Nonlinear σ model for localization in superconductors: Role of order-parameter phase fluctuations

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The phenomenon of Anderson localization of normal excitations in dirty superconductors is considered using the functional representation in terms of the nonlinear σ model. The derivation of the effective functional allows for the slow spatial variations of the superconducting order-parameter phase. The thermal fluctuations of the order-parameter phase are shown to violate the equivalence between particle-particle and particle-hole diffusion propagators due to the broken time-reversal symmetry in the presence of random local supercurrents. As a result, a new characteristic length $l_f \propto T^{-1/2}$ appears in a system. If the correlation length $l_c \propto |n-n_c|^{-\nu}$ connected with the Anderson transition is larger than l_f, the transport properties of normal excitations are the same as in normal metals in a weak magnetic field. At $l_c \ll \xi$ (ξ is a superconducting coherence length) they are the same as in normal metals without magnetic field. The behavior of diffusion propagators in a crossover region $\xi \ll l_c \ll l_f$ is discussed, and a crossover exponent is calculated.

I. INTRODUCTION

The formalism of the nonlinear σ model first introduced by Wegner[1] turned out to be the most powerful tool for the description of the Anderson localization and related phenomena in disordered normal conductors. Within the framework of this approach, it was possible to consider the Anderson transition from metal to insulator in $2+\epsilon$ dimensions and to calculate various critical exponents connected with this transition. Recently, in a series of works,[2] this formalism was generalized to consider the transport properties of normal excitations in disordered superconductors, including superconducting glass limits. In these works it was shown, in particular, that if a nonrandom superconducting order parameter Δ is assumed, the Anderson localization of normal excitations may occur in superconductors, with the critical behavior of the diffusion coefficient being the same as in the case of a normal metal.

It is well known, however, that the thermal fluctuations of the order-parameter phase are extremely important for low-dimensional systems and may even destroy the long-range superconducting order. In contrast to numerous works in which the influence of order-parameter phase fluctuations on superconducting properties has been investigated for clean superconductors, we consider here the effect of phase fluctuations on the disorder-generated localization of normal excitations with energies $\varepsilon > |\Delta|$.

In this paper we shall consider the low-temperature limit of the problem and put the temperature $T=0$ everywhere, but in the distribution function for slow spatial fluctuations of the superconducting order-parameter phase $\varphi(\mathbf{r})$:

$$f[\varphi]=\exp\left[-\frac{1}{\tilde{T}}\int(\nabla\varphi)^2 d^d r\right]. \tag{1.1}$$

For the same reason we shall not take into account the configurations of $\varphi(\mathbf{r})$ containing vortices and neglect thermal fluctuations of the modulus of the superconducting order parameter. Moreover, the disorder-induced fluctuations of Δ also will not be considered, so that the symbol Δ will denote the modulus of the superconducting order parameter averaged over the ensemble of disordered samples.

In Sec. II we review the derivation of a functional representation for the retarded and advanced Gor'kov Green's functions in terms of the anticommuting Grassmann variables $\Psi(\mathbf{r})$ and $\bar{\Psi}(\mathbf{r})$. The averaging over a random impurity potential is carried out using the replica trick, which is well defined for all the perturbative calculations.[3] Though such a representation is exact, it contains an information about small spatial scales of the order of the inverse Fermi momentum p_F, which is superfluous for our problem. More suitable for the problem of a long-range diffusion is another functional representation analogous to the nonlinear σ model which describes the Anderson localization in normal metals.[1] In this representation the functional integration is performed over a matrix field $Q_{ij}(\mathbf{r})$ conjugated to the direct product $\Psi(\mathbf{r})\otimes\bar{\Psi}(\mathbf{r})$, which varies slowly at scales p_F^{-1}. The small spatial scales are taken into account in the derivation of the effective functional $\mathcal{F}[Q,\varphi]$. This functional is the main object of the theory. It determines the weight function $\exp(-\mathcal{F}[Q,\varphi])$ in all the functional averages

$$\langle\cdots\rangle=\frac{\int\cdots\exp(-\mathcal{F}[Q,\varphi])\mathcal{D}Q}{\int\exp(-\mathcal{F}[Q,\varphi])\mathcal{D}Q}, \tag{1.2}$$

which give the observable values at a fixed spatial distribution of the order-parameter phase $\varphi(\mathbf{r})$. In Sec. III the derivation of the effective functional $\mathcal{F}[Q,\varphi]$ is carried out, with special attention paid to the cross terms containing both $\nabla\varphi$ and Q variables. They are the terms which describe the influence of the order-parameter

phase fluctuations on the transport properties of normal excitations. Because the observable values should be averaged over the thermal fluctuations of the phase φ, one should integrate $\exp(-\mathcal{F}[Q,\varphi])$ over $\varphi(\mathbf{r})$ with the distribution function (1.1). As a result, a new effective functional

$$\mathcal{F}[Q]=-\ln\int\exp(-\mathcal{F}[Q,\varphi])f[\varphi]\mathcal{D}\varphi \tag{1.3}$$

is obtained which contains all the effects of the order-parameter phase fluctuations.

In fact, the calculations of the functional $\mathcal{F}[Q]$ have been performed in Sec. IV as an expansion in powers of the temperature $\tilde{T}$. The zero-order functional $\mathcal{F}_0[Q]$ is shown to coincide mainly with the functional of the nonlinear σ model[3] describing the Anderson localization in normal metals. The only difference is a proper redefinition of the density of states. It means that the homogeneous superconducting order parameter does not change both the mobility edge and the critical behavior of the diffusion constant near it.

However, the linear in $\tilde{T}$ term in (1.3) changes drastically the internal symmetry of the functional $\mathcal{F}[Q]$. All the elementary propagators $\langle Q(\mathbf{q})\otimes Q(\mathbf{q}')\rangle_0$ corresponding to the functional $\mathcal{F}_0[Q]$ are diffusionlike and proportional to $(Dq^2-i\omega)^{-1}$, where D is a diffusion coefficient and ω is a frequency. The linear in $\tilde{T}$ term makes some of the propagators nondivergent at $\omega=q=0$. They correspond to the particle-particle diffusion propagators, which can be expressed in terms of the following product of the retarded and advanced electronic Green's functions G^R and G^A averaged over a random impurity potential:

$$C_\varepsilon(q,\omega)=\int\langle\langle G^R_{\varepsilon+\omega/2}(\mathbf{r},\mathbf{r}')G^A_{\varepsilon-\omega/2}(\mathbf{r},\mathbf{r}')\rangle\rangle_{\text{imp}}e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}d^dq$$
$$\propto\frac{1}{D_c(q^2+l_f^2)-i\omega}\ . \tag{1.4}$$

At the same time the ordinary (particle-hole) diffusion propagators remain divergent at $\omega=q=0$:

$$D_\varepsilon(q,\omega)=\int\langle\langle G^R_{\varepsilon+\omega/2}(\mathbf{r},\mathbf{r}')G^A_{\varepsilon-\omega/2}(\mathbf{r}',\mathbf{r})\rangle\rangle_{\text{imp}}e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}d^dq$$
$$\propto\frac{1}{D_dq^2-i\omega}\ . \tag{1.5}$$

Equation (1.4) and (1.5) show that a new characteristic length $l_f\propto T^{-1/2}$ appears due to the thermal fluctuations of the order-parameter phase. For the spatial scales larger than l_f only divergent particle-hole diffusion propagators should be taken into account, or equivalently, one should consider matrix field Q of the lower symmetry corresponding to the absence of the "massive" excitations described by the nondivergent propagators. The critical behavior of a diffusion coefficient near the Anderson transition point $n=n_c$ (n is a concentration of impurities) is a consequence of the divergent correlation length $l_c\propto|n-n_c|^{-\nu}$ describing the long-range correlations in the electronic wave functions. In the vicinity of the Anderson transition, the essential spatial scale is of the order of l_c. That is why the linear in T effect of the order-parameter phase fluctuations is important in the close vicinity of the Anderson transition where $l_c>l_f$. The situation is quite analogous to one which takes place in normal metals in the presence of a weak magnetic field or paramagnetic impurities.[3] In both latter cases the Anderson localization effects are weaker as a result of the broken time-reversal symmetry and the critical exponent β of the diffusion coefficient $D_d\propto|n-n_c|^\beta$ is 2 times smaller than in the presence of time-reversal symmetry.[1,3] In the case of a superconductor with no external magnetic field, time-reversal symmetry is also broken because of the local random supercurrents. Therefore, it is not surprising that in superconductors, in the close vicinity of the Anderson transition, the diffusion coefficient of normal excitations has the same critical behavior as for normal metals in a weak magnetic field.

However, even at $l_c<l_f$ the order-parameter phase fluctuations are important. Our results outlined in Sec. IV show that the second order in the T term in the functional $\mathcal{F}[Q]$ leads to the difference in the coefficients D_c and D_d in the diffusion propagators (1.4) and (1.5). This difference becomes essential at $l_c>\xi$, where $\xi=hv_F/\Delta$ is a coherence length. In a crossover region $\xi<l_c<l_f$, the coefficient D_c decreases slower than D_d with increasing l_c, their difference increasing critically $D_c-D_d\propto T^2|n-n_c|^{-2}$.

The physical meanings of the diffusion propagators (1.4) and (1.5) are quite different. The ordinary (particle-hole) diffusion propagator $D_\varepsilon(q)$ describes the diffusion of one-particle excitations with the energy ε. In a superconductor the coefficient D_d determines, for example, the electronic heat conductance and ultrasonic attenuation. The particle-particle diffusion propagator is known to describe the electron interference effects.[4] It determines the Bohm-Aharonov oscillations of the conductance of normal metal cylinders or rings with the variation of the magnetic flux[5] or the oscillations of the resistance of a *SNS* junction with the variation of the order-parameter phase difference.[6] Both particle-particle and particle-hole propagators contribute to the mesoscopic fluctuations of the one-electron density of states.[7] As is mentioned in the Conclusion, the density-of-states correlations for different energies, measured in an ensemble of small superconducting particles, can give information about the difference in the coefficients D_c and D_d.

II. FUNCTIONAL REPRESENTATION OF GOR'KOV GREEN'S FUNCTIONS

We start with the Gor'kov equations[8] for the matrix retarded and advanced Green's functions written in the form[9]

$$\mathcal{G}^{R,(A)}_\varepsilon=\begin{pmatrix} G^{R,(A)} & F^{R,(A)} \\ -F^{+R,(A)} & \bar{G}^{R,(A)}\end{pmatrix}, \tag{2.1}$$

$$\hat{H}^{R,(A)}\mathcal{G}^{R,(A)}_\varepsilon=1\ , \tag{2.2}$$

where

$$\hat{H}^{R,(A)}=\begin{bmatrix} -\varepsilon_{\pm}+\frac{1}{2m}(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A})^2-\varepsilon_F+U(\mathbf{r}) & -\Delta e^{i\varphi(\mathbf{r})} \\ \Delta e^{-i\varphi(\mathbf{r})} & \varepsilon_{\pm}+\frac{1}{2m}(\hat{\mathbf{p}}+\frac{e}{c}\mathbf{A})^2-\varepsilon_F+U(\mathbf{r}) \end{bmatrix} . \tag{2.3}$$

Here $\varepsilon_{\pm}=\varepsilon\pm i0$, $\mathbf{A}$ is a vector potential of the external magnetic field (if it is applied to superconductor), and $U(\mathbf{r})$ is a random impurity potential, which is supposed to be a Gaussian random variable with zero average and a correlation function

$$\langle\langle U(\mathbf{r})U(\mathbf{r}')\rangle\rangle_{\text{imp}}=\frac{1}{\pi\nu_0\tau}\delta(\mathbf{r}-\mathbf{r}') , \tag{2.4}$$

where ν_0 is the density of states for normal metal at the Fermi energy ε_F; $\tau=l/v_F$ and l is an impurity scattering mean free path. The Hamiltonian (2.3) describes the Andreev scattering converting the particlelike electronic excitations with the energies ε into the holelike excitations with the energy $-\varepsilon$. Therefore, the 2×2 matrix space corresponding to the matrices in Eqs. (2.1)–(2.3) will be referred to as the "$(+\varepsilon,-\varepsilon)$ space." Using the vector Grassmann fields $\chi=(\chi_{\varepsilon},\chi_{-\varepsilon})$ and $\kappa=(\kappa_{\varepsilon},\kappa_{-\varepsilon})$ and Pauli matrices τ_{α} ($\alpha=x,y,z$) as a basis in this space, one can represent the retarded and advanced Green's functions $\mathcal{G}_{\varepsilon}^{R,A}$ in a form of functional integrals:

$$\mathcal{G}_{\varepsilon}(\mathbf{r},\mathbf{r}')^{R(A)}=-i\langle\kappa(\mathbf{r})\otimes\chi(\mathbf{r}')\rangle , \tag{2.5}$$

where the functional average is defined by

$$\langle\kappa(\mathbf{r})\otimes\chi(\mathbf{r}')\rangle=\frac{\int\kappa(\mathbf{r})\otimes\chi(\mathbf{r}')\exp(iS_{\varepsilon}^{R(A)})\mathcal{D}\kappa\,\mathcal{D}\chi}{\int\exp(iS_{\varepsilon}^{R(A)})\mathcal{D}\kappa\,\mathcal{D}\chi} . \tag{2.6}$$

In Eqs. (2.5) and (2.6), the action $S^{R(A)}$ is given by

$$\begin{aligned} S_{\varepsilon}^{R(A)}&=\int\chi(\mathbf{r})\hat{H}^{R,(A)}\kappa(\mathbf{r})d^dr \\ &=\int\chi(\mathbf{r})\left[-\varepsilon_{\pm}\tau_z+i\,\Delta\tau_y e^{-i\varphi\tau_z}\right. \\ &\quad+\frac{1}{2m}\left[-\hat{\mathbf{p}}_l-\frac{e}{c}\mathbf{A}\tau_z\right]\left[\hat{\mathbf{p}}_r-\frac{e}{c}\mathbf{A}\tau_z\right] \\ &\quad\left.-\varepsilon_F+U\right]\kappa(\mathbf{r})d^dr , \end{aligned} \tag{2.7}$$

where the momentum operators $\hat{\mathbf{p}}_l$ and $\hat{\mathbf{p}}_r$ act to the left and right, respectively. These operators have different signs because of the integration by parts.

One can remove the term $\exp(-i\varphi\tau_z)$ from the action (2.7) making the unitary transformation of Grassmann fields:

$$\chi\to\chi e^{-i\varphi\tau_z/2},\quad \kappa\to e^{i\varphi\tau_z/2}\kappa . \tag{2.8}$$

Such transformations lead to the replacement of $\hat{H}$ in (2.7) by a new operator:

$$\mathcal{H}^{R,(A)}=-\varepsilon_{\pm}\tau_z+i\,\Delta\tau_y+\frac{1}{2m}\Pi_l\Pi_r-\varepsilon_F+U , \tag{2.9}$$

where

$$\Pi_l=-\hat{\mathbf{p}}_l+\frac{1}{2}\left[\nabla\varphi-\frac{2e}{c}\mathbf{A}\right]\tau_z , \tag{2.10}$$

$$\Pi_r=\hat{\mathbf{p}}_r+\frac{1}{2}\left[\nabla\varphi-\frac{2e}{c}\mathbf{A}\right]\tau_z . \tag{2.11}$$

Now it is seen that the order-parameter phase enters in the action as part of a guage field:

$$\mathbf{F}=\nabla\varphi-\frac{2e}{c}\mathbf{A} . \tag{2.12}$$

The next step is to introduce the bi-Grassmann fields

$$\Psi_{\pm\varepsilon}=\frac{1}{\sqrt{2}}\begin{bmatrix}\chi_{\pm\varepsilon}\\ \kappa_{\pm\varepsilon}\end{bmatrix}\quad\text{and}\quad\overline{\Psi}_{\pm\varepsilon}=\frac{1}{\sqrt{2}}(-\kappa_{\pm\varepsilon}\chi_{\pm\varepsilon}) , \tag{2.13}$$

instead of $\kappa_{\pm\varepsilon}$ and $\chi_{\pm\varepsilon}$. Such an extension makes it possible to use the quarternionic algebra for the elements of the matrix Q and to express explicitly the change in its symmetry due to the lack of time-reversal invariance. That is why the 2×2 matrix space corresponding to the definition (2.13) will be referred to as "quarterionic space."

Taking into account the anticommutative nature of Grassmann fields as well as the properties of Pauli matrices $\tau_z^T=\tau_z$, $\tau_y^T=-\tau_y$, and integrating by parts in Eq. (2.7), one can express the action (2.7) in the following form:

$$S_{\varepsilon}^{R(A)}=-\int\kappa(\mathbf{r})\overline{\mathcal{H}}^{R(A)}\chi(\mathbf{r})d^dr , \tag{2.14}$$

where

$$\overline{\mathcal{H}}^{R(A)}=-\varepsilon_{\pm}\tau_z-i\,\Delta\tau_y+\frac{1}{2m}\Pi_l'\Pi_r'-\varepsilon_F+U . \tag{2.15}$$

In Eq. (2.15) the operators $\Pi_{l,r}'$ can be obtained from (2.10) and (2.11) by replacing $\tau_z\to-\tau_z$. Combining (2.7) and (2.15) and using the definition (2.13), we have

$$\begin{aligned} S_{\varepsilon}^{R(A)}=\int\overline{\Psi}\left[-(\varepsilon_{\pm}\tau_z+i\,\Delta\tau_y\otimes\sigma_z)\right. \\ \left.+\frac{1}{2m}\hat{\Pi}_l\hat{\Pi}_r-\varepsilon_F+U\right]\Psi d^dr , \end{aligned} \tag{2.16}$$

where $\overline{\Psi}=(\overline{\Psi}_{\varepsilon},\overline{\Psi}_{-\varepsilon})$, $\Psi=(\Psi_{\varepsilon},\Psi_{-\varepsilon})^T$, and the operators $\hat{\Pi}_{l,r}$ are given by

$$\hat{\Pi}_l=-\hat{\mathbf{p}}_l-\tfrac{1}{2}\mathbf{F}\tau_z\otimes\sigma_z , \tag{2.17}$$

$$\hat{\Pi}_r=\hat{\mathbf{p}}_r-\tfrac{1}{2}\mathbf{F}\tau_z\otimes\sigma_z . \tag{2.18}$$

Here σ_{α} are Pauli matrices in the quarternionic space.

The expressions (2.16)–(2.18) generalize the functional representation for electronic Green's functions (see, e.g., Ref. 3) to allow for the superconducting order parameter. For the calculation of the low-frequency transport properties of normal excitations, one needs the value of the product $G^R_{\varepsilon+\omega/2}G^A_{\varepsilon-\omega}/2$ averaged over a random impurity potential. The functional representation for this product can be obtained from (2.5) and (2.6). The corresponding action is given by $S=S^R_{\varepsilon+\omega/2}+S^A_{\varepsilon-\omega/2}$, where $\omega \ll \varepsilon$ is a frequency (the Planck constant is omitted throughout the paper). Such a sum may be expressed in a form analogous to (2.16) if we label the fields Ψ and $\bar{\Psi}$ in the expression for S^R and S^A by the superscripts R and A, respectively, and enlarge the vector space Ψ to include both Ψ^R and Ψ^A fields. The difference in energies $\varepsilon\pm(\omega/2+i0)$ results in the additional [with respect to (2.16)] term in the action S, which absorbs $\pm i0$ in $\varepsilon_\pm$:

$$S_\omega=-\frac{\omega_+}{2}\int \bar{\Psi}\tau_z\otimes\rho_z\Psi\, d^dr \ . \tag{2.19}$$

In (2.19), $\omega_+=\omega+i0$ and ρ_z is the Pauli matrix in (R,A) space.

In averaging the product of Green's functions (2.5), we meet, as usual, with the problem of averaging the ratio of two functional integrals, each of them depending on a random impurity potential. This problem is solved here by the replica trick, which is known to be correct for the perturbative calculations made below (see, e.g., Refs. 1–4). After replicating, the fields $\bar{\Psi}$ and Ψ become $8N$-component vectors, where the number of replica components N should be set to zero in the final results. Now it is possible to average numerator and denominator in (2.5) separately. The averaging of $\exp(iS)$ over a Gaussian random potential $U(\mathbf{r})$ results in some interaction term $S_{\rm int}$ in the action instead of the last term in (2.16):

$$S=S_0+S_{\rm int}+S_\omega \ , \tag{2.20}$$

where S_ω is given by (2.19),

$$S_0=\int\bar{\Psi}\left[-(\varepsilon\tau_z+i\,\Delta\tau_y\otimes\sigma_z)+\frac{1}{2m}\hat{\Pi}_l\hat{\Pi}_r-\varepsilon_F\right]\Psi\, d^dr \ , \tag{2.21}$$

$$S_{\rm int}=\frac{i}{2\pi\nu_0\tau}\int(\bar{\Psi}\Psi)^2 d^dr \ . \tag{2.22}$$

In Eqs. (2.21)–(2.22) the $8N\times 8N$ matrix operators $\hat{\Pi}_{l,r}$ are determined by Eqs. (2.17), (2.18), and (2.12) (in all the expressions the unit matrices are omitted).

Now we show how to calculate the particle-particle and particle-hole diffusion propagators (1.4) and (1.5) using the functional representation (2.20)–(2.22). From Eqs. (2.5) and (2.8), it follows that

$$D_\varepsilon(\mathbf{r}-\mathbf{r}')=\langle(\chi^A_\varepsilon\kappa^R_\varepsilon)_{\mathbf{r}}(\chi^R_\varepsilon\kappa^A_\varepsilon)_{\mathbf{r}'}\rangle_S \tag{2.23}$$

and

$$C_\varepsilon(\mathbf{r}-\mathbf{r}')=\langle(\kappa^A_\varepsilon\kappa^R_\varepsilon)_{\mathbf{r}}(\chi^R_\varepsilon\chi^A_\varepsilon)_{\mathbf{r}'}\rangle_S e^{i[\varphi(\mathbf{r})-\varphi(\mathbf{r}')]} \ , \tag{2.24}$$

where the symbol $\langle\cdots\rangle_S$ denotes the functional average corresponding to the action S. It is seen that both $D_\varepsilon(\mathbf{r}-\mathbf{r}')$ and $C_\varepsilon(\mathbf{r}-\mathbf{r}')$ are the two-point correlation functions of the matrix

$$\mathbf{q}_{\alpha,\beta}=\Psi_\alpha\otimes\bar{\Psi}_\beta=\frac{1}{2}\begin{bmatrix}-\chi_\alpha\kappa_\beta & \chi_\alpha\chi_\beta\\ -\kappa_\alpha\kappa_\beta & \kappa_\alpha\chi_\beta\end{bmatrix} , \tag{2.25}$$

where α and β are all the matrix indices except those of the quarternionic space which are shown explicitly. One can confirm, using the rule for the conjugation of Grassmann variables $\chi^*=-\kappa$, $\kappa^*=\chi$,[10] that the matrix $\hat{q}_{\alpha,\beta}$ is a real-quarternionic matrix;[3] i.e., it can be represented as a linear combination of quarternions $\hat{\eta}_i$ with real coefficients a_i:

$$\hat{q}_{\alpha,\beta}=\sum_{i=0}^{3}a_i^{\alpha,\beta}\hat{\eta}_i \ , \tag{2.26}$$

where $\hat{\eta}_0=1$ and $\hat{\eta}_{1,2,3}=-i\sigma_{1,2,3}$. With the definition (2.25) we have

$$D_\varepsilon(\mathbf{r}-\mathbf{r}')=\frac{4}{N^2}\mathrm{tr}\langle(q^{A,R}_{\varepsilon,\varepsilon})_{11}(\mathbf{r})(q^{R,A}_{\varepsilon,\varepsilon})_{11}(\mathbf{r}')\rangle_S \ , \tag{2.27}$$

$$C_\varepsilon(\mathbf{r}-\mathbf{r}')=-\frac{4}{N^2}\mathrm{tr}\langle(q^{A,R}_{\varepsilon,\varepsilon})_{21}(\mathbf{r})(q^{R,A}_{\varepsilon,\varepsilon})_{12}(\mathbf{r}')\rangle_S \times e^{i\varphi(\mathbf{r})-i\varphi(\mathbf{r}')} \ , \tag{2.28}$$

where the symbol tr denotes the trace over replica indices. We have symmetrized the expressions (2.27) and (2.28) with respect to the replica indices using the replica symmetry of the functional S.

Equation (2.27) and (2.28) show that the particle-hole diffusion propagator D_ε is a correlation function of the diagonal quarternionic components a_1-ia_3 of the matrix $\hat{q}$. On the contrary, the particle-particle diffusion propagator C_ε is a correlation function of the off-diagonal quarternionic components $-ia_1\pm a_2$.

These correlation functions can also be expressed in terms of the generating functional

$$Z[\hat{h}]=\frac{\int\exp(iS_h)\exp(iS)\mathcal{D}\chi\,\mathcal{D}\kappa}{\int\exp(iS)\mathcal{D}\chi\,\mathcal{D}\kappa} \ , \tag{2.29}$$

where

$$S_h=\int\bar{\Psi}\hat{h}\Psi\, d^dr \ . \tag{2.30}$$

The second variational derivative of $Z[\hat{h}]$ with respect to the corresponding components of the infinitesimal matrix $\hat{h}$ gives the correlation functions (2.27) and (2.28).

It is important that the sum $\mathcal{S}=S_h+S$ determining the generating functional (2.29) can be entirely expressed in terms of the matrix $\hat{q}$. Using the definition (2.25) and taking into account the anticommutation of fields Ψ and $\bar{\Psi}$, one obtains, from Eqs. (2.20)–(2.22) and (2.30),

$$\mathcal{S}=-\frac{i}{2\pi\nu_0\tau}\int\mathrm{Tr}\hat{q}^2 d^dr-\int\mathrm{Tr}(\hat{q}\hat{h})d^dr + \int\mathrm{Tr}\left[\hat{q}\left[\varepsilon\tau_z+i\,\Delta\tau_y\otimes\sigma_z+\varepsilon_F -\frac{1}{2m}\hat{\Pi}_l\hat{\Pi}_r+\frac{\omega_+}{2}\tau_z\otimes\rho_z\right]\right]d^dr \ , \tag{2.31}$$

where the symbol Tr denotes the matrix trace. Thus the integrands of both the functional integrals in Eq. (2.29) are expressed in terms of $\hat{q}$. It is natural, therefore, to replace the integration over χ and κ by the integration over a new variable similar to $\hat{q}$.

III. DERIVATION OF THE NONLINEAR σ MODEL

The proper mathematical procedure of the change in variables of integration is based on the Hubbard-Stratonovich transformation. This procedure reduces to the multiplication of the numerator and the denominator in (2.29) by the following functional integral over a new matrix variable $\mathcal{P}$:

$$\int \exp\left[-a\int \mathrm{Tr}\mathcal{P}^2 d^2r\right]\mathcal{D}\mathcal{P} , \tag{3.1}$$

and to a proper shift of this variable in order to cancel both the interaction term proportional to $\hat{q}^2$ and the term (if any) proportional to $\hat{h}$ in the expression (2.31). The explicit form of this shift is

$$\mathcal{P}\to\mathcal{P}-\frac{1}{\sqrt{2\pi a\nu_0\tau}}\hat{q}+i\left[\frac{\pi\nu_0\tau}{2a}\right]^{1/2}\hat{h} , \tag{3.2}$$

where a is an arbitrary positive constant. (For future simplifying we choose it to be equal to $\pi\nu_0/8\tau$.) Then the Gaussian integration over χ and κ can be performed to obtain the effective functional $\mathcal{F}[\mathcal{P},\hat{h}]$:

$$\mathcal{F}[\mathcal{P},\hat{h}]=\frac{\pi\nu_0}{8\tau}\int \mathrm{Tr}\mathcal{P}^2 d^dr - \frac{i\pi\nu_0}{2}\int \mathrm{Tr}(\mathcal{P}\hat{h})d^dr-\frac{1}{2}\mathrm{Tr}\ln\hat{H}(\mathcal{P}) , \tag{3.3}$$

where $\mathcal{P}$, as well as $\hat{q}$ and $\hat{h}$, are $8N\times 8N$ real-quarternionic matrices,

$$\hat{H}(\mathcal{P})=-(\varepsilon\tau_z+i\,\Delta\tau_y\otimes\sigma_z)+\frac{1}{2m}\hat{\Pi}_l\hat{\Pi}_r - \varepsilon_F+\frac{i}{2\tau}\mathcal{P}-\frac{\omega_+}{2}\tau_z\otimes\rho_z . \tag{3.4}$$

The factor $\frac{1}{2}$ in the Tr ln term of Eq. (3.3) arises because of the bi-Grassmann nature (2.13) of the field Ψ.

It is seen from Eq. (3.3) that the infinitesimal field $\hat{h}$ enters in $\mathcal{F}[\mathcal{P},\hat{h}]$ in the same way as in the functional $\mathcal{S}[\hat{q},\hat{h}]$ [Eq. (2.31)]. Therefore, the differentiation of $Z[\hat{h}]$ with respect to $\hat{h}$ gives the expressions for the diffusion propagators (1.4) and (1.5) in a form analogous to Eqs. (2.27) and (2.28), with the fields $\mathcal{P}$ standing for $\hat{q}$ in all the functional averages:

$$\langle\mathcal{P}(\mathbf{r})\otimes\mathcal{P}(\mathbf{r}')\rangle_{\mathcal{F}}=\frac{\int\mathcal{P}(\mathbf{r})\otimes\mathcal{P}(\mathbf{r}')\exp(-\mathcal{F}[\mathcal{P}])\mathcal{D}\mathcal{P}}{\int\exp(-\mathcal{F}[\mathcal{P}])\mathcal{D}\mathcal{P}} , \tag{3.5}$$

where $\mathcal{F}[\mathcal{P}]=\mathcal{F}[\mathcal{P},0]$.

The advantage of the $\mathcal{P}$ representation is that the saddle-point approximation applied to the functional $\mathcal{F}(\mathcal{P})$ is parametrically justified. In the hydrodynamic limit of small frequencies ω and phase gradients $\mathbf{F}$, one can consider corresponding terms in (3.4) as perturbations and search for the spatially homogeneous saddle-point solution $\mathcal{P}_0$ at $\omega=\mathbf{F}=0$. In this limit the operator $\hat{H}$ in (3.4) is diagonal in the momentum representation. Replacing the integration over momenta $\mathbf{p}$ by the integration over the energetic variable $\xi=p^2/2m-\varepsilon_F$, we get, after the differentiation of (3.3) with respect to $\mathcal{P}$,

$$\mathcal{P}_0=\frac{i}{\pi}\int d\xi\left[\xi-\hat{E}+\frac{i}{2\tau}\mathcal{P}_0\right]^{-1} , \tag{3.6}$$

where

$$\hat{E}=(\varepsilon+i\delta\rho_z)\otimes\tau_z+i\,\Delta\tau_y\otimes\sigma_z , \tag{3.7}$$

and $\delta=+0$. In order to find the saddle-point matrix $\mathcal{P}_0$ from this equation, we first diagonalize the matrix $\hat{E}$:

$$\hat{E}=U_0E(\varepsilon)\otimes\tau_zU_0^{-1} , \tag{3.8}$$

where

$$E(\varepsilon)=[(\varepsilon+i\delta\rho_z)^2-\Delta^2]^{1/2} =(|\varepsilon^2-\Delta^2|)^{1/2}\begin{cases}1 & \text{if } \varepsilon^2-\Delta^2>0,\\ i\rho_z & \text{otherwise .}\end{cases} \tag{3.9}$$

The matrix $U_0=-U_0^{-1}$ is given by

$$U_0=(E_+-E_-)^{-1/2}(\sqrt{E_-}\tau_z+i\sqrt{E_+}\tau_y\otimes\sigma_z) , \tag{3.10}$$

where $E_\pm=\varepsilon\pm E(\varepsilon)$.

In deriving (3.9) we took into account that the function $(\varepsilon^2-\Delta^2)^{1/2}$ should be chosen to be analytical in the complex plane of the variable ε, with the cut being along the real axis from $-\Delta$ to Δ.

Multiplying Eq. (3.6) by U_0^{-1} and U_0 from the left and right, respectively, we have

$$\Lambda=\frac{i}{\pi}\int\frac{d\xi}{\xi-E(\varepsilon)\otimes\tau_z+(i/2\tau)\Lambda} , \tag{3.11}$$

where

$$\mathcal{P}_0=U_0\Lambda U_0^{-1} . \tag{3.12}$$

For large values of electronic bandwidth $J\sim\varepsilon_F \gg\Delta,\varepsilon,\tau^{-1}$, we can consider the limits of integration in (3.6) and (3.11) to be $\pm\infty$. In this limit the diagonal solution of Eq. (3.11) in all cases is given by

$$\Lambda=-\tau_z\otimes\rho_z . \tag{3.13}$$

This solution does not depend on the superconducting order parameter. At $\Delta=0$ (and $U_0=1$), it is evident that this is just the solution which provides the absence of a singularity in (3.3) and (3.4) in the upper half-plane of the complex variable ω.

Using the diagonal in the (R,A) space structure (3.10) of the matrix U_0 and taking into account Eqs. (3.8) and (3.9), one has, from (3.12) and (3.13),

$$\mathcal{P}_0=\begin{cases}-\rho_z\hat{s} & \text{if } \varepsilon^2-\Delta^2>0,\\ -i\hat{s} & \text{otherwise ,}\end{cases} \tag{3.14}$$

where the matrix $\hat{s}$ is introduced so that $\hat{s}^2=1$:

$$\hat{s}=\frac{\hat{E}}{(|\varepsilon^2-\Delta^2|)^{1/2}} . \tag{3.15}$$

It is important that, even at $\Delta\neq 0$, the solution (3.14) is not unique at $\varepsilon^2-\Delta^2>0$. Beyond the gap there is a whole manifold of solutions of the saddle-point equation (3.6). Indeed, multiplying Eq. (3.6) by the unitary matrices $U^\dagger$ and U from the left and right, respectively, one obtains the same equation for the quantity $U^\dagger\mathcal{P}_0U$, if U is a constant matrix commuting with $\hat{s}$:

$$[U,\hat{s}]=0 . \tag{3.16}$$

It means that at $\varepsilon^2>\Delta^2$ all matrices

$$\mathcal{P}=U^\dagger\mathcal{P}_0U=-\hat{s}\mathcal{Q} , \tag{3.17}$$

where

$$\mathcal{Q}=U^\dagger\rho_zU \tag{3.18}$$

are also the solutions of the saddle-point equation (3.6). However, for $\varepsilon^2<\Delta^2$ the solution $\mathcal{P}_0$ is unique. The reason is that in this case the matrix $\mathcal{P}_0$ commutes with U and $U^\dagger\mathcal{P}_0U=\mathcal{P}_0$.

In order to find the symmetry of the matrix U obeying (3.16), we first represent it in a form

$$\begin{aligned} &U=e^{-\mathcal{W}/2} , \\ &\mathcal{W}^\dagger=-\mathcal{W} , \end{aligned} \tag{3.19}$$

where $\mathcal{W}$ is a $8N\times 8N$ anti-Hermitian real-quarternionic matrix. As usual,[1,2,4] the matrix $\mathcal{W}$ should be chosen to be off diagonal in the (R, A) space in order to exclude the matrices U which commute with ρ_z and, hence, give $\mathcal{P}=\mathcal{P}_0$ [see (3.17)]. The new requirements resulting from (3.16) may be expressed in terms of the components of the matrix $\mathcal{W}$ in the $(+\varepsilon,-\varepsilon)$ space:

$$\begin{aligned} &\mathcal{W}_{\varepsilon,-\varepsilon}=-\sigma_z\mathcal{W}_{-\varepsilon,\varepsilon}\sigma_z , \\ &\Delta(\mathcal{W}_{\varepsilon,\varepsilon}\sigma_z-\sigma_z\mathcal{W}_{-\varepsilon,-\varepsilon})=2\varepsilon\mathcal{W}_{\varepsilon,-\varepsilon} . \end{aligned} \tag{3.20}$$

In the derivation of Eq. (3.20), we have omitted the infinitesimal term in (3.7) proportional to ρ_z because it is essential only at $\varepsilon^2-\Delta^2<0$ when the saddle-point solution does not depend on U. The first requirement [Eq. (3.20)] together with the anti-Hermiticity condition $\mathcal{W}^\dagger_{\varepsilon,-\varepsilon}=-\mathcal{W}_{-\varepsilon,\varepsilon}$ gives

$$\mathcal{W}^\dagger_{\varepsilon,-\varepsilon}=\sigma_z\mathcal{W}_{\varepsilon,-\varepsilon}\sigma_z . \tag{3.21}$$

Performing the Hermitian conjugation of the second equation [Eq. (3.20)], one obtains, with the use of Eq. (3.21),

$$\Delta(\mathcal{W}_{\varepsilon,\varepsilon}\sigma_z-\sigma_z\mathcal{W}_{-\varepsilon,-\varepsilon})=-2\varepsilon^*\mathcal{W}_{\varepsilon,-\varepsilon} . \tag{3.22}$$

For real ε considered here, the comparison of Eqs. (3.20) and (3.22) leads to

$$\begin{aligned} &\mathcal{W}_{\varepsilon,-\varepsilon}=0 , \\ &\Delta(\mathcal{W}_{\varepsilon,\varepsilon}\sigma_z-\sigma_z\mathcal{W}_{-\varepsilon,-\varepsilon})=0 . \end{aligned} \tag{3.23}$$

In the normal state $\Delta=0$, the second equation [Eq. (3.23)] is trivial and the diagonal parts $\mathcal{W}_{\varepsilon,\varepsilon}$ and $\mathcal{W}_{-\varepsilon,-\varepsilon}$ are independent. At $\Delta\neq 0$ only matrices $\mathcal{W}$ with connected diagonal elements correspond to a degenerate saddle-point:

$$\mathcal{W}_{-\varepsilon,-\varepsilon}=\sigma_z\mathcal{W}_{\varepsilon,\varepsilon}\sigma_z . \tag{3.24}$$

The conditions (3.23) and (3.24) are valid for any power of $\mathcal{W}$. Therefore, they are also valid for the matrix $\mathcal{Q}$ [Eq. (3.18)] which determines the manifold of the saddle-point solutions [Eq. (3.17)]:

$$\mathcal{Q}_{\varepsilon,-\varepsilon}=0, \quad \mathcal{Q}_{-\varepsilon,-\varepsilon}=\sigma_z\mathcal{Q}_{\varepsilon,\varepsilon}\sigma_z . \tag{3.25}$$

The meaning of the matrix $\mathcal{Q}$ is clear from (3.17). The variable $\mathcal{Q}$ corresponds to the genuine normal excitations in a superconductor, while the variable $\mathcal{P}$ describes the diffusion of electronlike and holelike excitations which can be converted into each other as a result of the Andreev scattering. That is why the off diagonal in the $(+\varepsilon,-\varepsilon)$ space terms are present in the matrix $\mathcal{P}$, while they are absent in the matrix $\mathcal{Q}$. The diagonal parts $\mathcal{Q}_{\varepsilon,\varepsilon}$ and $\mathcal{Q}_{-\varepsilon,-\varepsilon}$ of the matrix $\mathcal{Q}$ describe the diffusion of normal excitations with the energy ε and $-\varepsilon$ (with respect to the Fermi energy). We shall see below that in the absence of the order-parameter phase fluctuations, the diffusion of both excitations is independent, but the independence breaks down when one takes into account these fluctuations. Nevertheless, if we consider the effect of phase fluctuations perturbatively, we need only one component

$$\mathcal{Q}_{\varepsilon,\varepsilon}=Q \tag{3.26}$$

for the full description. The symmetry of matrix Q in (3.26) is just the same as in a usual σ model for localization in normal metals.[4] From (3.18) and (3.25), it follows that Q is a Hermitian real-quarternionic matrix obeying the constraints

$$\begin{aligned} &Q^2=1 , \\ &\mathrm{Tr}Q=0 . \end{aligned} \tag{3.27}$$

Equations (3.17) and (3.18) together with (3.25)–(3.27) describe the degenerate saddle-point solution at $\nabla\varphi=\omega=0$. The small terms in (3.4) proportional to $\nabla\varphi$ and ω as well as nonzero ∇U break down the degeneracy. Our goal is to obtain the explicit expression for the expansion of the effective functional (3.3) in powers of these perturbations.

In order to do this we make the identical transformation by multiplying the operator $\hat{H}$ in (3.3) by $U(\mathbf{r})$ and $U^\dagger(\mathbf{r})$ and take into account the nonzero commutator $[\hat{\xi},U]$, where $\hat{\xi}=\hat{p}^2/2m-\varepsilon_F$. As a result, we obtain, using (3.17),

$$\hat{H}(U)=\hat{\xi}-\hat{E}+\frac{i}{2\tau}\mathcal{P}_0^\dagger\hat{V} , \tag{3.28}$$

where the perturbation $\hat{V}$ is given by

$$\hat{V}=-U\left[\frac{\hat{\mathbf{p}}}{2m}(\mathbf{F}\tau_z\otimes\sigma_z+2i\nabla U^\dagger U)+\frac{\omega}{2}\tau_z\otimes\rho_z\right]U^\dagger . \tag{3.29}$$

After such a transformation the resolvent $\mathcal{R}$

$=[\hat{\xi}-\hat{E}+(i/2\tau)\mathcal{P}_0]^{-1}$ becomes independent on coordinates and diagonal in the momentum representation. In the case of interest $\varepsilon^2>\Delta^2$, we have

$$\mathcal{R}=[f_1(\bar{\xi})+if_2(\bar{\xi})\rho_z]\times\left\{\xi+\hat{s}\left[(\varepsilon^2-\Delta^2)^{1/2}+\frac{i}{2\tau}\rho_z\right]\right\}, \tag{3.30}$$

where $\bar{\xi}=2\tau\xi$, $b=2\tau(\varepsilon^2-\Delta^2)^{1/2}$ and

$$f_1(\bar{\xi})+if_2(\bar{\xi})=(\bar{\xi}^2+1-b^2-2ib)^{-1}. \tag{3.31}$$

Now one can expand the effective functional (3.3) in powers of $\hat{V}$:

$$\mathcal{F}=-\tfrac{1}{2}\mathrm{Tr}\ln(1+\mathcal{R}\hat{V}). \tag{3.32}$$

This expansion reduces to the calculation of the ordinary integrals over momenta and combining these integrals with (3.18) to express the result in terms of Q rather than U. Such an expansion to the first order in $\hat{V}$ gives

$$\mathcal{F}_1=\frac{i\pi\nu_0\omega}{8}\int \mathrm{Tr}(U^{\dagger}\rho_z U\rho_z[\hat{s},\tau_z]_+)d^dr. \tag{3.33}$$

Using the definition of Q [Eq. (3.18)] and calculating the anticommutator $[\hat{s},\tau_z]_+$, we obtain

$$\mathcal{F}_1=\frac{i\pi\nu_s\omega}{4}\int \mathrm{Tr}(Q\rho_z)d^dr, \tag{3.34}$$

where, as expected, the coefficient ν_s coincides with the well-known expression for the density of states in a superconducting state:[11]

$$\nu_s=\frac{\nu_0\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}\theta(\varepsilon^2-\Delta^2). \tag{3.35}$$

The part of $\hat{V}$ proportional to $\mathbf{p}$ gives a nonzero result only in the second-order expansion

$$\mathcal{F}_2=\frac{\pi\nu_0 v_F^2\tau}{8d}\int \mathrm{Tr}\left[-[C_+,Q]^2+\frac{1}{1+b^2}[C_-,Q]_+^2+\frac{4ib}{1+b^2}C_-^2Q\right], \tag{3.36}$$

where

$$C_+=i\nabla U^{\dagger}U+\frac{\mathbf{F}}{4}(\tau_z\otimes\sigma_z+\hat{s}\tau_z\otimes\sigma_z\hat{s})=i\nabla U^{\dagger}U+\frac{\mathbf{F}}{2}\frac{\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}\hat{s}\sigma_z, \tag{3.37}$$

$$C_-=\frac{\mathbf{F}}{4}(\tau_z\otimes\sigma_z-\hat{s}\tau_z\otimes\sigma_z\hat{s})=\frac{\mathbf{F}}{2}\frac{\Delta}{(\varepsilon^2-\Delta^2)^{1/2}}\hat{s}\tau_x, \tag{3.38}$$

and the symbols $[\ ,\ ]$ and $[\ ,\ ]_+$ denote a commutator and an anticommutator, respectively. With the aid of the commutation properties $[\hat{s},\tau_x]_+=0$ and $[\hat{s},Q]=0$, it is easy to show that $\mathrm{Tr}(C_-^2Q)=0$. Using also the expression

$$[\nabla U^{\dagger}U,Q]=\nabla Q, \tag{3.39}$$

which follows from (3.18), we have

$$\mathcal{F}_2=\frac{1}{t_0}\int \mathrm{Tr}\left[\nabla Q-\frac{i\nu_s}{2\nu_0}\hat{s}[\mathbf{F}\sigma_z,Q]\right]^2d^dr-\frac{z_0}{4t_0}\int Tr[\mathbf{F}\tau_x,Q]_+^2d^dr, \tag{3.40}$$

where the coefficient z_0 is given by

$$z_0=\frac{\Delta^2}{(1+b^2)(\varepsilon^2-\Delta^2)}. \tag{3.41}$$

The coefficient t_0 does not depend on Δ, and it is related to the diffusion coefficient D_0 in a normal state far away from the Anderson transition:

$$t_0^{-1}=\frac{\pi\nu_0D_0}{8}. \tag{3.42}$$

Equation (3.34) and (3.40) give the lowest-order expansion of the functional (3.3) in powers of ∇Q, $\mathbf{F}$, and ω. It is seen from (3.40) that the gradient of phase $\nabla\varphi$ is present in the effective functional even if one takes the formal limit $\Delta\to 0$. In a normal metal, such a dependence on $\nabla\varphi$ is unphysical, because at $\Delta=0$ one can perform the unitary transformation of the matrix Q which removes this dependence:

$$Q\to\exp\left[\frac{i}{2}\varphi(\mathbf{r})\tau_z\otimes\sigma_z)\right]Q\exp\left[-\frac{i}{2}\varphi(\mathbf{r})\tau_z\otimes\sigma_z)\right]. \tag{3.43}$$

However, at any $\Delta\neq 0$ the transformation (3.43) cannot be performed as it violates the relationships (3.23) and (3.25) between $Q_{\varepsilon,\varepsilon}$ and $Q_{-\varepsilon,-\varepsilon}$. It means that the effect of the order-parameter phase fluctuations is present even if we set $\Delta=0$ in (3.40). The reason is that the gauge invariance is broken at $\Delta\neq 0$, and the symmetry conditions (3.23) and (3.25) reflect this breakdown. Of course, one can restrict oneself to the consideration of massless modes [given by the constraints (3.23)] only for small momenta $k<k_0$. Therefore, the effective functional (3.42) describes only very slow spatial variations of Q and φ, with the upper momentum cutoff k_0 being of the order of the inverse coherence length:

$$k_0=\left[\frac{\Delta}{hv_F}\right]\frac{1}{(1+4\varepsilon^2\tau^2)^{1/2}}. \tag{3.44}$$

Thus the effect of the phase fluctuations can be seen at small Δ only very close to the Anderson transition point, when the correlation length l_c is larger than ξ.

One can take into account the symmetry (3.25) and obtain the explicit expression of the functional $\mathcal{F}$ in terms of the field Q defined by (3.26):

$$\tfrac{1}{2}\mathcal{F}[Q,\varphi]=\mathcal{F}_0+\mathcal{F}_{\varphi},$$

where

$$\mathcal{F}_0=\frac{1}{t_0}\int \mathrm{Tr}(\nabla Q)^2d^dr+\frac{i\omega\pi\nu_s}{4}\int \mathrm{Tr}(\rho_zQ)d^dr, \tag{3.45}$$

$$\mathcal{F}_\varphi=-\frac{\nu_s^2}{4t_0v_0^2}\int \mathrm{Tr}[\mathbf{F}\sigma_z,Q]^2d^dr$$
$$-\frac{z_0}{4t_0}\int \mathrm{Tr}[\mathbf{F}\sigma_z,Q]_+^2d^dr\ . \tag{3.46}$$

The particle-hole and particle-particle diffusion propagators (1.4) and (1.5) can be expressed using Eqs. (2.27), (2.28), (3.5), and (3.17) in terms of the following correlation functions:

$$D_\varepsilon(\mathbf{r}-\mathbf{r}')=\frac{(\pi\nu_s)^2}{N^2}\mathrm{tr}\langle Q_{11}^{AR}(\mathbf{r})Q_{11}^{RA}(\mathbf{r}')\rangle_{\mathcal{F}}\ , \tag{3.47}$$

$$C_\varepsilon(\mathbf{r}-\mathbf{r}')=-\frac{(\pi\nu_s)^2}{N^2}\mathrm{tr}\langle Q_{21}^{AR}(\mathbf{r})Q_{12}^{RA}(\mathbf{r}')\rangle_{\mathcal{F}}e^{i\varphi(\mathbf{r})-i\varphi(\mathbf{r}')}\ , \tag{3.48}$$

where ν_s is given by (3.35) and the subindices 1 and 2 correspond to the quarternionic space.

IV. EFFECT OF ORDER-PARAMETER PHASE FLUCTUATIONS

The comparison of the functional (3.45) with that describing the Anderson localization in normal metals[1,3] shows that in the absence of the order-parameter phase fluctuations and an external magnetic field the effective functional is the same as in normal metals, except for the proper change in the density of states.[2] The last two terms $\mathcal{F}_\varphi$ in $\mathcal{F}[Q,\varphi]$ describe the effect of the order-parameter phase fluctuations. Omitting the inessential N-proportional constant, we have, from (3.27) and (3.46),

$$\mathcal{F}_\varphi=-\frac{\gamma_0}{t_0}\int(\nabla\varphi)^2\mathrm{Tr}(\sigma_zQ)^2d^dr\ , \tag{4.1}$$

where the coefficient γ_0 is given by

$$\gamma_0=\left[\frac{\nu_s}{\nu_0}\right]^2+z_0=\frac{1}{\varepsilon^2-\Delta^2}\left[\varepsilon^2+\frac{\Delta^2}{1+b^2}\right]\ . \tag{4.2}$$

Two different contributions to γ_0 correspond to two terms in (3.46). The contribution of the first term in (3.46) describes the effect of a broken gauge invariance. As seen from (4.2), this contribution does not depend on disorder and it remains constant at $\varepsilon\gg\Delta$. On the contrary, the second contribution, proportional to z_0, vanishes in clean superconductors and for $\varepsilon\gg\Delta$. The strong divergence of both contributions at $\varepsilon^2=\Delta^2$ manifests the breakdown of our perturbative consideration in a gap region.

In order to calculate the effect of the phase fluctuations on the diffusion propagators (1.4) and (1.5), one should average Eqs. (3.47) and (3.48) over the thermal fluctuations of $\varphi(\mathbf{r})$ with the distribution function (1.1). The averaging of $D_\varepsilon(\mathbf{r}-\mathbf{r}')$ over such fluctuations reduces in the replica limit $N\to0$ to the replacement of the functional $\mathcal{F}[Q,\varphi]$ by a new functional $\mathcal{F}[Q]$ according to Eq. (1.3). The expression (3.48) for the particle-particle diffusion propagator contains an additional factor $\exp(i\varphi-i\varphi')$. In the limit of small temperature $\tilde{T}$, one can neglect the correlations between two φ-dependent factors in (3.48) and average them independently over the thermal phase fluctuations. The result can be expressed in a form analogous to (3.48), where the $\langle QQ\rangle$ correlation function should be calculated with the functional $\mathcal{F}[Q]$ and the additional factor is replaced by an averaged one:

$$K(\mathbf{r}-\mathbf{r}')=\int\exp[i\varphi(\mathbf{r})-i\varphi(\mathbf{r}')f[\varphi]\mathcal{D}\varphi\ , \tag{4.3}$$

which behaves like the spin correlation function in the classical XY model. Above two dimensions, $d-2=\varepsilon>0$, this correlation function tends to unity at $|\mathbf{r}-\mathbf{r}'|\gg\tilde{T}^{1/\varepsilon}$ and does not change the particle-particle diffusion propagator at scales of interest $|\mathbf{r}-\mathbf{r}'|>\xi$. In two dimensions the function $K(\mathbf{r}-\mathbf{r}')$ is known to decrease at large distances, the decrease being changed from power law to exponential as the temperature is increased.[12] In what follows we concentrate, however, on the behavior of the correlation functions $\langle QQ\rangle$ governed by the functional $\mathcal{F}[Q]$ [Eq. (1.3)].

The linear in $\tilde{T}$ term in the functional $\mathcal{F}[Q]$ [Eq. (1.3)] is given by

$$\mathcal{F}_f=-\frac{1}{2t_0l_f^2}\int\mathrm{Tr}(\sigma_zQ)^2d^dr\ , \tag{4.4}$$

where the characteristic length l_f is defined by

$$l_f=(\gamma_0\tilde{T}k_0^d)^{-1/2}\ . \tag{4.5}$$

Using the expression for the quantity $\tilde{T}$,[13] we have

$$l_f=\xi\left[\frac{\gamma_0}{\varepsilon_F\tau}\left[\frac{T}{\Delta}\right]\left[\frac{\Delta}{\varepsilon_F}\right]^{d-2-1/d}\right]^{-1/2}\ . \tag{4.6}$$

To make clear the physical meaning of the term $\mathcal{F}_f$, we expand Q in (3.45) and (4.4) up to the quadratic terms in the matrix $W=\mathcal{W}_{\varepsilon,\varepsilon'}$:

$$Q=e^{W/2}\rho_ze^{-W/2}=\left[1+W+\frac{W^2}{2}+\cdots\right]\rho_z\ . \tag{4.7}$$

Using the property $[W,\rho_z]_+=0$, one obtains the following expression for the corresponding functional $\mathcal{F}^{(2)}=\mathcal{F}_0^{(2)}+\mathcal{F}_f^{(2)}$ quadratic in W:

$$\mathcal{F}_0^{(2)}=-\frac{1}{t_0}\int\mathrm{Tr}(\nabla W)^2d^dr+\frac{i\omega\pi\nu_s}{8}\int\mathrm{Tr}(W^2)d^dr\ , \tag{4.8}$$

$$\mathcal{F}_f^{(2)}=-\frac{1}{2t_0l_f^2}\int\mathrm{Tr}(W^2-\sigma_zW\sigma_zW)d^dr\ . \tag{4.9}$$

The functional $\mathcal{F}^{(2)}$ determines the free-diffusion propagators (with no account for the Anderson localization effects). The particle-hole and particle-particle free-diffusion propagators are given by the correlation functions

$$D_\varepsilon^{(0)}(\mathbf{r}-\mathbf{r}')=-\frac{(\pi\nu_s)^2}{4N^2}\mathrm{Tr}\langle W_d(\mathbf{r})W_d(\mathbf{r}')\rangle_{\mathcal{F}^{(2)}} \tag{4.10}$$

and

$$C_{\varepsilon}^{(0)}(\mathbf{r}-\mathbf{r}')=\frac{(\pi\nu_s)^2}{4N^2}\mathrm{Tr}\langle W_c(\mathbf{r})W_c(\mathbf{r}')\rangle_{\mathcal{F}^{(2)}}K(\mathbf{r}-\mathbf{r}') , \tag{4.11}$$

where $K(\mathbf{r}-\mathbf{r}')$ is defined by (4.3) and the matrices W_d and W_c are the diagonal and off-diagonal quarternionic components of the matrix W, respectively. If we represent the matrix W as a sum $W=W_d+W_c$ and take into account the relationship $[\sigma_z,W_d]=0$ and $[\sigma_z,W_c]_+=0$, we see that the terms in (4.9) proportional to $\mathrm{Tr}(W_d^2)$ cancel each other. On the other hand, the terms proportional to $\mathrm{Tr}(W_c^2)$ have the same signs and they are present in (4.9). It means that the particle-particle diffusion propagator has no divergence at $\omega=q=0$, as was mentioned in (1.4). On the contrary, the ordinary (particle-hole) diffusion propagator remains divergent, as it should be according to the probability conservation law[3]

$$D_d(q=0,\omega)\propto\frac{1}{\omega+i\delta} . \tag{4.12}$$

Though the functional $\mathcal{F}[Q]$ given by Eqs. (3.45) and (4.4) is quadratic in Q, it is strongly nonlinear in the variable W because of the constraint (3.27) and the parametrization (4.7). The higher powers of W which appear in the further expansion of the functional $\mathcal{F}[Q]$ describe the interaction of the free-diffusion modes. Such an "interaction" is present even in the case of noninteracting electrons, and it is connected with the spatial correlations in the electronic wave functions at different realizations of the impurity potential. It is this interaction which leads to the Anderson localization.

Near the Anderson transition point $n=n_c$, the correlation length $l_c\propto|n-n_c|^{-\nu}$ determining the spatial correlations in the electronic wave functions becomes very large.[4] It means that in the vicinity of the Anderson transition the essential spatial scales q^{-1} are of the order of l_c. Therefore, in the region $l_c>l_f$, one can neglect the nondivergent free-diffusion propagators in calculating the effect of the interacting diffusion modes. Formally, it is equivalent to the additional constraint

$$W_c=0 . \tag{4.13}$$

With the symmetry condition (4.13), the functional (4.4) reduces to the inessential constant and the functional (3.45) is equivalent to the nonlinear σ model defined on a coset $Q\in\mathrm{U}(2N)/\mathrm{U}(N)\otimes\mathrm{U}(N)$. This model is known[1,2,4] to describe systems with broken time-reversal symmetry, e.g., the localization in a weak magnetic field. It is not surprising that the order-parameter phase fluctuations lead to the model of this kind because at nonzero $\nabla\varphi$ there are supercurrents in a system which violate time-reversal symmetry.

It is interesting to calculate the behavior of transport coefficients in the crossover region $\xi<l_c<l_f$. In this region the q^2 terms are the leading ones in all the free-diffusion propagators [see, e.g., Eqs. (1.4) and (1.5)] and we can neglect the gradientless terms in the effective functional $\mathcal{F}[Q]$. Instead, we will be interested in the $(\nabla Q)^2$ terms arising in averaging of $\exp(-\mathcal{F}_\varphi)$ over $\varphi(\mathbf{r})$. Such a term appears in the second-order expansion of $\mathcal{F}[Q]$ in powers of $\tilde{T}$:

$$\overline{\mathcal{F}}\propto\frac{\alpha_0}{t_0}\int\mathrm{Tr}(\sigma_z Q\sigma_z\nabla Q)\mathrm{Tr}(\sigma_z Q\sigma_z\nabla Q)d^dr , \tag{4.14}$$

where at $T\ll\Delta$ the coefficient α_0 is given by[13]

$$\alpha_0=\left[\frac{\gamma_0\tilde{T}}{t_0}\right]^2=\left[\frac{\gamma_0 T}{\Delta}\right]^2\left[\frac{\varepsilon_F}{\Delta}\right]^{2/d} . \tag{4.15}$$

We have carried out the renormalization-group (RG) analysis of the nonlinear σ model containing this term in addition to the functional $\mathcal{F}_0$ [Eq. (3.45)]. The calculations, which we published elsewhere, are similar to those made in Ref. 14. The main result of these calculations is that the vertex [Eq. (4.14)] is a part of the combination of vertices $\mathcal{F}_\alpha$, which is relevant in the RG transformations:

$$\mathcal{F}_\alpha=\frac{\alpha}{t_0}\int V(Q)d^dr , \tag{4.16}$$

where

$$\begin{aligned}V(Q)=&\mathrm{Tr}(\sigma_z Q\sigma_z\nabla Q)\mathrm{Tr}(\sigma_z Q\sigma_z\nabla Q)\\&+\tfrac{1}{4}\mathrm{Tr}(\sigma_z Q)^2\mathrm{Tr}(\sigma_z\nabla Q)^2\\&-\mathrm{Tr}[(\sigma_z Q)^2(\sigma_z\nabla Q)^2]-\tfrac{3}{2}\mathrm{Tr}(\sigma_z Q\sigma_z\nabla Q)^2 .\end{aligned} \tag{4.17}$$

The renormalization of the coefficient α can be expressed in terms of the renormalized value of the coefficient t^{-1} attached to the gradient term in the functional $\mathcal{F}_0$ [Eq. (3.45)]:

$$\alpha=\alpha_0\left[\frac{t}{t_0}\right]^2 . \tag{4.18}$$

At small α_0 the renormalization of the coefficient t does not depend on α and it is the same as in the case of the Anderson localization in normal metals in the presence of time-reversal invariance. In the latter case the value t^{-1} is known to be proportional to the diffusion coefficient D, which decreases near the Anderson transition.[1-4] Using the relationship $D\propto l_c^{-\varepsilon}$,[4] we have

$$\frac{t}{t_0}=\left|\frac{n-n_c}{n_c}\right|^{-\nu\varepsilon} . \tag{4.19}$$

In the framework of the RG approach,[3] the calculation of the diffusion propagators (3.47) and (3.48) reduces to the calculation of the free-diffusion propagators (4.10) and (4.11) corresponding to the renormalized functional $\mathcal{F}[Q]$ expanded up to the quadratic in W terms (functional $\mathcal{F}_R^{(2)}$). The interaction between the free-diffusion modes is actually taken into account in a process of the renormalization. The functional $\mathcal{F}_R^{(2)}$ consists of the renormalized functionals (4.8) and (4.9) (with the coefficients t instead of t_0) and the quadratic in W part of the functional (4.16). It can be verified that the first two terms in (4.17) make no contribution to the functional $\mathcal{F}_R^{(2)}$ in the replica

limit $N \to 0$. However, there is a contribution made by the last two terms in (4.17), which have been generated in the RG transformations:

$$\mathcal{F}_\alpha^{(2)} = -\frac{3\alpha}{2t_0}\int \mathrm{Tr}(\nabla W)^2 d^d r + \frac{\alpha}{t_0}\int \mathrm{Tr}(\sigma_z \nabla W)^2 d^d r \ . \tag{4.20}$$

If we again express $\mathcal{F}_R$ in terms of the diagonal and off-diagonal quaternionic components of the matrix W, we obtain the following expressions for the coefficients attached to the terms $\mathrm{Tr}(\nabla W_d)^2$ and $\mathrm{Tr}(\nabla W_d)^2$:

$$D_d \propto \frac{1}{t}\left[1+\frac{\alpha}{2}\frac{t}{t_0}\right] \tag{4.21}$$

and

$$D_c \propto \frac{1}{t}\left[1+\frac{5\alpha}{2}\frac{t}{t_0}\right] . \tag{4.22}$$

These coefficients are evidently proportional to the coefficients D_d and D_c defined in (1.4) and (1.5). Equations (4.21) and (4.22) show that in the crossover region $\xi < l_c < l_f$ the coefficients D_d and D_c are different from the diffusion coefficient D in a normal metal under the same conditions:

$$D_d = D\left[1+\frac{\alpha_0}{2}\left|\frac{n-n_c}{n_c}\right|^{-3\nu\epsilon}\right] , \tag{4.23}$$

$$D_c = D\left[1+\frac{5\alpha_0}{2}\left|\frac{n-n_c}{n_c}\right|^{-3\nu\epsilon}\right] . \tag{4.24}$$

Moreover, the coefficient D_d is no longer equal to the coefficient D_c, the difference increasing critically:

$$\frac{D_c - D_d}{D} = 2\alpha_0\left|\frac{n-n_c}{n_c}\right|^{-3\nu\epsilon} . \tag{4.25}$$

V. CONCLUSION

Now we discuss the possibilities of the experimental observation of the peculiarities in the diffusion propagators (1.4) and (1.5). The most usual way to measure the diffusion coefficient $D_d(\varepsilon)$ in dirty superconductors is the investigation of the thermal conductivity or the ultrasonic attenuation. Because of the electron-phonon interaction, the equation for the phonon dispersion law acquires an additional term:

$$\omega - \omega_0(q) = \tfrac{1}{2} g^2 \omega_0(q)\,\Sigma(q,\omega) , \tag{5.1}$$

where $\omega_0(q) = sq$, s is a sound velocity, and g is a constant of electron-phonon interaction. The imaginary part of the self-energy Σ, which determines the low-frequency ultrasonic attenuation, is expressed in terms of the diffusion propagator $D_\varepsilon(q,\omega)$ as follows:[8]

$$\mathrm{Im}\Sigma(q,\omega) = \frac{\omega}{\pi}\int_0^\infty \mathrm{Re} D_\varepsilon(q,\omega)\frac{df_0}{d\varepsilon}d\varepsilon , \tag{5.2}$$

where $f_0(\varepsilon)$ is a Fermi distribution function. At a sufficiently low frequency of phonons, $sq \ll s^2/D_d$, Eq. (5.2) reduces to

$$\mathrm{Im}\Sigma = \frac{2q^2}{\omega}\int_\Delta^\infty \nu_s(\varepsilon) D_d(\varepsilon)\frac{df_0}{d\varepsilon} , \tag{5.3}$$

where it is taken into account that the density of states ν_s is zero at $\varepsilon < \Delta$. At low temperatures $T \ll \Delta$, the ultrasonic attenuation is exponentially small because of the exponentional decrease of the function $df_0/d\varepsilon \propto \exp(-\varepsilon/T)$.[15] For the same reason the main contribution to the integral (5.3) is made by the energy interval $\varepsilon - \Delta < T \ll \Delta$. In this region the second term in the expression for the diffusion constant (4.23) is dominant because of the divergence of the coefficient α_0 at $\varepsilon - \Delta \to 0$:

$$\alpha_0 \sim \left[\frac{T}{\varepsilon-\Delta}\right]^2\left[\frac{\varepsilon_F}{\Delta}\right]^{2/d} . \tag{5.4}$$

It means that, unlike the case of normal metal, the attenuation constant increases rather than decreases as $n \to n_c$ in all the crossover region $\xi < l_c < l_f$. In the close vicinity of the Anderson transition $l_c > l_f$, the diffusion coefficient and attenuation constant should decrease, though considerably slower than in the case of normal metal. Therefore, a peak might appear in the dependence of the ultrasonic attenuation on the impurity concentration near the Anderson transition point. Of course, one should not take this statement too seriously, because in the vicinity of a gap region our perturbative calculations cease to be valid. In the nonperturbative results, the divergence in (5.4) has to be cut at small $\varepsilon - \Delta$, with the cutoff occuring when the second term in (4.25) is of the order of unity. Nevertheless, we can conclude that the effect of the order-parameter phase fluctuations on the electronic part of the ultrasonic attenuation is not negligible even at low temperatures.

The next possible application of the results obtained is connected with the reproducible fluctuations of the electronic density of states in an ensemble of small superconducting particles with sizes $L < \min(\sqrt{D/T}, l_f)$ (mesoscopic fluctuations). As follows from the results of Ref. 7, the correlation function $\langle\langle \nu(\varepsilon)\nu(\varepsilon+\omega)\rangle\rangle_{\mathrm{imp}}$ is proportional to

$$\langle\langle \nu(\varepsilon)\nu(\varepsilon+\omega)\rangle\rangle_{\mathrm{imp}} \propto \frac{1}{D_d(\varepsilon)/L^2+\omega^2} + \frac{1}{D_c(\varepsilon)/L^2+\omega^2} . \tag{5.5}$$

Therefore, one can find both the coefficients $D_d(\varepsilon)$ and $D_c(\varepsilon)$ by measuring the correlation function (5.6) at different energies ε and different shifts in energy ω.

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