuss whether there is an upper bound to K. As the transport cross section of a "hard potential" of radius R_0 is $\gtrsim R_0^2\pi$, it is easily seen that η itself is not bounded. But as the relation (30) between K and η resulted from an expansion in the distance a, one is *not* allowed to draw conclusions on the boundedness of K using Eq. (30). This can be seen explicitly for the case of a one-dimensional potential discussed in Sec IV.

IV. EXACT RESULT FOR ONE DIMENSION

As shown in Sec. III, the exponent K depends on the *dissipative* behavior of the electron gas in the small-distance regime. The behavior of K in the opposite limit $k_Fa\to\infty$ is known for the three- (and two-) dimensional case. Using $j_0(k_Fa)$, where $j_0(x)=\sin x/x$ is the spherical Bessel function of order zero, as an expansion parameter Yamada, Sakurai, and Takeshige[5] have shown that K approaches *twice* the value corresponding to *inserting* the local potential into the unperturbed Fermi sea. The corresponding expansion parameter in the 1D case would be $\sin(k_Fa)$, which does not go to zero in the limit $a\to\infty$. Therefore, the 1D long-distance behavior is very different from the 3D (and 2D) case, and it is worth studying in more detail. It turns out that it is very simple to produce an explicit result for K for arbitrary distances a.

First, a potential $V_i(x)$ is considered, which is different from zero in the interval $[x_-,x_+]$. For wave vector $k>0$, the scattering wave function has the form

$$\psi_{k+}(x)=\frac{1}{\sqrt{2\pi}}\begin{cases}e^{ikx}+Ae^{-ikx}, & \text{for } x<x_- ,\\ Be^{ikx}, & \text{for } x>x_+ .\end{cases} \quad (34)$$

The reflection and transmission amplitudes A and B directly determine the scattering matrix $\underline{S}_i(\varepsilon)$ (where $k=\sqrt{2m\varepsilon}/\hbar$):

$$\underline{S}_i(\varepsilon)=\begin{bmatrix}B & -A^*(B/B^*)\\ A & B\end{bmatrix} . \quad (35)$$

This follows by comparison with the Lippman-Schwinger equation[15] and the explicit form of $\langle x|G_0(\varepsilon_k+i0)|x'\rangle$. The values of the additional quantum number α are ±1 corresponding to right- or left-moving plane waves. If the potential is moved by a distance a, the scattering wave function has the same form as in Eq. (34) in a coordinate system centered at $x=a$. This corresponds to the replacements $A\to Ae^{2ika}$, $B\to B$, i.e.,

$$\underline{S}_f\underline{S}_i^{-1}=\begin{bmatrix}1+|A|^2(e^{-2ika}-1) & BA^*(1-e^{-2ika})\\ -AB^*(1-e^{2ika}) & 1+|A|^2(e^{2ika}-1)\end{bmatrix} . \quad (36)$$

The eigenvalues of this matrix are given by

$$\lambda_{1,2}=1-2R\sin^2(ka)\pm i\{1-[1-2R\sin^2(ka)]^2\}^{1/2}$$
$$=\exp\left[\pm i\tan^{-1}\left[\frac{2[\tilde{R}(1-\tilde{R})]^{1/2}}{1-2\tilde{R}}\right]\right] , \quad (37)$$

where $R\equiv|A|^2$ is the reflection coefficient, and

$\tilde{R} \equiv R \sin^2(ka)$. Using a well-known identity for $\tan^{-1}(x)$, the final result for K reads

$$K = \left[\frac{1}{\pi} \tan^{-1}[\tilde{R}/(1-\tilde{R})]^{1/2} \right]^2 . \tag{38}$$

For arbitrary potentials, this exponent is a universal function of the single variable $\tilde{R} = R \sin^2(k_F a)$. Of the *three* independent real variables determining $\underline{S}(\varepsilon_F)$, only *one* ($|A|$) enters. The friction coefficient in one dimension for $v \ll v_F$ is given by[13,16]

$$\eta = \frac{2\hbar k_F^2 R(k_F)}{\pi} . \tag{39}$$

Therefore, K can be expressed in terms of the friction coefficient for arbitrary distances a. In the limit $k_F a \ll 1$, one recovers

$$K = \frac{\eta a^2}{2\pi\hbar} + O((k_F a)^4) . \tag{40}$$

The general result that Eq. (38) shows for the case of spinless fermions discussed in this paper is that $K \leq \frac{1}{4}$ for arbitrary distances a ($K \leq \frac{1}{2}$ if spin is included). K is a periodic function of $k_F a$ and, in the limit $a \to \infty$, one does not obtain the "incoherent" result of two or more dimensions.

It is instructive to compare the result for K with the overlap of the unperturbed Fermi sea with the ground state with the potential present. The corresponding exponent K_+ is given by $\mathrm{tr}[\underline{\delta}(\varepsilon_F)/\pi]^2/2$. With $B \equiv |B| e^{i\varphi_B}$, the eigenvalues of the phase-shift matrix corresponding to $\underline{S}(\varepsilon)$ given in Eq. (35) are

$$\delta_{1,2} = [\varphi_B \pm \arctan(|A|/|B|)]/2 . \tag{41}$$

This leads to

$$\mathrm{tr}[\underline{\delta}(\varepsilon_F)/\pi] = \varphi_B(\varepsilon_F)/\pi = \Delta N \tag{42}$$

and

$$K_+ = \tfrac{1}{2}\mathrm{tr}[\underline{\delta}(\varepsilon_F)/\pi]^2 = \tfrac{1}{4}\left[[\varphi_B(\varepsilon_F)/\pi]^2 + \left[\frac{1}{\pi} \arctan\left\{ \frac{|A|}{|B|} \right\} \right]^2 \right] , \tag{43}$$

where ΔN is the number of electrons moved to "infinity" in the case of a repulsive potential. The different dependence on the scattering potential in K and K_+ is most clearly seen when the potential $V(x)$ is *smooth* and has a maximum value $V_{\max}$ smaller than ε_F. Then the reflection amplitude $|A(\varepsilon_F)|$ is negligible and leads to $K_+ = (\Delta N)^2/4$. When the range of the smooth potential is increased, ΔN also increases. The value of the *exponent K_+ is not bounded in contrast to K*. This shows the very different nature of K and K_+ in one dimension. This can easily be overlooked if one only works with very short-range potentials. Potentials of this type are considered in Sec. V for electrons on a lattice.

V. TIGHT-BINDING MODEL

In this section the electrons are assumed to move on a d-dimensional lattice. With each lattice site a localized state $|\mathbf{m}\rangle$ is associated. The range of the hopping matrix elements is irrelevant for the following discussion. The overlap exponent of the many-body ground states $|i\rangle$ with an impurity potential $\hat{V}_i = V|\mathbf{m}_i\rangle\langle\mathbf{m}_i|$ and $|f\rangle$ with an impurity potential $\hat{V}_f = V|\mathbf{m}_f\rangle\langle\mathbf{m}_f|$ can quite easily be calculated. Using Eq. (7), one obtains

$$\hat{S}_f(\varepsilon)\hat{S}_i^{-1}(\varepsilon) = 1 - 2\pi i \delta(\varepsilon - H_0)[|\mathbf{m}_f\rangle\langle\mathbf{m}_f| t(\varepsilon+i0) - |\mathbf{m}_i\rangle\langle\mathbf{m}_i| t(\varepsilon-i0)] + 4\pi^2 |t(\varepsilon+i0)|^2 \delta(\varepsilon-H_0)|\mathbf{m}_f\rangle\langle\mathbf{m}_f|\delta(\varepsilon-H_0)|\mathbf{m}_i\rangle\langle\mathbf{m}_i| , \tag{44}$$

where $t(\varepsilon+i0)$ is the c number which determines the single-impurity T matrix,

$$t(\varepsilon+i0) = \frac{V}{1 - G_{00}(\varepsilon+i0)V} , \tag{45}$$

and $G_{00}(z) \equiv \langle\mathbf{m}_i|(z-H_0)^{-1}|\mathbf{m}_i\rangle = \langle\mathbf{m}_f|(z-H_0)^{-1}|\mathbf{m}_f\rangle$ is the site-independent diagonal element of the unperturbed resolvent. In terms of the projection operator,

$$P \equiv |\mathbf{m}_i\rangle\langle\mathbf{m}_i| + |\mathbf{m}_f\rangle\langle\mathbf{m}_f| , \tag{46}$$

the right-hand side of Eq. (44) has the form $1 + \hat{A}\hat{P}$. As the expression (14) for K involves the *trace* $\mathrm{Tr}[\ln^2(1+\hat{A}\hat{P})]$, the exponent K is determined by the 2×2 matrix $\langle\mathbf{m}_\lambda|\hat{S}_f\hat{S}_i^{-1}|\mathbf{m}_{\lambda'}\rangle$ ($\lambda = i, f$). Its eigenvalues can easily be calculated,

$$\lambda_{1,2} = 1 - 2\pi i|t|^2(\rho_{00}^2 - |\rho_{fi}|^2) \pm 2i[\pi^2|t|^2(\rho_{00}^2 - |\rho_{fi}|^2) - \pi^4|t|^4(\rho_{00}^2 - |\rho_{fi}|^2)^2]^{1/2} , \tag{47}$$

where

$$\rho_{00}(\varepsilon) = \langle\mathbf{m}_i|\delta(\varepsilon-H_0)|\mathbf{m}_i\rangle = \langle\mathbf{m}_f|\delta(\varepsilon-H_0)|\mathbf{m}_f\rangle ,$$
$$\rho_{fi}(\varepsilon) = \langle\mathbf{m}_f|\delta(\varepsilon-H_0)|\mathbf{m}_i\rangle ,$$

and the optical theorem

$$\mathrm{Im}\, t(\varepsilon+i0) = -\pi^2 |t(\varepsilon+i0)|^2 \rho_{00}(\varepsilon)$$

has been used. This leads to a result for K which has the same form as in the one-dimensional case (38):

$$K=\left[\frac{1}{\pi}\tan^{-1}\sqrt{C/(1-C)}\right]^2 , \tag{48}$$

with

$$C=\pi^2|t(\varepsilon_F+i0)|^2[\rho_{00}^2(\varepsilon_F)-|\rho_{fi}(\varepsilon_F)|^2] . \tag{49}$$

The scattering properties of the single impurity $V|\mathbf{m}\rangle\langle\mathbf{m}|$ are determined by the phase shift

$$\exp[2i\delta(\varepsilon)]=1-2\pi i\rho_{00}(\varepsilon)t(\varepsilon+i0) ,$$

i.e.,

$$\sin^2\delta(\varepsilon)=\pi^2\rho_{00}^2(\varepsilon)|t(\varepsilon+i0)|^2 .$$

With the definition $x\equiv|\rho_{fi}(\varepsilon_F)|^2/\rho_{00}^2(\varepsilon_F)$, the constant C can therefore be written as

$$C=(1-x)\sin^2\delta(\varepsilon_F) . \tag{50}$$

This shows that the result (48) for K has exactly the same form as in the single-phase-shift model of Yamada, Sakurai, and Takeshige,[5] when $j_0^2(k_Fa)$ is replaced by $|\rho_{fi}(\varepsilon_F)|^2/\rho_{00}^2(\varepsilon_F)$. On the lattice, the distance dependence of K is determined by

$$\rho_{fi}(\varepsilon_F)=\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{m}_f-\mathbf{m}_i)}\delta(\varepsilon_F-\varepsilon_{\mathbf{k}}) , \tag{51}$$

where $\varepsilon_{\mathbf{k}}$ is the energy dispersion of the unperturbed band. Even for models with nearest-neighbor hopping matrix elements, only a few analytical results for $\rho_{fi}(\varepsilon)$ are available. For nearest-neighbor sites $\mathbf{m}_i$ and $\mathbf{m}_f$ on a simple (hyper)cubic lattice, the equation of motion for the resolvent $(z-H_0)^{-1}$ leads, e.g., to the result $\rho_{10}(\varepsilon)=(\varepsilon-\varepsilon_0)\rho_{00}(\varepsilon)/(2dt_{10})$, where ε_0 is the unperturbed site energy, and t_{10} is the hopping matrix element. For large distances, $\rho_{fi}(\varepsilon)$ goes to zero in two and more dimensions, i.e.,

$$K\rightarrow[\delta(\varepsilon_F)/\pi]^2=2K_+ \quad \text{as } |m_i-m_j|\rightarrow\infty , \tag{52}$$

where K_+ is the exponent for inserting a single impurity into the unperturbed Fermi sea, in agreement with the general result of Yamada, Sakurai, and Takeshige.[5]

VI. COMPARISON OF THE RESULTS

Various exact results for the overlap exponent K have been presented, and its dependence on the distance a and the number of spatial dimensions has been discussed. The nature of the *small-* distance behavior is *independent* of the spatial dimension. It is determined by the *dissipative* properties of the electron gas. The relation (30) with the friction coefficient shows via its dependence on the transport cross section $\sigma_{\rm tr}(\varepsilon_F)$ that the proportionality factor between K and a is small for smooth potentials for which forward scattering dominates. The *large*-distance behavior drastically *depends* on the number of spatial dimensions. In one dimension, K is an oscillatory function of the distance, and K is smaller than $\frac{1}{4}$ for all distances for the model of spinless fermions discussed in this paper ($K\leq\frac{1}{2}$ if spin is included). For dimension 2 or greater, K approaches $2K_+$ in the limit $a\rightarrow\infty$, where K_+ is the orthogonality exponent for introducing a single local potential into the unperturbed Fermi sea. As the exponent K_+ is not bounded, the exponent K is also *not bounded* in two or more dimensions.[10] An exact calculation of K for arbitrary distances a in two or more dimensions appears to be very difficult.[5–8] The exact result for K presented for a short-range potential on a lattice for arbitrary distances unfortunately does not a show a transition to $K>\frac{1}{4}$, as $K_+\leq\frac{1}{8}$ for arbitrary strength of the short-range potential. Therefore, additional work is needed to cover the generic case.

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APPENDIX

In this Appendix the relation $K=\eta a^2/(2\pi\hbar)$ between the exponent K and the friction coefficient η is derived *without* the use of the YY relation Eq. (14).

A large but *finite* system with periodic boundary conditions is considered, i.e., the exact one-electron states $|\mathbf{k}_m+\rangle_{(i)}$ and $|\mathbf{k}_n+\rangle_{(f)}$ are normalized by a Kronecker δ,

$${}_{(\alpha)}\langle\mathbf{k}_m+|\mathbf{k}_n+\rangle_{(\alpha)}=\delta_{mn} , \tag{A1}$$

where $\alpha=i,f$. The overlap of the many-body ground states is given by

$$\langle f|i\rangle=\mathrm{Det}\underline{A} , \tag{A2}$$

with

$$(\underline{A})_{mn}={}_{(f)}\langle\mathbf{k}_m+|\mathbf{k}_n+\rangle_{(i)} , \tag{A3}$$

where the matrix indices run over the *occupied* one-electron states. In the following the occupied states are labeled by $\underline{m},\underline{n},\ldots$, while unoccupied states are labeled by $\bar{r},\bar{s},\ldots$. If one is only interested in $|\langle f|i\rangle|$, it is convenient to calculate $|\langle f|i\rangle|^2=\mathrm{Det}(\underline{A}\,\underline{A}^\dagger)$, where

$$(\underline{A}\,\underline{A}^\dagger)_{\underline{m}\,\underline{n}}=\delta_{\underline{m}\,\underline{n}}-\sum_{\bar{r}}{}_{(f)}\langle\mathbf{k}_{\underline{m}}+|\mathbf{k}_{\bar{r}}+\rangle_{(i)}{}_{(i)}\langle\mathbf{k}_{\bar{r}}+|\mathbf{k}_{\underline{n}}+\rangle_{(f)} . \tag{A4}$$

Using the time-independent Schrödinger equation for $H_i=H_0+V(\hat{\mathbf{x}}-\mathbf{x}_i)$, and $H_f=H_0+V(\hat{\mathbf{x}}-\mathbf{x}_f)$, and the fact that energy eigenvalues are independent of the position of the potential, one obtains

$${}_{(f)}\langle\mathbf{k}_{\underline{m}}+|\mathbf{k}_{\bar{r}}+\rangle_{(i)}=\frac{{}_{(f)}\langle\mathbf{k}_{\underline{m}}+|(V_f-V_i)|\mathbf{k}_{\bar{r}}+\rangle_{(i)}}{\varepsilon_{\mathbf{k}_{\underline{m}}}-\varepsilon_{\mathbf{k}_{\bar{r}}}} . \tag{A5}$$

The determinant can now be evaluated using $\ln[\mathrm{Det}(\underline{A}\,\underline{A}^\dagger)]=\mathrm{tr}\ln(\underline{A}\,\underline{A}^\dagger)$, where the trace runs over

the *occupied* one-electron states. For small distances, $a=|\mathbf{x}_f-\mathbf{x}_i|$, the overlaps ${}_{(f)}\langle \mathbf{k}_{\underline{m}}+|\mathbf{k}_{\bar{r}}+\rangle_{(i)}$ are linear in a, and therefore up to quadratic terms in a, one can approximate $\ln \mathrm{tr}(\underline{A}\,\underline{A}^{\dagger})$ by $\mathrm{tr}(\underline{A}^{\dagger}\underline{A}-\underline{1})$. This leads to

$$\ln|\langle f|i\rangle|^2=-\sum_{\underline{m},\bar{r}}\frac{\left|{}_{(i)}\left\langle \mathbf{k}_{\underline{m}}+\left|\mathbf{a}\cdot\frac{\partial V_i}{\partial \mathbf{x}}\right|\mathbf{k}_{\bar{r}}+\right\rangle_{(i)}\right|^2}{(\varepsilon_{\mathbf{k}_{\underline{m}}}-\varepsilon_{\mathbf{k}_{\bar{r}}})^2}\times[1+O(k_F a)]\ . \tag{A6}$$

The expression on the right-hand side can be brought into a representation-independent form:

$$\ln|\langle f|i\rangle|^2=-\int_0^{\varepsilon_F}d\varepsilon\int_{\varepsilon_F}^{\infty}d\varepsilon'\frac{G(\varepsilon,\varepsilon')}{(\varepsilon-\varepsilon')^2}\ , \tag{A7}$$

with

$$G(\varepsilon,\varepsilon')=\mathrm{Tr}\left[\delta(\varepsilon-H_i)\mathbf{a}\cdot\frac{\partial V_i}{\partial \mathbf{x}}\delta(\varepsilon'-H_i)\mathbf{a}\cdot\frac{\partial V_i}{\partial \mathbf{x}}\right]\ , \tag{A8}$$

where Tr denotes the trace in the *full* Hilbert space of one-electron states. In the infinite-volume limit, the function $G(\varepsilon,\varepsilon')$ becomes a smooth function of the energy variables. The integral on the right-hand side of (A7) is logarithmically divergent with a strength determined by $G(\varepsilon_F,\varepsilon_F)$. The energy separation (inversely proportional to the volume) $\delta\varepsilon\sim 1/\mathcal{V}\sim 1/N$ presents a cutoff, and one obtains the Anderson power-law behavior with the exponent (to order a^2)

$$K=G(\varepsilon_F,\varepsilon_F)/2\ . \tag{A9}$$

This can be compared to the general expression for the energy-transfer rate in the adiabatic limit,[17,18]

$$\dot{\varepsilon}_t=\pi\hbar\,\mathrm{Tr}[\delta(\varepsilon_F-H_t)\dot{V}_t\delta(\varepsilon_F-H_t)\dot{V}_t]\ , \tag{A10}$$

where $H_t=H_0+V_t$ is an arbitrary slowly time-dependent Hamiltonian for the electron gas. The use of $V_t=V(\mathbf{x}-\mathbf{v}t)$, and the definition $\dot{\varepsilon}_t\equiv\eta_{ij}v_iv_j$, leads with (A8) and (A9) to the relation (32):

$$K=\frac{a_ia_j}{2\pi\hbar}\eta_{ij}\ . \tag{A11}$$

The *explicit* expression (29) for the friction coefficient can be obtained from (A10), e.g., by using Eq. (51) of Ref. 11.

The advantage of the simple derivation of the relation between the orthogonality exponent and the friction coefficient presented in this Appendix lies in the fact that no use is made of relation (11). The proof of this relation by Yamada and Yosida[3] is very complicated for arbitrary nonspherical potentials.

The relation between the exponent K and the friction coefficient η can be shown to hold also for *interacting* electrons, if one assumes the same qualitative behavior for the overlap as for noninteracting electrons. This can be done without performing an explicit calculation of either K or η. A formal expression for the friction coefficient of interacting electrons can be obtained by the bootstrap derivation of d'Agliano *et al.*[11] or by using an expansion around the adiabatic limit.[18] This yields, at $T=0$,

$$\eta_{\alpha\beta}=\pi\hbar\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\lim_{\mathcal{V}\to\infty}\left\langle\phi_0^{(i)}\left|\frac{\partial V_i}{\partial x_\alpha}\delta(\varepsilon-(H_i-E_0))\frac{\partial V_i}{\partial x_\beta}\right|\phi_0^{(i)}\right\rangle, \tag{A12}$$

where $|\phi_0^{(i)}\rangle\equiv|i\rangle$ is the many-body ground state with the impurity potential V_i. Here H_i and V_i are operators in Fock space. With use of the completeness relation (for finite volume) and the Schrödinger equation, the overlap between the initial and final ground state $|\phi_0^{(f)}\rangle$ can be expressed as

$$|\langle\phi_0^{(f)}|\phi_0^{(i)}\rangle|^2=1-\sum_{n\ (\neq 0)}|\langle\phi_n^{(f)}|\phi_n^{(i)}\rangle|^2=1-\int_0^{\infty}\frac{F(\varepsilon)}{\varepsilon}d\varepsilon\ , \tag{A13}$$

with

$$F(\varepsilon)=\frac{1}{\varepsilon}\langle\phi_0^{(i)}|(V_f-V_i)\delta(\varepsilon-(H_f-E_0))(V_f-V_i)|\phi_0^{(i)}\rangle\ . \tag{A14}$$

If one now *assumes* that the overlap has the same functional form as for noninteracting electrons, i.e., in the small-distance limit,

$$|\langle\phi_0^{(f)}|\phi_0^{(i)}\rangle|^2=\exp\{-a^2(c_0+c_1\ln N)[1+O(k_Fa)]\}\ , \tag{A15}$$

the result for c_1 can be read off in the limit $c_1a^2\ln N\ll 1$. Up to quadratic terms in a, $F(\varepsilon)$ is given by

$$F(\varepsilon)=a_ia_j\frac{1}{\varepsilon}\left\langle\phi_0^{(i)}\left|\frac{\partial V}{\partial x_i}\delta(\varepsilon-(H_i-E_0))\frac{\partial V}{\partial x_j}\right|\phi_0^{(i)}\right\rangle, \tag{A16}$$

and the term proportional to $\ln N$ in Eq. (A15) is given by $\lim_{\varepsilon\to 0}\lim_{\mathcal{V}\to\infty}F(\varepsilon)$. Comparison with (A12) leads again to the relation (32) or (A11):

$$K=\frac{a_i a_j}{2\pi\hbar}\eta_{ij} \ . \tag{A17}$$

In a more elaborate proof, one would start from the generalization of the YY relation (11) to interacting electrons, which involves the self-energy at the chemical potential,[3] and use a Ward identity for the friction coefficient.[19]

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