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Directed paths in a random potential

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The properties of directed paths in random media are explored, with emphasis on the low-temperature phase. Scaling arguments, numerical simulations, and exact results are all utilized. Some of the results presented concern the existence of large-scale low-free-energy excitations in the low-temperature phase, sample-to-sample entropy variations which are much larger than the free-energy variations, and concomitant sensitivity of the optimal configuration to temperature changes, analogously to spin glasses. Nevertheless, it is argued that at fixed temperature, the possible states of a directed path in an infinite system with one end fixed are simply parametrized by its average orientation. Possibilities for the behavior in high dimensions are examined and some of the pathologies of the system on Cayley trees are discussed.

I. INTRODUCTION

Thermodynamic phases in which quenched randomness plays the dominant rather than a subsidiary role are still, in general, poorly understood. In the best studied case of spin glasses, the behavior of the ordered phase is highly controversial.[1,2] It is thus important to try to understand simpler models which share some of the complicating features of spin glasses. The simplest nontrivial model that exhibits a randomness-dominated phase is a directed path (or directed polymer) in a locally random potential in $D=d+1$ dimensions with Hamiltonian

$$\mathcal{H}=\int\left[\frac{\kappa}{2}\left[\frac{\partial \mathbf{y}}{\partial x}\right]^2+V(x,\mathbf{y}(x))\right], \tag{1.1}$$

with $\mathbf{y}$ the d-dimensional transverse coordinate representing the displacements of the directed path which cost energy given by the first term in Eq. (1.1).

The second term is a random potential that we take to be Gaussian with short-range correlations:

$$\overline{V(x,\mathbf{y})V(x',\mathbf{y}')}=\delta(x-x')\Gamma(\mathbf{y}-\mathbf{y}'), \tag{1.2}$$

where Γ decays rapidly for separation larger than a short-distance cutoff a, and the overbar denotes averaging over realizations of the random potential. Discrete versions of the model with either x or $\mathbf{y}$ or both restricted to a lattice are also of interest: in the continuum, it may be necessary to impose a cutoff in the x direction, for example, by a $(\partial^2 y/\partial x^2)^2$ term, however, a natural such short-distance cutoff is also provided by the scale a in the y direction.

This model, and others closely related to it, arise in many contexts, both in equilibrium statistical mechanics[3] and dynamical problems[4] which exhibit structures similar to the transfer-matrix representation of Eq. (1.1). Various partial pieces of information are known about the random directed-path models.[5] In transverse dimensions $d>2$, it has been proven[6] that there exists a high-temperature "free" phase in which the disorder is irrelevant at long length scales and the directed path executes a random walk in $\mathbf{y}$ as a function of x. However, it has also been proven that this phase cannot persist to low temperatures.[7] In dimensions $d\le 2$ and for low temperatures in $d>2$, the directed path instead exhibits a "pinned" phase in which the dominant competition is between the elastic stiffness and the random potential with thermal fluctuations playing a subsidiary role.[3] In one dimension, exact results for some of the exponents are known[3]—in particular, the wandering exponent ζ defined by

$$\overline{\langle[\mathbf{y}(x)-\mathbf{y}(x')]^2\rangle}\sim|x-x'|^{2\zeta} \tag{1.3}$$

is found to be given by

$$\zeta(d=1)=\tfrac{2}{3}. \tag{1.4}$$

Bethe-lattice (actually, Cayley-tree[8]) and hierarchical[9] versions of Eq. (1) have also been studied analytically and numerical work[10,11] in $d>1$ has been carried out. In addition some general results on the thermal fluctuations of $\mathbf{y}$ have been derived.[12] Nevertheless, a more comprehensive understanding of the behavior, particularly in the pinned phase, is lacking.

In this paper we will present several conjectures, supported by numerical and other evidence, for the behavior of the random directed path in various dimensions. One of the main new results is that the sample to sample variations in the entropy $\Delta S_L=S_L-\overline{S_L}$ for a string of length L are *normal* in the pinned phase with

$$\overline{(\Delta S_L)^2}\sim L. \tag{1.5}$$

This is in contrast to the variations in the free energy

$$\overline{(\Delta F_L)^2} \sim L^{2\theta} , \tag{1.6}$$

with $\theta = 2\zeta - 1 < \frac{1}{2}$. This behavior is very similar to that of the low-temperature phase of spin glasses conjectured earlier by us.[2] The fact that for large L, $|\Delta S_L|$ is typically $\gg |\Delta F_L|$ has various interesting consequences, which make the pinned phase highly nontrivial.

We also discuss additional properties of thermal fluctuations and sample-to-sample variations in both the high- and low-temperature phases. Various possibilities for the behavior in high dimensions are discussed and critiqued in this paper, as are (briefly) some very recent papers.

The remainder of the paper is organized as follows: In the next section some properties of the high temperature phase are summarized, and in Sec. III the main conjectures and results for the low-temperature phase are introduced. Section IV discusses various subtle properties of the low-temperature phase, particularly the uniqueness of the "state" of a system and sensitivity of the state to temperature changes. Analytical and numerical results in one dimension are given in Sec. V, which support the general picture of Sec. III. Finally, Sec. VI discusses possible behavior in higher dimensions and critiques the applicability of results on the Cayley tree. Some new calculations on the Cayley tree are relegated to the Appendix.

II. HIGH-TEMPERATURE FREE PHASE

In the high temperature or weak disorder phase, which only exists for $d>2$, a simple argument shows that the directed path behaves like a random walk on long length scales, i.e., the disorder is irrelevant.[6,13]

To avoid the effects of ends, we consider a directed path of length L with periodic boundary conditions in the x direction and d-dimensional cross section W^d with periodic boundary conditions also in the $\mathbf{y}$ directions. If we choose $W \sim L^{1/2}$ then in the absence of disorder the behavior is roughly the same as in a section of length L of an infinite system, at least as far as the scaling properties we are interested in. Consider a typical configuration of the directed path in the absence of disorder. The typical lowest-order perturbative effect of the random potential will be to change the energy of this configuration by a random amount of order $L^{1/2}$ due to a sum of a large number of independent terms. If we now average over all the configurations of the unperturbed path, this will strongly suppress the overall contribution to the free energy since it is the average of a large number of at-least-somewhat independent contributions. A very crude estimate is to consider each segment of the path as running freely over the W^d possible positions, yielding a suppression factor of $W^{-d/2}$ from the central-limit theorem. A simple, but more convincing multiscale argument yields the same result. Thus we may guess that the typical random contribution to the free energy is of order[13]

$$F_R \sim L^{1/2} W^{-d/2} \sim L^{(2-d)/4} . \tag{2.1}$$

In dimensions $d>2$, this is small at large length scales and one concludes that the thermal averaging of disorder renders it irrelevant. Rigorous results of Imbrie and Spencer[6] show that this argument is essentially correct. Indeed, they find that with probability one for large L the total free energy in this regime is *independent* of the realization of the disorder up to $O(1)$, with random corrections of the form Eq. (2.1). A similar argument[13] shows that both the entropy and energy will have small sample-to-sample variations of the same form as F_R. The presence of the disorder also produces *nonrandom* contributions of order L to the entropy, energy, and free energy.

Note that if we had used a directed path with one or both ends fixed instead of periodic boundary conditions, there would be a random $O(1)$ contribution to the free energy and other thermodynamic quantities arising from points within distances of order $a^2\kappa/T$ from the fixed ends.[13] These would dominate over the bulk contribution Eq. (2.1) for dimension $d>2$.

From the irrelevance of the disorder, we thus see that the transverse fluctuations of the directed path will scale the same way as for a free random walk with $\zeta = \frac{1}{2}$. It is useful, however, to decompose the wandering in the presence of randomness into two parts, the thermal fluctuations

$$C_T(x-\mathbf{x}') \equiv \overline{\langle [\mathbf{y}(x)-\mathbf{y}(x')]^2 \rangle - \langle \mathbf{y}(x)-\mathbf{y}(x') \rangle^2} \tag{2.2}$$

and the part caused by the disorder,

$$C_R(x-x') \equiv \overline{\langle \mathbf{y}(x)-\mathbf{y}(x') \rangle^2} . \tag{2.3}$$

In the absence of disorder, $\overline{\langle \mathbf{y}(x)-\mathbf{y}(x')\rangle^2}$ is zero; however, even a small amount of randomness will give some bias to the wandering of a directed path, yielding a nonzero C_R.

A simple estimate[13] again yields the correct results. On small length scales with $y \sim a$, a perturbative calculation will yield a C_R proportional to the mean-square disorder $\overline{V^2}$, i.e., reduced by this amount from the thermal fluctuations. On longer scales, we must take into account the renormalization of the disorder due to thermal fluctuations. From the comparison of the random part of the free energy,[13] Eq. (2.1) with the thermal free energy T, we expect that the effective $V_L \sim L^{(2-d)/4}$. Renormalizing to scale L and then calculating the perturbative effects of the renormalized disorder, we thus expect a fraction V_L^2 of the fluctuations to contribute to C_R, implying that

$$C_R \sim |x-x'|^{2\zeta_R} , \tag{2.4}$$

with

$$\zeta_R = \frac{1}{2} + \frac{2-d}{4} = \frac{4-d}{4} . \tag{2.5}$$

Higher moments of $\langle \mathbf{y}(x)-\mathbf{y}(x')\rangle$ can be shown to scale as the appropriate power of ζ_R, and so we can conclude that a typical configuration of the disorder (sample) will have a random thermally averaged displacement on length scale L typically growing as L^{ζ_R}. The fluctuations about this mean are thus much larger than the mean for dimension $d>2$ since $\zeta_T > \zeta_R$.

More detailed discussion of the propreties of the high-temperature phase is contained in Ref. 13. This phase will occur for weak disorder or high temperatures in dimensions $d>2$. Note, however, that there is really only one parameter in the problem since we can rescale the x coordinate in such a way as to make the temperature or the potential strength fixed. For our later purposes it is more convenient to do the latter and to measure the temperature in units of

$$T^{*}=[\tilde{\Gamma}\kappa a^{2-d}]^{1/3} , \qquad (2.6)$$

where

$$\tilde{\Gamma}\equiv\int\Gamma(\mathbf{y})d^{d}y . \qquad (2.7)$$

Note the appearance of $2-d$ powers of the length which corresponds to the critical dimension of $d_c=2$ discussed above. For discussing the exact results in one dimension in Sec. V we will use a different rescaling.

III. LOW-TEMPERATURE PINNED PHASE

In the preceding section, we saw that even at high temperatures (or weak disorder) the effects of the disorder grow with length scale for $d\leq 2$ and a perturbative treatment is inappropriate. We expect that this will also be true for low temperatures in any dimension. Indeed a renormalization-group calculation shows that in the marginal case of two dimensions, the effects of disorder grow logarithmically with length scale.[5,13] In $d=2+\epsilon$ dimensions, an unstable critical fixed point is found, presumably separating the high- and low-temperature phases. In this section we discuss the properties of the low temperature phase.

As for any low temperature phase, it is natural to start by considering the ground-state properties. As originally conjectured by Huse and Henley,[3] the typical displacements in the ground state also grow with a wandering exponent ζ:

$$C_R(x)\equiv\overline{\langle[\mathbf{y}(x')-\mathbf{y}(x'+x)]^2\rangle}\sim|x|^{2\zeta} , \qquad (3.1)$$

where the thermal average $\langle\ \rangle$ now represents the ground-state-energy minimization. We expect that the displacements will be at least as large as in the free phase so that $\zeta\geq\frac{1}{2}$.

In order to be more precise we impose the boundary condition that $\mathbf{y}(0)=\mathbf{0}$ and consider the displacements of the other points. At a distance x, we then expect that typically $y(x)\sim x^{\zeta}$ in the ground state. It is useful to consider terminating the system at distance $x=L$, and consider the properties as a function of the boundary condition, $\mathbf{y}(L)\equiv\mathbf{y}_L$, in particular, the ground-state energy [=free energy at $T=0$] $F_L(\mathbf{y}_L)$.

The distribution of $F_L(\mathbf{y}_L)$ for the continuum model is strongly constrained by the independence of the random potential $V(x,\mathbf{y})$ for different values of x. If we make a change of variables to $\mathbf{y}'(x)=\mathbf{y}(x)-\boldsymbol{\alpha}x$ for any constant vector $\boldsymbol{\alpha}$, the probability distribution of the random potential, as a function of $\mathbf{y}'$, is the same as it was originally as a function of $\mathbf{y}$.[12] The free energy also has the same form except for an extra piece

$$\Delta F_{\alpha}=\tfrac{1}{2}|\boldsymbol{\alpha}|^2L+\boldsymbol{\alpha}\cdot[\mathbf{y}'(L)-\mathbf{y}'(0)] . \qquad (3.2)$$

By choosing $\boldsymbol{\alpha}=\mathbf{y}_L/L$, we conclude that the distribution

$$\text{Prob}[F_L(\mathbf{y}_L)]=\text{Prob}[F_L(\mathbf{0})+|\mathbf{y}_L|^2/(2L)] . \qquad (3.3)$$

We can thus see that the function $F_L(\mathbf{y}_l)$ is the sum of a parabolic part, which is the same as in the absence of disorder, and a random part, which is a function of $\mathbf{y}_L$ with a distribution *independent* of $\mathbf{y}_L$, but of course, in general, containing nontrivial correlations. If we conjecture that the typical variations of this function are of order L^{θ} as y_L varies over distances of order L^{ζ}, this implies that the minimum of $F_L(\mathbf{y}_L)$—which will be attained if we relax the boundary condition on $\mathbf{y}_L$—will typically occur for a value of $\mathbf{y}_L$ such that $\mathbf{y}_L^2/L\sim L^{\theta}$. Since with a free-boundary condition, we expect $|\mathbf{y}_L|\sim L^{\zeta}$, and so this yields an exact equality between the exponent[3]

$$\theta=2\zeta-1 . \qquad (3.4)$$

[Note that the same argument can be used for the midpoint of a directed path fixed at $\mathbf{y}=\mathbf{0}$ at both ends.] We also expect that the sample-to-sample variations in the free energy $\Delta F_L=F_L-\overline{F_L}$ with $F_L\equiv\min\{F_L(\mathbf{y}_L)\}$ are of the same order so that

$$\Delta F_L\sim L^{\theta} . \qquad (3.5)$$

Indeed, we will see that this result is required for consistency.

It is instructive to examine the form of the correlations of

$$\Phi_L(\mathbf{y}_L)\equiv F_L(\mathbf{y}_L)-\frac{1}{2}\frac{(\mathbf{y}_L)^2}{L} \qquad (3.6)$$

as a function of y_L and L.[14] In particular for $y_L\gg L^{\zeta}$, we expect the optimum paths will be roughly independent of those with $y_L\lesssim L^{\zeta}$ except near their common origin, as shown in the illustration in Fig. 1.[15] Thus we expect that the characteristic scale of

$$\Phi_L(\mathbf{y}_L)-\Phi_L(\mathbf{0})\sim L^{\theta} , \qquad (3.7)$$

for $y_L\gg L^{\zeta}$. Indeed for $y_L\sim cL^{\zeta}$ with c a large constant, we already expect the $\Phi_L(\mathbf{y}_L)$ to be approximately independent (with correlations decaying, as we will argue later, as a power of y_L). From this one could obtain a

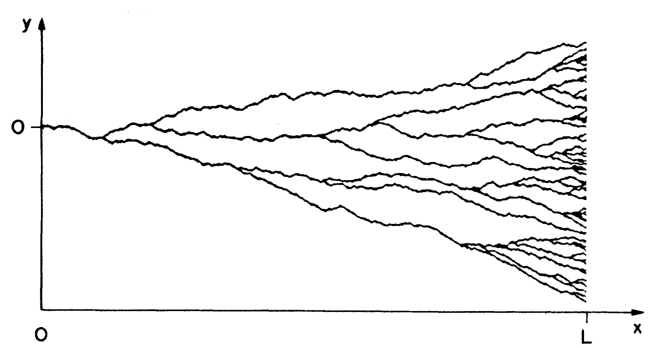

FIG. 1. Lowest energy (optimal) paths from the origin to each point along the vertical line $x=L$ for $d=1$. From Kardar and Zhang (Ref. 15).

contradiction if the sample-to-sample variations were larger than $\sim L^{\theta}$, since the minimum of $F_L(\mathbf{y}_L)$ would then be further from the origin than the presumed L^{ζ}. Thus L^{θ} must set the scale for *both* the variations of F_L for a given sample and the variations between samples. This argument assumes a form of scaling for $\phi_L(\mathbf{y}_L)$ roughly like that found in one-dimension (see below).

For $y_L \ll L^{\zeta}$, by contrast, we expect $\Phi_L(\mathbf{y}_L)-\Phi_L(0)$ to be much smaller and independent of L. The most likely form is a power-law behavior independent of L:

$$\Phi_L(\mathbf{y}_L)-\Phi_L(0)\sim|y_L|^{\sigma} . \tag{3.8}$$

For consistency, the scaling relation

$$\sigma=\frac{\theta}{\zeta}=2-\frac{1}{\zeta} \tag{3.9}$$

must hold. As we will see below, for the one-dimensional case, σ is the exponent that can be directly computed, as can the correlations in $\Phi_L(\mathbf{y}_L)$ for small separations. Numerical evidence[16] in 1D also bears out the scaling form: The function

$$\phi_L(\boldsymbol{\eta})\equiv\frac{1}{L^{\theta}}[\Phi_L(\boldsymbol{\eta}L^{\zeta})-\overline{\Phi}_L(0)] , \tag{3.10}$$

with the transverse lengths scaled by L^{ζ} and free energies scaled by L^{θ} tends to an L-independent limiting probability distribution (which is manifestly stationary in $\boldsymbol{\eta}$) for large L. The variations of $\phi(\boldsymbol{\eta})-\phi(\boldsymbol{\eta}')$ are of order unity for $|\boldsymbol{\eta}-\boldsymbol{\eta}'|\gg 1$ and of order $|\boldsymbol{\eta}-\boldsymbol{\eta}'|^{\theta/\eta}$ for $|\boldsymbol{\eta}-\boldsymbol{\eta}'|\ll 1$.

Note that although we have used the invariance properties of the distributions Eq. (3.3) that arise from the δ-function correlations in the disorder along the directed path, universality should imply that the results for large L will be insensitive to short-scale correlations.

In Fig. 1 the schematic form of the minimal-energy paths through the origin are sketched as a function of their end points y_L. Note the tree structure.[15] The optimal paths to two ends points a distance $\Delta_L \sim L^{\zeta}$ apart will typically be joined up to a distance of order $(\Delta y_L)^{1/\zeta}$ from the end. We will return later to further consequences of this picture.

Two very recent papers[14] on the one-dimensional model have conjectured a result (based in one case on the replica method[17]) for the distribution of Φ_L for large L, which has correlations of $\Phi_L(\mathbf{y}_L)$ of the form Eq. (3.8) even for $y_L \gg L^{\zeta}$. The argument given above of approximate independence, which might quite possibly be made rigorous, casts serious doubts on these conjectures.

A. Thermal fluctuations

We now consider the behavior at small positive temperatures, which we expect will hold, qualitatively, at all temperatures in the pinned phase.

The most important observation is that, as for most low-temperature phases, the temperature is formally irrelevant. Indeed, a comparison of the thermal free-energy scale T with the characteristic scale of variations of the zero-temperature free energy L^{θ} implies that the renormalization-group eigenvalue of temperature is

$$\lambda_T=-\theta , \tag{3.11}$$

which is negative unless $\zeta\le\frac{1}{2}$. The case $\zeta=\frac{1}{2}$ is marginal and we will return to it later. For the present discussion we assume $\zeta>\frac{1}{2}$. We thus expect that for a given end point $\mathbf{y}_L$, the directed path fixed at the origin will have a preferred configuration with free energy $F_L(\mathbf{y}_L)$ and qualitatively similar behavior to that at zero temperature but with fluctuations about the preferred configuration.

Although it is formally irrelevant, the temperature nevertheless affects some of the correlation functions in a nontrivial way. For example, the thermal correlation function C_T Eq. (2.2) vanishes at zero temperature. Thus its low-T behavior is controlled by the dangerously irrelevant thermal fluctuations, similar to the behavior in other low-temperature phases, particularly random ones.[2]

If we remove the restriction that the directed path end at $\mathbf{y}_L$, we can in principle calculate the thermal fluctuations in $\mathbf{y}_L$ in terms of the free energy $F_L(\mathbf{y}_L)$. (We can similarly consider the midpoint of the directed path.) This free energy will have some absolute minimum at y_m and then vary on a scale $\sim|y-y_m|^{\sigma}$ (we have dropped the L subscript) for $a \ll y \ll L^{\zeta}$. The only appreciable contributions to the thermal fluctuations will come from y's for which $F_L(y)-F_L(y_m)\lesssim T$. For y near y_m, one expects many such "active" minima, giving rise to local contributions to the entropy. However, for larger y, the minima are unlikely to be close in free energy to the absolute minimum since the characteristic scale of $F_L(y)-F_L(y_m)\sim|y-y_m|^{\sigma}\gg T$. From the structure discussed earlier, each factor-of-2 scale in y will be approximately independent. Therefore we may consider the probability that there is an active minimum in a range say $|y-y_m|\sim z$. This will, naively, be of order T/z^{σ} since the depth of the mininum will be uniformly distributed on scales $\ll z^{\sigma}$, except for the constraint that arises from choosing the lowest minimum (at y_m) as the reference. For $z \ll L^{\zeta}$, the contribution to the mean-square thermal fluctuations

$$C_T(L)\equiv\overline{\langle \mathbf{y}_L^2\rangle-\langle \mathbf{y}_L\rangle^2} , \tag{3.12}$$

with fixed left boundary condition, will be of order $Tz^{2-\sigma}$. This is dominated by the larger values of z since $\sigma<2$. For $z \gg L^{\zeta}$, on the other hand, low free-energy minima are unlikely since the y^2/L term in Eq. (3.6) for $F_L(y_L)$ dominates the random term. Thus the dominant scale for the thermal fluctuations will be $|\mathbf{y}-\mathbf{y}_m|\sim L^{\zeta}$, where the minima occur with probability T/L^{θ}, yielding

$$C_T(L)\sim TL^{2\zeta-\theta}\sim TL , \tag{3.13}$$

i.e., $\zeta_T=\frac{1}{2}$, the same as in the unpinned phase.[16] Note that other moments of y_L will scale in a different manner since the correlation function C_T (as often happens for random systems) is dominated by rare events that occur with probability T/L^{θ}. Thus the mth moment (for even m) is

$$\overline{\langle (y_L - \langle y_L \rangle)^m \rangle} \sim T L^{m\zeta - \theta} \tag{3.14}$$

for $m > \sigma$, while for $m < \sigma$ the moments are dominated by small scale fluctuations and are thus independent of L. The result for the second moment C_T of Eq. (3.13) can be derived exactly using the same statistical invariance of the Hamiltonian that we used in deriving the scaling law $\theta = 2\zeta - 1$. The mean-square correlation function C_T is found[12] to be *identical* to that without disorder, which can be seen by adding a field $h(x)$ conjugate to $y(x)$ to the Hamiltonian, changing variables as in Eq. (3.2), and then differentiating with respect to $h(L)$. This derivation, however, does not elucidate the important physics that underlies the result.

Unfortunately, one *cannot* calculate all other moments [e.g., Eq. (3.14)] exactly; only the combinations of correlation functions that are equal to derivatives of the *average* free energy with respect to the $h(x)$ can be obtained,[12] for example,

$$\overline{\langle (y_L - \langle y_L \rangle)^4 \rangle} - 3\overline{\langle (y_L - \langle y_L \rangle)^2 \rangle^2} = 0 \ . \tag{3.15}$$

As we will show later, special cancellations occur in these quantities, which are nevertheless useful in constraining possible interpretations of the results.

An alternative definition of θ could have been given in terms of the probability $p_T(L)$ that there exists a second path with free energy within order of T of the optimum $F_L(y_m)$ and with end point separated by $|y - y_m| \sim L^{\zeta}$, via $p_T(L) \sim T/L^{\theta}$. The argument leading to Eq. (3.13) and the exact result[12] imply that this definition also yields the scaling law $\theta = 2\zeta - 1$.

B. Entropy variations

From the above discussion, we can see that there will be local contributions to the entropy from small-scale fluctuations of the directed path. These will give rise to a finite temperature correction to the free-energy density. On long scales, however, the fluctuations will be rare and not contribute appreciably to the entropy. The sample-to-sample variations in the entropy ΔS_L, can be estimated by assuming that the small scale contributions to S_L from widely separated regions are approximately independent of each other. They will then give rise to a normally distributed random contribution to ΔS_L of order

$$\Delta S_L \sim L^{1/2} \ . \tag{3.16}$$

To check that ΔS_L is dominated by the small-scale fluctuations, we can estimate the contribution from fluctuations of scale l with associated displacements $\sim l^{\zeta}$. We have

$$\overline{\Delta S_L^2} \sim \int \frac{L}{l} \frac{T}{l^{\theta}} \frac{dl}{l} \tag{3.17}$$

arising from L/l approximately independent sections of length l, which will contribute $O(1)$ to S if they are active: this occurs with probability $\sim T/l^{\theta}$. Each scale is approximately independent of scales twice as large or small, yielding the dl/l factor. The integral is dominated by small scales, supporting the previous conclusion, Eq. (3.15).

In spite of the larger variations of the entropy implied by Eq. (3.16), we still expect, since the controlling fixed point represents the zero-temperature pinned phase, that $\Delta F_L \sim L^{\theta} \ll \Delta S_L$ as long as $\theta < \frac{1}{2}$, which is believed to hold in all dimensions. This implies that the energy variations at positive temperature must also be of the form $\Delta E_L \sim L^{1/2}$ but with contributions to the free energy that almost cancel the entropic contributions $T\,\Delta S_L$. This behavior occurs because it is the coarse-grained *free* energy that is minimized by the optimum coarse-grained path of the directed path (with only rare large-scale fluctuations around it), while the energy and entropy are *not* independently optimized. The small-scale fluctuations determine the coarse-grained free energy.

We will present numerical evidence in Sec. V in support of this conjectured behavior for the one-dimensional case.

IV. SENSITIVITY TO BOUNDARY CONDITIONS, "STATES," AND SENSITIVITY TO TEMPERATURE CHANGES

From the illustration in Fig. 1 and the discussion above, we have seen that the ground state (and preferred "state" at positive temperature) is sensitive to the boundary condition at the end. We do not, however, expect this sensitivity to be too extreme. To see this let us consider two minimal paths from the origin to two points at a distance L separated by $\Delta y_L \sim L^{\zeta}$. Typically, these paths will be joined at a distance of order a fraction of L from the origin. What is the probability that, instead, they are only joined up to a separation point a distance $x_s \ll L$ from the origin? The hypothesized scale invariance of the tree in Fig. 1 suggests that given that the paths have not joined at distance l there is some probability $p_s(l)$ that the paths still have not joined at distance $l/2$. Since the typical separations of paths that have not yet joined at l are order l^{ζ}, the definition of p_s appears scale invariant so we expect p_s to be roughly independent of length scale. It should also be true that at widely separated length scales, the probabilities are nearly independent. Thus the probability that $x_s \ll L$ will be roughly given by

$$\text{Prob(paths separate before } x_s) \sim (p_s)^{k \ln(L/x_s)} \sim \left[\frac{x_s}{L} \right]^{k \ln(1/p_s)} \tag{4.1}$$

with some constant k yielding a power-law dependence on x_s/L with possibly a non-trivial exponent, $\rho = -k \ln p_s$.

If, on the other hand, the pair of end points are separated by a much larger distance $\Delta y_L = \alpha L$ with α a nonzero angle, then the expected separation will occur at a finite distance $x_s \sim \alpha^{-1/(1-\zeta)}$, since at this separation, αx_s and the random component of $y(\alpha_s)$ are comparable. In this case, however, the free-energy *density* of the paths generally differs because of the radically different boundary conditions—thus we can hardly expect the paths not to differ over a positive fraction at their length.

For the intermediate case with $\Delta y_L = \gamma L^{\zeta}$ with $\gamma \gg 1$, a similar argument implies that the paths will typically separate at a distance $x_s \sim \gamma^{-1/(1-\zeta)} L$. Note that this result suggests a form for the decay of the correlations of the free energies $\Phi_L(\mathbf{y}_L)$ introduced in the previous section: the correlations between the free energy of paths with different end points will be at least as large as the mean-square free-energy variations of the section of the path over which they overlap. Thus we expect that the truncated correlations

$$\overline{\Phi_L(\mathbf{y}_L+\Delta\mathbf{y}_L)\Phi_L(\mathbf{y}_L)}-(\overline{\Phi_L})^2 \sim [x_s(\Delta y_L)]^{\theta} \sim \left[\frac{L}{\Delta y_L}\right]^{2\theta/(1-\zeta)} \quad (4.2)$$

for $\Delta y_L \gg L^{\zeta}$.

We now return to the general problem of separation probabilities and define $P_s(x_s, L, \Delta y_L)$ to be the probability that two minimal paths with a common origin that terminate at a distance L at end points separated by Δy_L first separate at a distance less than x_s. We conjecture that for *any fixed* x_s

$$\lim_{L\to\infty} P_s(x_s, L, \Delta y_L = CL^{\lambda}) = 0 \quad (4.3)$$

for all constants C and all $\lambda < 1$. This implies that in the appropriate thermodynamic limit, given an average angle and a constraint that it must pass through a given point (e.g., the origin), there is, with probability 1, a *unique ground-state path.*

It is useful to estimate the form of the decay of the probability P_s in Eq. (4.3). For $\lambda > \zeta$ (i.e., greater than typical end-point separation, we expect that, from similar arguments to above, $P_s \sim [x_s / L^{(1-\lambda)/(1-\zeta)}]^{\rho}$, as $L \to \infty$. For $\lambda < \zeta$, on the other hand, so that the end-point separation is small, the probability will decay rapidly even for $x_s \sim L/2$. A simple guess is that this probability is just given by the probability that nearby "leaves" (separated by L^{λ}) of the tree are connected to different primary branches. This is of order L^{λ}/L^{ζ}. So for $\lambda < \zeta$, we conjecture that $P_s \sim L^{\lambda-\zeta}(x_s/L)^{\rho}$, as $L \to \infty$.

At positive temperature, we can consider, instead of ground-state paths, the correlation functions of the section of the directed paths between the origin and x_s as a function of the boundary condition at $x = L$. The analogous statement to Eq. (4.3) is that with probability one, the correlation functions involving only points with $x < x_s$ at a fixed temperature are, in the limit $L \to \infty$ with $\Delta y_L = CL^{1-\epsilon}$, the same for the two boundary conditions for all C and positive ϵ. Nevertheless, we will see that the configurations are extremely sensitive to temperature changes.

The relative insensitivity to boundary conditions far away is directly analogous to that predicted (by us) for spin glasses with short (or finite) -range interactions.[18] The sensitivity to temperature changes was also predicted for spin glasses.[19,20] The random directed path is the simplest random system with a non-trivial randomness-dominated low temperature phase in which related issues arise.

A. Replica-symmetry breaking

Several recent papers[14,16] have argued that some kind of "replica-symmetry breaking" or "existence of many states" occurs in the random direct path. Indeed, it has been claimed that the very existence of low-lying excitations on long length scales is equivalent, in some undefined sense, to "many states." Thus far, however, there has been no real definition of "replica-symmetry breaking" in finite dimensions, either here or for spin glasses,[1] and certainly none that is related to the existence, in any well-defined sense, of "many states." Indeed, it appears quite likely that this much ballyhooed but ill-defined concept will turn out not to have physical consequences in finite-dimensional systems of a qualitatively different nature from those that we have discussed here and in earlier papers on spin glasses.[2,18] If, on the other hand, there do really turn out to be many states in spin glasses in some finite dimension, it is still unclear what the special properties of the infinite-range model would have to do with this. Indeed difficulties of interpretation of "replica-symmetry breaking," etc., already occur for spin glasses on Bethe lattices.[21]

For the case at hand of the random directed path, some definition of many states is needed that differs from the conventional ones appropriate for bulk systems. This is because the directed path occupies an infinitesimal fraction of the space in the thermodynamic limit. For example, if neither end of the directed path is constrained, then in a large system with both ends of the directed path taken to infinity, it will pass infinitely far from the origin (or any other prechosen point) with probability one. From arguments like those given above, this is true no matter what sequence of boundary conditions is used to define the thermodynamic limit. Thus, in the usual sense of bulk systems there is *no* "state" of an infinite system with a directed path passing through any prechosen finite region of it.

Some other definition is therefore needed to address the questions analogous to those in spin glasses. In the above discussion, we have chosen a particular such definition by fixing one end of the directed path at the origin. It was concluded that, nevertheless, there is still a unique "state" of the directed path for each macroscopic angle.

It is not inconceivable, however, that with some other definition, or alternate consistent picture of the behavior, many states with the same macroscopic angle might somehow exist. It seems to us to be incumbent upon those who so believe to come up with such a definition and scenario rather than just claiming,[14,16] in effect, that the large-scale low-free-energy excitations appearing in the present picture are different "states." For the case of spin glasses in which there are precise definitions of states,[18] the presence of such large-scale excitations certainly does not imply many states,[18] but perhaps large-scale low-free-energy excitations are all that "replica-symmetry breaking" implies.

B. Sensitivity to temperature changes

We now return to concrete questions and consider what happens to a given sample (i.e., configuration of the random potential) as the temperature is changed by a small amount, ΔT. For definiteness, we again consider a path fixed at the origin but free at its other end, a distance L away. With a given end point $\mathbf{y}_L$, we can estimate the effect of a small temperature change on the free energy:

$$F_L(T+\Delta T, \mathbf{y}_L) \simeq F_L(T, \mathbf{y}_L) - \Delta T\, S_L(T, \mathbf{y}_L) \ . \qquad (4.4)$$

where S_L is the entropy. If we consider two different end points with $\Delta y_L \sim L^{\zeta}$, then for a finite fraction of their lengths, the two direct-path configurations at temperature T will typically not overlap as discussed earlier. Thus, from the above arguments we expect that

$$F_L(T+\Delta T, \mathbf{y}_L + \Delta\mathbf{y}_L) - F_L(T+\Delta T, \mathbf{y}_L) \sim L^{\theta} - \Delta T\, L^{1/2} \ , \qquad (4.5)$$

where the second term arises from the random local contributions to the entropy *difference* between the two paths. [We have ignored temperature-dependent prefactors that can depend on the small scale details of the model. Note that in addition $S_L(T, \mathbf{y}_L)$ will contain a nonrandom piece linear in L that corresponds to the change in the average free-energy density with temperature.] We thus see that when $\Delta T \gtrsim L^{\theta - 1/2}$ the change in the free-energy difference Eq. (4.5) is comparable to the original free-energy differences, and the expansion in ΔT breaks down. With the end point free to adjust to minimize the free energy, we thus expect the optimal displacement of the end of the directed path at temperature $T+\Delta T$ to be virtually independent of that at temperature T, provided

$$\Delta T \gg L^{\theta - 1/2} \ . \qquad (4.6)$$

We can turn this around to conclude that directed paths of length L at temperatures T and $T+\Delta T$ will, if

$$L \gg L_{\Delta} \sim (\Delta T)^{-1/(1/2-\theta)} \ , \qquad (4.7)$$

have optimal configurations that differ radically. (In these expressions [Eqs. (4.6) and (4.7)] there are prefactors whose values are unknown, but our numerical work in one dimension (Sec. V) suggests they can be surprisingly large.)

This temperature sensitivity of the low-temperature phase of random directed paths appears at first sight quite remarkable and is analogous to similar effects in spin glasses.[2,19,20] It has important consequences for the equilibrium of such systems as they are cooled,[2] which we will not delve into here.

In a certain sense, the temperature sensitivity is like an infinite sequence of infinitesimal first-order phase transitions. From a renormalization-group point point of view, it can be readily understood in terms of the scale and temperature dependence of the effective Hamiltonians that govern the system.[2,19] Since the zero-temperature random-directed-path fixed point is stable against temperature variations (for $\theta > 0$), we expect that the *statistical* properties of the coarse-grained Hamiltonian at finite temperature will be the same (up to an overall free-energy scale) as those at zero temperature. However, the *actual* coarse-grained Hamiltonian, which can be minimized to obtain the optimal configuration of a *specific* long but finite directed path, will *not* be similar to the coarse-grained version of the microscopic Hamiltonian of the same system at zero temperature. This divergence of the renormalization-group flows under small temperature changes has been studied explicitly for hierarchical spin-glass models by Bray and Moore.[20]

V. RESULTS IN ONE DIMENSION

In this section we present various analytical and numerical results for directed paths in one transverse dimension that lend some support to the general picture presented in the previous sections.

A. Analytic results

It is convenient to consider a continuum model with no cutoff in either direction, i.e., with correlation function of the Gaussian random potential

$$\overline{V(x,\mathbf{y}), V(x',\mathbf{y}')} = \delta(x-x')\tilde{\Gamma}\delta(\mathbf{y}-\mathbf{y}') \ , \qquad (5.1)$$

where $\tilde{\Gamma} = \int \Gamma(\mathbf{y}) d^d y$ and the limit $a \to 0$ is taken. This model is only well defined at positive temperature, and infinite contributions to the free energy from small-scale fluctuations need to be subtracted. It can readily be seen that the thermal fluctuations provide an effective short-distance cutoff in the y direction at

$$a_T \sim \left[\frac{T^3}{\kappa \tilde{\Gamma}} \right]^{1/(2-d)} \qquad (5.2)$$

and a concomitant cutoff in the x direction,

$$b_T \sim \frac{a_T^2 \kappa}{T} \ . \qquad (5.3)$$

Note the appearance of $2-d$ in Eq. (5.1); this is related to the irrelevance of weak disorder in dimensions $d > 2$ as noted earlier. The cutoff-free model is therefore only well defined in dimensions $d < 2$. We will thus concentrate on $d = 1$.

In one dimension, for length scales smaller than b_T, the thermal fluctuations with displacements of the directed path $< a_T$ effectively average out the random potential. The effects of the random potential only become important for scales $L \gg b_T$; it is hence this limit we are interested in. As we will see, the zero-temperature limit, which is believed to dominate the behavior, is thus rather subtle because of the T dependence of the cutoffs. [For fixed L, it will turn out that we must measure displacements y in units of $(\tilde{\Gamma}/\kappa T)^{1/3}$ and energies in units of T to take the limit $T \to 0$, at fixed L.[22]] The advantage of the continuum model is that some exact results can be obtained,[3] in particular the exponent σ from which the other exponents can be extracted by the scaling laws discussed in the previous sections.

In $1+1$ dimensions, we can straightforwardly derive a differential equation for the restricted partition function $Z(x,y)$ for a given realization of a directed path ending at distance x at a displacement y.[3] This is simply

$$Z(x,y) \equiv e^{-F(x,y)/T} \tag{5.4}$$

with, in the earlier notation $F(x,y)=F_x(y_x)$ for a path starting at the origin. We obtain, after integrating, the transfer operator from x to $x+dx$ over $y(x)$,

$$\frac{\partial Z}{\partial x} = \frac{T}{2\kappa}\frac{\partial^2 Z}{\partial y^2} - \frac{V(x,y)Z}{T} \ . \tag{5.5}$$

For the free energy up to distance x, we then have

$$\frac{\partial F}{\partial x} = \frac{T}{2\kappa}\frac{\partial^2 F}{\partial y^2} - \frac{1}{\kappa}\left[\frac{\partial F}{\partial y}\right]^2 + V(x,y) \ . \tag{5.6}$$

Taking one derivative and defining

$$u \equiv \frac{2}{\kappa}\frac{\partial F}{\partial y} \ , \tag{5.7}$$

we obtain

$$\frac{\partial u}{\partial x} = \frac{T}{2\kappa}\frac{\partial^2 u}{\partial y^2} - u\frac{\partial u}{\partial y} + \frac{2}{\kappa}\frac{\partial V}{\partial y} \ , \tag{5.8}$$

which is Burgers's equation with conservative noise. Some properties of this equation are known, in particular the invariant distribution[5]

$$P_\infty\{F(y)\} \sim \exp\left[-C\int dy \left[\frac{\partial F(y)}{\partial y}\right]^2\right] \ , \tag{5.9}$$

with $C=T^3/2\tilde{\Gamma}\kappa$. At this point, however, it is not known how to solve Burgers's equation with noise for a fixed initial condition, nor how to calculate correlation functions involving more than one different "time" x. Indeed, even the relevance to the problem at hand of the invariant measure is somewhat problematical. Fortunately, a natural interpretation is available in terms of the discussion of Sec. III.[14]

For separations Δy_L much less than L^ζ, we expect the correlations in the free energy $F_L(y_L)$ of paths from the origin to end points y_L to be independent of L, with

$$F_L(y_L) - F_L(y_L') \sim |y_L - y_L'|^\sigma \ . \tag{5.10}$$

Thus we conjecture that, in general, the distribution of such ΔF_L's will be independent of L and be given by the explicit distance independent invariant measure Eq. (5.9). From this we conclude that in one dimension the statistics of $F_L(y_L)$ as a function of y_L are just those of a random walk, so that $\sigma=\frac{1}{2}$. Somewhat surprisingly, this is true even for separations smaller than the effective thermal cutoff a_T given by Eq. (5.2). From σ we obtain the results

$$\zeta = \tfrac{2}{3} \tag{5.11}$$

and

$$\theta = \tfrac{1}{3} \tag{5.12}$$

by the scaling laws Eqs. (3.4) and (3.9). These exponents have been conjectured by Huse and Henley[3] on the basis of direct numerical calculations at zero temperature for a lattice model. The numerical calculations thus provide supporting evidence for the universality of the exponents. The exponents have also since been derived by other, formal, methods such as via Bethe ansatz for a system of parallel directed paths with short-range interactions[23] whose scaling behavior can also be predicted on the basis of the general scaling arguments given in this paper. (See also Ref. 17.)

It would, of course, be highly desirable to be able to obtain further results directly from the Burgers equation with noise; unfortunately, at this juncture, this does not appear to be easy.

The scaling laws have recently also been derived from the replica method for a single directed path using somewhat questionable assumptions.[14] Note, however, as mentioned earlier, that some of the other results obtained by this method appear to be erroneous.

B. Entropy variations

In order to test our expectation that the entropy and energy variations diverge faster with L than the free-energy variations we have examined a 1D model using the transfer-matrix technique. The model is on a lattice with Hamiltonian

$$H = \sum_{x=1}^{L} \{ V(x,y) + E_s |y(x+1) - y(x)| \} \ , \tag{5.13}$$

where the random potentials $\{V(x,y)\}$ are independent and distributed uniformly in the interval (0,1), x and $y(x)$ are integers, and the nearest-neighbor displacements are restricted to satisfy $|y(x+1)-y(x)| \le 1$. We work at temperature $T=1$ and the step energy of $E_s=4$ was

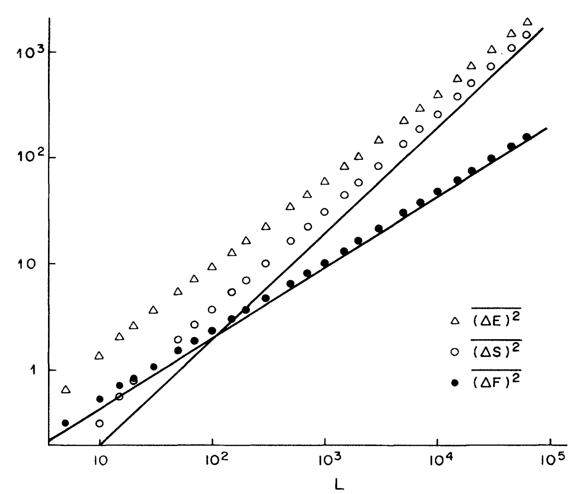

FIG. 2. Sample-to-sample variances of energy, entropy, and free energy for directed paths of length L on a lattice with one end pinned at temperature $T=1$. The slopes of the solid lines on this log-log plot are $\frac{2}{3}$, as expected for $\overline{(\Delta F)^2}$, and unity, as expected for large L, for $\overline{(\Delta E)^2}$, and $\overline{(\Delta S)^2}$. See text for precise statement of the model and parameters used.

chosen to maximize the relative entropy variations for small L. Systems of size up to 800 000 in the y direction were studied with periodic boundary conditions imposed in the y direction. For each y, the free energy $F(T=1, y_0=y, L)$ and the expectation values of the energy E, and the entropy S were calculated for strings of length L with one end fixed at y and the other end free. The variances (mean-square fluctuations) of these three quantities are shown in Fig. 2 on a log-log plot, along with the expected asymptotic slopes of $2\theta=\frac{2}{3}$ for $\overline{(\Delta F)^2}$ and unity for $\overline{(\Delta S)^2}$ and $\overline{(\Delta E)^2}$. Note that at the largest length of $L=61\,000$ the variances of the energy and entropy exceed that of the free energy by roughly 1 order of magnitude, and so the free-energy differences are near cancellations of significantly larger energy and entropy differences, as expected. Note also that the entropy variance does not exceed the free-energy variance until $L\simeq 25$, even though we have chosen parameters that nearly minimize the position of this crossing. As argued above, a consequence of this stronger divergence of the entropy variance should be sensitivity of the state to temperature changes. Thus we have compared the free energies at two different temperatures in the same sample.

Let $F(T,y,L)$ be the free energy of a string of length L in a given sample at temperature T with one end fixed at y and the other free at L. The deviation of this from its average over samples is

$$\Delta F(T,y,L)=F(T,y,L)-\overline{F(T,y,L)}\ . \tag{5.14}$$

We have measured the correlation between ΔF at $T=1$ and $T=2$ for $E_s=5$, constructing the correlation function

$$G_F(L)\equiv\frac{\overline{[\Delta F(1,y,L)\Delta F(2,y,L)]}^2}{\overline{[\Delta F(1,y,L)]^2}\,\overline{[\Delta F(2,y,L)]^2}}\ , \tag{5.15}$$

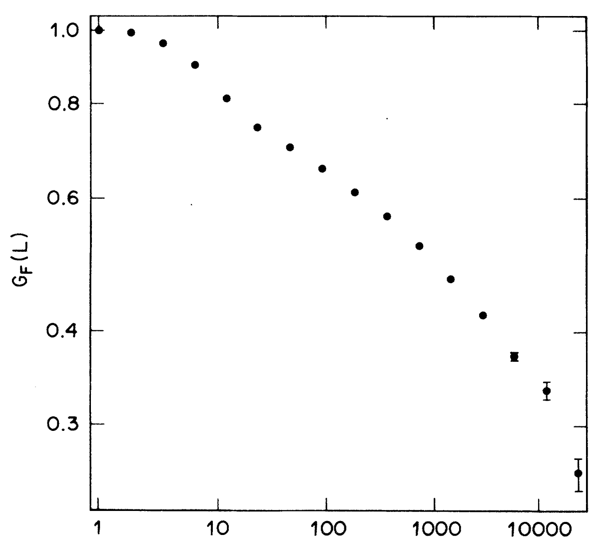

FIG. 3. Correlation $G_F(L)$, as defined by Eq. (5.15), between the free energies of directed paths of length L at two temperatures differing by a factor of 2. See text for precise definition of the model and parameters used.

which is shown in Fig. 3 on a log-log plot. This correlation function should decay to zero at large L, reflecting the system's sensitivity to temperature changes. The length scale on which this happens would naively be of the order of that where the crossing of $\overline{(\Delta S)^2}$ and $\overline{(\Delta F)^2}$ in Fig. 2 occurs ($L\simeq 50$ for these T,E). In fact, the decay of $G_F(L)$ is seen to be much slower, but does appear to be accelerating on this log-log plot at the largest lengths studied. For large L, a simple expectation is that the states at the two temperatures overlap strongly only near the pinned end over some length, L_0. This would result in an asymptotic decay of the correlations as

$$G_F(L)\approx\left[\frac{L_0}{L}\right]^{4\theta}\ . \tag{5.16}$$

The behavior in Fig. 3 is not of this form, but it could be headed for such a decay law at somewhat larger L, albeit with a surprisingly large L_0. Understanding of the source of the large length scales that appear must await further progress.

VI. PINNED PHASE IN HIGHER DIMENSIONS

In dimensions $d+1$ greater than $1+1$, there are no exact results for the low-temperature phase. As discussed in Sec. II, we do, however, expect a pinned phase for strong disorder or low temperatures in dimensions $d>2$. In exactly two dimensions, the disorder is marginally relevant[5,13,24] and the system will always be in the pinned phase with the behavior dominated by disorder for length scales larger than $\xi_T\sim\exp(4\pi T^3/\tilde{\Gamma}\kappa)$.

The exponent θ that characterizes the pinned phase has been calculated numerically by various authors for $d=2$, 3, and 4 dimensions. The recent numerical results of Kim and Kosterlitz[11] for a related growth model (see also references cited therein) are consistent with their conjecture $\theta=1/(d+2)$, although we see no reason to believe that this result is exact. Another approach that gives an estimate of $\theta(d)$ is the Migdal-Kadanoff approximate renormalization group. As discussed by Derrida and Griffiths[9] and by Halpin-Healy,[25] this is an uncontrolled approximation for d-dimensional Euclidean lattices, but is exact for certain hierarchical lattices. We have performed numerical calculations on such hierarchi-

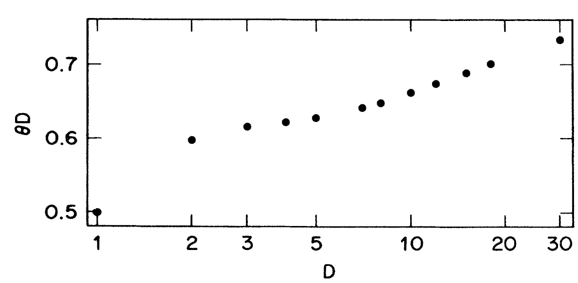

FIG. 4. Free-energy exponent θ, multiplied by $D=d+1$, as calculated within the Migdal-Kadanoff approximation as a function of D. This is a semilogarithmic plot. The exponents are exact (up to numerical solution of the fixed-point equations) for the Berker hierarchical lattices.

cal lattices to calculate θ. We show θD, where $D=d+1$, versus D on a semilog plot in Fig. 4. It is clear that θ does not vanish here for any finite D but it appears that it may behave as $\theta \sim (\ln D)/D$ for large D. The (Berker) hierarchical lattices[9,25] for which this is the exact θ do have loops on all length scales, and thus they may represent more accurately the large but finite D real lattices than does the Cayley tree or Bethe lattice (discussed below), which has no such large loops.

It can be seen that there is a tendency for θ to decrease with increasing dimensions. A lower bound of zero is implied, at least for a positive temperature phase, by the requirement that the thermal eigenvalue $\lambda_T = -\theta$ be nonnegative.

If θ were negative, thermal fluctuations would be relevant and no pinned phase could exist except at zero temperature. Indeed, it appears extremely unlikely that such a situation would exist at all. First, note that the elastic energy cost of a directed path with a uniform angle giving rise to a displacement $L^{1/2}$ is only of order 1, independent of L. If θ were negative then small-scale distortions that give rise to energies of order unity would dominate over the large-scale contributions to the energy and thus control the optimal position of the directed path, apparently leading back to $\zeta=\frac{1}{2}$ and $\theta=0$. Thus we expect that even at *zero temperature*, $\theta \geq 0$ and $\zeta \geq \frac{1}{2}$. This result may well be provable. An important question is whether θ is strictly positive in any finite dimension but asymptotically vanishing as $d \to \infty$ (as in the above Migdal-Kadanoff approximation) or whether there is an upper critical dimension for the pinned phase d_c above which θ vanishes and $\zeta=\frac{1}{2}$. The remainder of this section is devoted to this equation.

A. High-dimensional limits?

For many conventional critical phenomena one of the most productive routes to understanding the critical behavior has been to analyze some high-dimensional limit and then attempt to expand about it, for example, by ϵ expansions about an upper critical dimension. This program has been attempted for various systems with quenched randomness and appears to work well for properties of phase transitions from entropy-dominated disordered phases to energy-dominated ordered phases. For example, the *critical* behavior of random exchange, random field, and spin-glass magnets can be analyzed in ϵ expansions about 4 or 6 dimensions.[26–28] Even there, however, some crucial features are missed, for example, the critical dynamics of random-field Ising magnets.[29] In other random systems, the situation is much worse; it appears that there may well be no useful high-dimensional limit of the phase transitions at all. Examples are the quantum-mechanical localization transitions for either noninteracting fermions[30] or interacting bosons.[31]

What we are faced with in trying to understand the pinned phase of random directed paths is something rather different: the properties of a *phase* (rather than a transition), which is dominated in a nonperturbative way by the randomness. The best studied problem in this class is the ordered phase of spin glasses about which there is still a great deal of controversy,[1,2] particularly about whether there is an upper critical dimension above which the ordered phase of spin glasses becomes equivalent, in some sense, to the infinite-range Sherrinton-Kirkpatrick model. We have previously argued that this is *not* the case.

An important issue that arises for both spin glasses and random directed paths[2,18] is which models are appropriate to analyze the high-dimensional limit, and concomitantly which properties of these models should be taken seriously as predictions for the behavior of systems in large but finite dimensions. For the random directed path there is currently only one candidate model that may in some sense exactly reproduce the high-dimensional limit: the directed path on a Cayley tree, studied recently by Derrida and collaborators.[8,32] Before summarizing the picture that emerges from these studies, it is worth considering the successes and failures in other problems of models on relatives of the Cayley trees that have no boundaries, i.e., Bethe lattices.

Generally, the thermodynamic properties of conventional phase transitions in high dimension are well approximated by those on Bethe lattices, since in high-temperature expansions the scarcity of loops in high dimensions is well approximated by the absence of loops on the Bethe lattice. The properties of ordered phases (and also correlation functions) fare rather more poorly. Indeed, various pathologies are found on Bethe lattices, notably the presence of infinitely many infinite clusters for percolation (which does not occur in *any* finite dimension),[33] and the exponential decay of two-point correlation functions *at* criticality, which must be interpreted with great care. The status of more subtle random problems on Bethe lattices, such as spin glasses[21] and localization,[30] are still controversial, although certain pathological features definitely occur.

B. Random directed paths on a Cayley tree

We now describe the model of a random directed path on a Cayley tree, properties of its solution, following Derrida and collaborators,[8,32] and problems that arise.

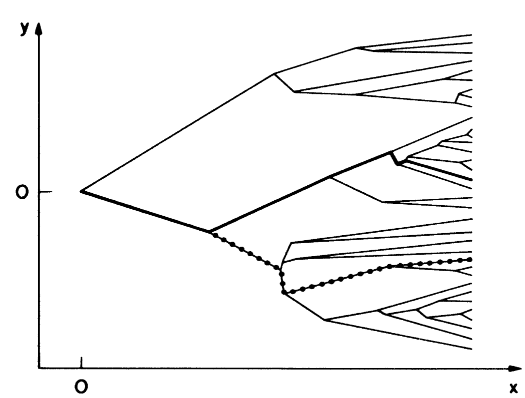

FIG. 5. Randomly branching Cayley tree. The lowest-energy path is shown by the bold line; the lowest-energy path that branches off from the ground state at a given distance from the origin is shown by the dotted line.

The tree is arranged as shown in Fig. 5 with the root (which we take to be the origin) at the left with branches up and down as one moves to the right. The simplest case has only twofold branchings. The directed path is fixed at the root and has a free boundary a distance of L steps along the tree. Its energy is the sum of the random potentials $V(x,y)$ (with mean zero and variance Γ) of all the bonds it traverses.

For analytical calculations it is more convenient to consider a randomly branching tree that has a probability dx/b of a twofold branching in distance dx along each branch. There is a random potential with δ-function correlations in x at each point along the branches so that the typical random potential of a section of a branch grows as the square root of its length. Derrida and Spohn[8] (DS) have investigated various properties of this system. They find that it has two phases. the high-temperature free phase occurs for $T>T_c$. Below the phase transition at T_c, the model exhibits a pinned, almost frozen, phase in which the directed path coincides with the single optimal path on the tree with *nonzero probability* in the limit $L\rightarrow\infty$. In the whole of this phase, the free-energy density f is equal to the ground-state-energy density $\overline{\epsilon}_0$ thus the system does not have extensive entropy.[8] Indeed the entropy turns out to be of order unity arising, with high probability, primarily from alternate paths along the tree which branch off from the ground state at *small* distances from the root. This can be seen explicitly via computation of the distribution of overlaps of paths on identical copies of the tree.[8]

To understand the low-temperature phase; it thus suffices to study the ground state and its low-energy excitations. The exponent θ can be defined by considering the width of the distribution of ground-state energies E_L for systems of length L. DS[8] have shown that the distribution of $E_L-\overline{E}_L$ attains a limiting form as $L\rightarrow\infty$ with width of order unity: i.e., $\theta=0$. If we interpret up and down steps on the tree as steps in some direction on a finite-dimensional lattice, then the symmetry of the tree immediately implies that the distribution of steps in the ground state is the same as a random walk along the tree, so that $\zeta=\frac{1}{2}$. The energy variations are thus consistent with the scaling law $\theta=2\zeta-1$. Note that, in general, the structure of the tree implies that $\langle y(x)^2\rangle$ is independent of temperature. Extension of the model to include small-scale loops[32] (up to a finite maximum size) allows for some entropy at positive temperatures, but preserves, because of the basic tree structure, the features that yield $\zeta=\frac{1}{2}$ and $\theta=0$.

Another feature of random directed paths on the Cayley-tree is the finite-size correction to the ground-state energy with a free-boundary condition at the far end. Unless there are exact cancellations from the effects of the two ends, we expect, in finite dimensions, that

$$\overline{E_L-\overline{\epsilon}_0 L}\sim L^{\theta}\ , \tag{6.1}$$

where the ground-state-energy density is defined by

$$\overline{\epsilon}_0=\lim_{L\rightarrow\infty}(\overline{E}_L/L)\ . \tag{6.2}$$

In the case $\theta=0$, L^{θ} is replaced by $\ln L$ arising from combinations of l^{θ} from each factor of 2 in length scale l away from the origin. This nonrandom logarithmic dependence on L is found for the Cayley tree by DS.[8,34]

So far so good. A closer examination of some of the properties on the Cayley tree unfortunately reveals various pathologies. The most obvious of these is an immediate consequence of the tree structure: if we fix both ends of the directed path, the path is totally constrained (or, with small-scale loops allowed, constrained at long scales) and hence will have a higher energy *density* than the unconstrained ground state: This cannot occur in finite dimensions. More subtle are the properties of the unconstrained ground state itself.

In addition to the ground-state energy of the whole directed path to the right of the origin, it is instructive to also consider the ground-state energy of *sections* of the ground-state directed path. We define $E(x',x;L)$ to be the energy of the section of the ground-state configuration between distances x' and x of the origin, with the other end of the path free at a distance L. In the Appendix, we calculate the asymptotic behavior of this quantity. For $L\gg x\gg b$ (the mean branching distance), we find that the distribution of $E(x,x')$ is L independent, with, for example,

$$E(0,x;L)-\overline{\epsilon}_0x\rightarrow E(0,x)-\overline{\epsilon}_0x\sim x^{1/2}\ , \tag{6.3}$$

with *negative* mean value and variations of order $x^{1/2}$—a somewhat surprising result in light of the expectation that $\theta=0$.

This behavior—which suggests an alternate definition of θ with value $\frac{1}{2}$—is incompatible with the structure on a real finite-dimensional lattice discussed in the earlier sections. Specifically, if the last portion of length x of a much longer directed path had energy that varied by an amount of order $x^{1/2}$, alternate paths would exist with $\zeta=(1+\theta)/2=\frac{3}{4}$ rather than $\frac{1}{2}$. Further calculations on the Cayley tree (as in the Appendix) show that the energy of sections of length $l\ll L$ at distance x much greater than l from either end actually has a normal distribution with

$$E(x,x+l)-\overline{\epsilon}_0 l\sim l^{1/2}\ . \tag{6.4}$$

Again, the difference between this result and that for the total energy variations suggests that the role played by the constraints at the end of the directed path is much more special on the Cayley tree than on real lattices.

We now study excitations from the ground state. Specifically, we consider the minimal energy of paths that branch off from the ground-state path at position x as shown in Fig. 5. A simple argument shows that if the ground-state path has anomalously low energy (which, from the above discussion, it typically will), the side branches will be typical; i.e., they will have essentially the same distribution for large x as ground-state paths in a system of length $L-x$. Thus the lowest-energy path branching off from the ground state at a distance x from the root will typically have energy $\Delta(x)\sim x^{1/2}$ above the ground state.

By examining the distribution of $E(0,x;L)$ in more detail, we can understand why the entropy is finite at low temperatures. From the Appendix we conclude that the probability that the difference in energy $\Delta(x)$ between the ground state and the minimal side branch at x is of order one (or smaller) is of order $1/x^{3/2}$. Thus the probability that there are *no* low-energy branches beyond a distance x is bounded by

$$\text{Prob}(\Delta(x')>c \quad \text{for all } x'>x)>1-O(x^{-1/2}) \tag{6.5}$$

for all finite c. Since higher-energy branches will only contribute amounts of order $e^{-\Delta/T}$ to the entropy, we can conclude that, at low temperatures, the entropy arises almost entirely from branches in the part of the tree nearest to the origin, leading to a total entropy of order one. This behavior is again very different from that expected on any real lattice, for which all the way along the minimal path we expect there to be some low-energy "detours" of all possible sizes.

This difference can be seen by examining the source of contributions to the low-temperature thermal fluctuations of a section of the directed path,

$$C_T(x,x';L)\equiv\overline{\langle[y(x)-y(x')]^2\rangle-\langle y(x)-y(x')\rangle^2}\ , \tag{6.6}$$

where $y(x)$ is the number of up steps minus the number of down steps out to a distance x. For $x=0$ the dominant contributions will come from the active branches that branch off near the origin, yielding, since the minimal paths on these side branches are virtually independent of each other for large x',

$$C_T(0,x';L)\sim Tx'\ , \tag{6.7}$$

which is the same form as the exact result for finite-dimensional continuum models. For $|x-x'|\ll x\ll L$

$$C_T(x,x';L)\sim T|x-x'| \tag{6.8}$$

valid again both for the Cayley tree[8] and real lattices. In this case, however, the source of the behavior is very different: On real lattices, we expect a significant contribution from active detours between x and x', while on Cayley trees there will typically be *no* active branches between x and x', so that all the dominant contributions will arise from the branches near the origin. Unfortunately, higher-order correlation functions of the fluctuations of the y's are needed to see this difference directly; these cannot be computed exactly in finite dimensions.

On the Cayley tree, the side branches near the origin give rise with high probability to low-free-energy excitations with end points separated by $O(L^{\zeta})$. This behavior is *not* possible in finite dimensions for $\zeta>\frac{1}{2}$, as it would give rise to a C_T that grows more rapidly than L. It also suggests that the nontrivial "overlap" function $P(q)$ found for the Cayley tree[8] (which arises from these excitations) *cannot* exist with $\zeta>\frac{1}{2}$.

Because of these and other related differences, it appears unlikely that the Cayley tree represents a sensible high-dimensional limit, at least as far as the correlation functions and exponents in which we are interested are concerned. The expansions of Derrida and Cook[32] which insert finite sections of high dimensional lattices in place of the nodes of the tree are thus unlikely to lead to the correct form for the long-distance correlation functions due to difficulties in exchanging the $L\to\infty$ and $d\to\infty$ limits. They may, however, yield the correct high-dimensional expansion for the free-energy density and the transition temperature, although even this has not been convincingly demonstrated.

C. Can $\zeta=\frac{1}{2}$ in high dimensions?

We now turn to the general question of whether it is consistent to have a low temperature phase with $\zeta=\frac{1}{2}$ in sufficiently high dimensions.[35] At this point we must distinguish between whether or not logarithmic terms appear, particularly in the free-energy variations. It is hard to rule out, on any general grounds, behavior such as

$$W\sim L^{1/2}(\ln L)^{\tilde{\zeta}}\ , \tag{6.9}$$

with $\tilde{\zeta}$ positive, although we do not know of any reason to expect this. We thus restrict the discussion to consideration of pure $W\sim L^{1/2}$ behavior in a pinned phase.

Since this phase is, by assumption, dominated by the randomness, the amplitude of the wandering should go to a constant as $T\to 0$. We therefore expect that, in some sense, the distribution of free-energy variations will be scale independent even at zero temperature since with $\zeta=\frac{1}{2}$, $\theta=0$. In this marginal case, however, there are important subtleties.

We first consider a system of width $W\sim L^{1/2}$ with, say, periodic boundary conditions in both directions. Here there are no end effects, and so we expect the distribution of ΔF_L to be scale independent. However if instead, as we have done all along, we fix one or both ends in a wide strip of length L, there will be end effects that raise the energy for a fixed end, and lower it for a free end. By fixing only the left end, we can then consider both of these effects by examining the zero-temperature energy $F_L(\mathbf{y}_L)$ as a function of the other end point $\mathbf{y}_L$. The statistics of the stationary (in $\mathbf{y}_L$) distribution of the nondeterministic part of the free energy $\Phi_L(\mathbf{y}_L)$ [Eq. (3.6)] will yield, by the arguments of Sec. III, the behavior of the minima of $F_L(\mathbf{y}_L)$. For consistency with $\zeta=\frac{1}{2}$, we expect that on each length scale there will be minima of Φ_L that differ by of order unity. Thus if we first consider the absolute minimum of F_L, there will be typically one other such minimum at distance of order 1, another at distance of order 2, 4, etc., with typically a comparable number in a given excitation energy range in each factor of 2 in length scale. By the assumption of scale invariance, however, there will also be secondary low-energy excitations at each length scale above the primary excitations; these will on average be a factor of 2 higher energy above the ground state. For a typical end point $\mathbf{y}_L$, on the other hand, the energy will be much higher: essentially a factor of order unity higher for each length scale, yielding

$$F_L(\mathbf{y}_L)-F_L^{\min}\sim\ln L\pm\sqrt{\ln L}\ , \tag{6.10}$$

with a *deterministic* logarithmic part and variations only of order $\sqrt{\ln L}$.[13,35] Thus the variations of $\Delta F_L(\mathbf{y}_L)$ with fixed $\mathbf{y}_L$ will be of order $\sqrt{\ln L}$ (as will variations in the minimum due to fixing the other end), but the differences between low-lying minima will be much smaller; for example,

$$\min_{|\mathbf{y}_L|>cL^{1/2}}\{F_L(\mathbf{y}_L)\}-\min_{|\mathbf{y}_L|<cL^{1/2}}\{F_L(\mathbf{y}_L)\}=O(1)\ . \qquad (6.11)$$

In order for a low-temperature phase with $\zeta=\frac{1}{2}$ to exist, one would need to have temperature be marginally irrelevant at the zero-temperature fixed point, or alternatively a fixed line might exist at low temperatures, perhaps with some continuously variable exponents (such as ρ from Sec. IV). We will now consider the consistency of these possibilities.

It is again useful to consider the thermal fluctuations, for small T, about the ground state of a system of length L. From the above picture, the number of excitations with energy of order 1 grows as $\ln L$. Thus, naively, it appears that the temperature will be marginally relevant. A better argument is certainly needed to make this convincing: this would first have to address the consistency of a zero temperature $\theta=0$ fixed point with the above, or perhaps some other, structure. If this were possible, then a low-temperature renormalization-group expansion might be attempted. The simplest possibility is an unstable zero-temperature fixed point with the flows running off to high temperatures. This is inconsistent, however, with the proof of Cook and Derrida[7] that there *must* be a phase transition in any dimension.

An alternative possibility is a fixed line with temperature exactly marginal. At positive temperature, structure roughly like that described above has been found to occur (at least in a $d=2+\epsilon$ expansion) at the *critical point* separating the high and low temperature phases.[13,36] It is, of course, unstable to any small temperature change. Nevertheless one might conjecture that it could expand into a critical low-temperature phase in high enough dimensions with a transition analogous to the Kosterlitz-Thouless transition separating this critical phase from the high-temperature phase.

The third possibility of marginally irrelevant temperature suggests behavior qualitatively like the low-temperature phase with $\theta>0$ considered in earlier sections. This does, however, give rise to problems.

We consider, as before, the thermal fluctuations with one end fixed at the origin and the other free. The mean-square thermal fluctuations $C_T(L)$ are likely to be dominated by the rare configurations for which there is a thermally active path with end point differing by $\sim L^{1/2}$ from the lowest free-energy one. If, as expected, the probability of this occurring is of order $T/L^{\theta}\sim T$, then this yields the correct result $C_T(L)\sim TL$. The higher-order truncated correlation functions are more problematical, however. From the invariance properties of the distribution of disorder Eq. (3.3), it follows that[12]

$$Z(h_L)\equiv \mathrm{Tr}\exp(-\beta H+h_L y_L) \qquad (6.12)$$

has the same distribution as

$$Z(0)\exp\left[+\frac{h_L^2 LT}{2\kappa T}\right] \qquad (6.13)$$

implying

$$\overline{\ln Z(h_L)-\ln Z(0)}=\frac{h_L^2 LT}{2\kappa T} \qquad (6.14)$$

so that all truncated correlations, which can be obtained from derivatives of the *average* free energy, are equal to those for the nonrandom system. In particular,

$$C_T^{(4)}(L)\equiv\overline{\langle (y_L-\langle y_L\rangle)^4\rangle-3\langle (y_L-\langle y_L\rangle)^2\rangle^2}=0\ . \qquad (6.15)$$

On a lattice, there may be correction terms that are smaller by powers of L; however, the cancellation of the naively expected leading $TL^{-\theta}L^{4\zeta}$ part of Eq. (6.15) should still occur. This cancellation is straightforward to see for $\theta>0$. The contribution to $C_T^{(4)}(L)$ involves the integral over the density of states $\rho(\Delta)$ for scale L excitations,

$$C_T^{(4)}(L)\propto\int_0^{\infty}d\Delta\,\rho(\Delta)\left[\mathrm{sech}^2\left[\frac{\Delta}{2T}\right]-\frac{3}{2}\mathrm{sech}^4\left[\frac{\Delta}{2T}\right]\right]$$
$$=0+O(T/L^{\theta})\ . \qquad (6.16)$$

For $\theta>0$, $\rho(\Delta)$ is *constant* for $\Delta\ll L^{\theta}$ giving rise to the cancellation above. Higher-order cumulants similarly cancel, and, collectively, these conditions give rise to a series of equalities for the moments of $u\equiv\mathrm{sech}^2(\Delta/2T)$. If we consider, for the $\theta=0$ case at hand, only a single excitation, then these series of moment conditions should completely specify the distribution of the bounded variable u. Since the integrals for the $\theta=0$ case are no longer dominated by atypically small Δ, one can see that the only distribution, which satisfies the condition, is

$$\tilde{\rho}(u)du\propto\frac{du}{u\sqrt{1-u}}\ , \qquad (6.17)$$

which is not normalizable. This problem can be fixed for $\theta>0$ by changing slightly the form of the distribution for u close to zero; i.e., cutting it off for large Δ, giving rise only to correction terms. For $\theta=0$, however, this is not legitimate since the characteristic scale of the distribution $\rho(\Delta)$ is the same as the part that dominates the moments of u. Thus, within the approximation of only a single dominant excitation, we appear to have arrived at a contradiction for $\theta=0$.

Further work is clearly needed to see whether a contradiction can be obtained without this assumption, but it appears that some very special correlations would need to occur in order to yield a consistent picture of a pinned phase with $\theta=0$. At this point, it is tempting to conjecture that in *any* dimension, $W_L/L^{1/2}\to\infty$, as $L\to\infty$.

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APPENDIX

In this Appendix we derive some results for the ground state and excitations on the Cayley tree following a variant of the method of Derrida and Spohn (DS).[8] We follow here the notation of DS, which differs from that used in the bulk of this paper. We consider a randomly branching tree with t denoting the distance from the leaves of the tree. Each branch of the tree has a probability dt of a twofold branching in any interval dt. At every point on the tree there is a random potential with correlations

$$\overline{V(t)V(t')}=\Gamma\delta(t-t') \tag{A1}$$

along one branch and no correlations between branches. We are interested in the ground-state energy $E(t)$ of a path ending at a root a distance t from the leaves, but free at the leaf end. We define $G_t(x)=\text{Prob}[E(t)>-x]$; this is the zero-temperature limit of the quantity used by DS. A differential equation for G is straightforwardly derived

$$\frac{\partial G}{\partial t}=\frac{\Gamma}{2}\partial_x^2 G+b(G^2-G) , \tag{A2}$$

when the G^2 arises from the minimizing over two branches when they split.

We will also be interested in keeping track of the energy of sections of the ground-state path far away from the root. In particular we define $E_m(\hat{t},t)$ to be the energy up to time $\hat{t}$ of the path that has minimum energy with its root at time t. The joint probability

$$H_t(x,y;\hat{t})\equiv\text{Prob}[E(t)>-x \text{ and } E_m(t,\hat{t})>-y] \tag{A3}$$

then keeps track of the requisite information. In a similar way to the derivation of Eq. (A2), one finds that

$$\frac{\partial H}{\partial t}=\frac{1}{2}\partial_x^2 H-H+2G_t(x)H -2\int_{-\infty}^{x} H_t(z,y,\hat{t})\partial_z G_t(z)dz , \tag{A4}$$

where here and henceforth we have rescaled "time" and energy to set $\Gamma=b=1$. Note that y and $\hat{t}$ are merely carried through as parameters in Eq. (A4); this is because the branch, which has lower energy at a split, is only determined by the total energies up to that point, but not by how that energy is divided among the earlier times. We must supplement Eq. (A4) with an initial condition at $\hat{t}=t$,

$$H_{\hat{t}}(x,y;\hat{t})=G_{\hat{t}}[\min(x,y)] ; \tag{A5}$$

$\hat{t}$ and y then enter through this initial condition.

We are primarily interested in the behavior very far from the leaves. In this limit, $G_t(x)$ approaches a moving steady-state form,[8]

$$G_t(x)\approx w[x-m(t)] , \tag{A6}$$

where

$$m(t)=ct+O(\ln t) , \tag{A7}$$

with "velocity" $c=\sqrt{2}$, which is just the negative of the ground-state-energy density, $\bar{\epsilon}_0$. For our purposes, it can be seen that we may drop the $\ln t$ correction to $m(t)$; this is equivalent to taking a distribution of energies on the leaves of the fixed-point form $w(x)$. We may now change variables to

$$\tilde{x}=x-ct, \quad \tilde{y}=y-c\hat{t} , \tag{A8}$$

and define

$$\tilde{G}_t(\tilde{x})=G_t(x-ct) , \tag{A9}$$

and similarly for $\tilde{H}$. The fixed point $w(\tilde{x})$ then satisfies the equation

$$\tfrac{1}{2}\partial_{\tilde{x}}w+C\partial_{\tilde{x}}w+w^2-w=0 , \tag{A10}$$

with the boundary conditions $w(+\infty)=1$, $w(-\infty)=0$. It can be seen that w tends exponentially to these limits. Thus on a scale (in x) much larger than one, w is essentially a step function.

From Eq. (A4), we find that $\tilde{H}$ satisfies a linear equation; it can be written in a convenient form for the density

$$\psi(\tilde{x},\tilde{y};\hat{t}-t)\equiv\partial_{\tilde{x}}\partial_{\tilde{y}}\tilde{H}_t(\tilde{x},\tilde{y};\hat{t}) , \tag{A11}$$

yielding

$$\frac{\partial\psi}{\partial t}=\frac{1}{2}\partial_{\tilde{x}}^2\psi+\sqrt{2}\partial_{\tilde{x}}\psi+[2w(\tilde{x})-1]\psi\equiv\mathcal{L}\psi , \tag{A12}$$

with initial condition

$$\psi(\tilde{x},\tilde{y};0)=\delta(\tilde{x}-\tilde{y})\partial_{\tilde{y}}w(\tilde{y}) . \tag{A13}$$

If $\tilde{y}$ is large and positive, the flow under Eq. (A12) amplifies the initial peak in ψ by an almost constant amount $2w-1\approx 1$, moves it to the left with velocity $c=\sqrt{2}$, and smears it out. When the peak gets past 0, it rapidly decays. Thus at long times $t-\hat{t}$, the conditional probability distribution with $\tilde{x}\sim 0$ (which corresponds to a typical ground-state energy up to t) will be peaked at values of y that are large enough that the decay of the peak does not dominate. This implies that the typical energy $E_m(t,\hat{t})-\bar{\epsilon}_0\hat{t}$ will be large and negative for $t-\hat{t}$ large.

It is instructive to solve an approximate problem with $2w-1$ replaced by 1 for $x>0$ and $-\infty$ (i.e., an absorbing wall) for $x<0$. In order to conserve probability we must change the initial condition to

$$\psi(x,y;0)=\delta(x-y)\phi_0(x) , \tag{A14}$$

with

$$\phi_0(x)=2xe^{-\sqrt{2}x} , \tag{A15}$$

the steady-state solution to the linear equation

$$\hat{\mathcal{L}}\phi_0=0 , \tag{A16}$$

with $\hat{\mathcal{L}}$ the linear operator on $[0,\infty]$ with $2w-1$ replaced by unity in Eq. (A12). We will henceforth drop the tildes on x and y. The long-time behavior of ψ under $\hat{\mathcal{L}}$ can be found straightforwardly from

$$\psi(x,y;t)=2\int_0^\infty \frac{dk}{\pi}\sin(kx)\sin(ky) \times e^{-\sqrt{2}x}e^{\sqrt{2}y}e^{-k^2/2t}\phi_0(y) . \tag{A17}$$

Integrating over x yields the distribution of the contribution $E_m(\hat{t}+t,\hat{t})$ to the energy of the minimal path ending at $\hat{t}+t$ from times less than $\hat{t}$:

$$h(y,t) \equiv \int dx\, \psi(x,y,t)$$
$$\approx \frac{2y^2}{\sqrt{2\pi}} \frac{1}{t^{3/2}} e^{-y^2/2t}[1+O(1/t, y^2/t^2)] \qquad (A18)$$

where the $e^{\pm\sqrt{2}y}$ factors have canceled, leaving diffusivelike behavior. The typical y is thus $\sim\sqrt{t}$, which is large, as guessed above. Because of this domination at large times by large y, the approximation of replacing $\mathcal{L}$ by $\hat{\mathcal{L}}$ can be shown to be asymptotically correct in the limit $t\to\infty$ and $y\sim\sqrt{t}$. For y of order one (with a tail extending to negative y) there will be a small extra weight in h totaling $\sim 1/t^{3/2}$.

We thus see that the contribution to the energy of the minimal path at distances more than t from the root will typically be of order $\sqrt{t}$ *lower* than a typical minimal path that ends at a distance t from the root. Paths branching off from such anomalous sections of the ground-state path, as in Fig. 5, will, on the other hand, have *typical* minimal energies (their distribution can also be calculated). This implies, as claimed in the text, that low-energy side branches will only be common near the root. The probability of finding such a branch a distance t from the root will be of order $t^{-3/2}$, essentially just the probability that the minimal branch has typical energy.

Similar techniques can be used to evaluate the contribution to the energy of sections of the minimal path far from the origin, i.e.,

$$E_m(t,\hat{t})-E_m(t,\hat{t}'), \qquad (A19)$$

with

$$\hat{t}-\hat{t}' \ll t-\hat{t} \ll t. \qquad (A20)$$

This found to be simply Gaussian with variance given by $\Gamma|\hat{t}-\hat{t}'|$. Note that for a non-Gaussian random potential, the distribution of the energy of such sections of the minimal path will still be Gaussian, but with variance given by that of the integral of the random potential constrained so that the energy density is equal to that of the ground state paths.

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