Nonlocal elastic properties of flux-line lattices in anisotropic superconductors in an arbitrarily oriented field

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The real-space anisotropic interaction between arbitrarily curved London vortices is calculated for a uniaxially anisotropic superconductor. From this we derive the elastic energy of a distorted flux-line lattice (FLL) in a uniaxially anisotropic superconductor for inductions $B \ll B_{c2}$ and arbitrary field direction. Avoiding the continuum description of the FLL, we obtain the exact elastic matrix, which is periodic in Fourier space and from which all elastic moduli of the FLL may be extracted. In the continuum limit, we give explicit expressions for the various nonlocal tilt and bulk moduli for the two cases $\mathbf{B}\perp\hat{\mathbf{c}}$ and $\mathbf{B}\|\hat{\mathbf{c}}$; here $\hat{\mathbf{c}}$ is the symmetry axis of the uniaxial crystal perpendicular to the basal plane. These results complement previous local theories and extend previous nonlocal treatments.

I. INTRODUCTION

Recently, there has been much interest in the properties of flux-line lattices (FLL) in high-temperature superconductors. A number of varieties of the mixed state of type-II superconductors have been proposed, such as vortex liquid,[1,2] vortex glass,[3] and even vortex plasma.[4] In particular, there exist a number of theoretical estimates[2,5,6] that a vortex lattice may melt well below the mean-field phase boundary where superconductivity disappears. The main features that make thermal fluctuations important in the high-T_c compounds are the small coherence lengths ξ (Ref. 5) and the large mass anisotropies.[6] It was recognized a number of years ago[7] that in order to obtain an adequate description of the elastic properties of the FLL, the *nonlocality* of its elastic response must be accounted for. This means that some of the elastic moduli of the FLL depend on the length scale of the elastic strain. In atomic lattices one usually considers the elastic response as *local* since a nearest-neighbor ion-ion interaction is a reasonable description of the interactions in the lattice. In a FLL the situation is quite different: The vortex-vortex interaction extends over many vortex separations $a \approx (\Phi_0/B)^{1/2}$, where $\Phi_0 = 2.07\times10^{-7}$ G cm^2 is the flux quantum and B the magnetic induction. Typically, the range is given by the magnetic penetration depth λ. Thus nonlocality becomes increasingly important as the reduced induction $b = B/B_{c2}$ increases; here B_{c2} is the upper critical field.

The criterion for the applicability of *local* elasticity theory is $(M_z/M)^{1/2}\kappa^2 b \ll 1$,[6] where M_z and M are the quasiparticle masses along the $\hat{\mathbf{c}}$ direction and basal plane, respectively. $\kappa = \lambda_{ab}/\xi_{ab}$ is the Ginzburg-Landau (GL) parameter with λ_{ab} and ξ_{ab} the in-plane penetration depth and coherence length, respectively. We have $\lambda_c/\lambda_{ab} = \xi_{ab}/\xi_c = (M_z/M)^{1/2}$ in an obvious notation. Particularly in the Bi-Sr-Ca-Cu-O compounds, with large $(M_z/M)^{1/2} \simeq 60$ and $\kappa \sim 100$, the local limit becomes of mere academic interest. In the local limit, the elastic tilt and bulk moduli of the FLL become completely independent of the crystal anisotropies, and in fact of *all* material properties of the underlying superconductor as soon as the vortex fields start to overlap (i.e., for $B \geq 2B_{c1}$, where B_{c1} is the lower critical field); the small, material- and angle-dependent corrections are proportional to the small reversible magnetization. This little recognized fact was explicitly demonstrated in Ref. 6 for the case $\mathbf{B}\|\hat{\mathbf{c}}$, where $(M_z/M)^{1/2}$, the only manifestation of underlying material properties, was shown to cancel out of the description of the flux-line *lattice*. This must be so, as the local tilt and bulk moduli are simply magnetic energy densities associated with compressing flux into the compound. Thus in the limit of negligible magnetization, i.e., for $(M_z/M)^{1/2}\kappa^2 b \ll 1$, there is only one bulk and one tilt modulus, given by

$$c_{11}(0) = c_{44}(0) = \frac{B^2}{4\pi} . \quad (1)$$

This result is independent of the direction of the applied field with respect to the crystal $\hat{\mathbf{c}}$ axis. In the local limit, material properties only enter the elastic description of the FLL through various shear and rotation moduli (rotation of the FLL with respect to the crystal about the applied field[8]).

The derivation of the elastic energy in Refs. 6, 7(c), and 7(d) [the main results of Refs. 7(c) and 7(d) are summarized in 7(a)] employed complicated variational techniques, solving the GL equations for the spatially varying order parameter and magnetic field of a distorted FLL, and obtaining finally the elastic moduli in the continuum limit. All results were valid in the entire field range $(2\kappa^2)^{-1} < b < 1$. However, as shown in Ref. 7(b), this complicated mathematical apparatus can be avoided entirely if one is interested only in the field regime $b \ll 1$. In this case, the vortex *cores* do not overlap and spatial variations of the order parameter outside the core may thus be neglected. Actually, for this condition to apply $b \leq 0.2$ suffices. The limit $b \leq 0.2$ should be adequate for the high-T_c compounds due to the large values of B_{c2}, and it is the one we will consider in this paper. There will, nevertheless, still be a wide range of fields $[(M_z/M)^{1/2}\kappa^2]^{-1} \ll b \ll 1$, where the vortex *fields* overlap strongly (but not the *cores*) and where nonlocality is thus pronounced, but the simple London theory still applies.

It is the purpose of this paper to compute the general elastic energy of the FLL in an arbitrarily tilted magnetic field. The full periodicity and geometric anisotropy of the FLL itself is accounted for since we avoid the continuum limit. Therefore, one may also reproduce various (local) anisotropic shear and rotation moduli from our theory, as was previously done by Kogan and Campbell[8] within a local approach (i.e., for uniform strain). When avoiding the continuum limit in describing the elastic properties of the FLL, it is essential to start from the correct *equilibrium* configuration[8–10] in order to avoid spurious results, e.g., a negative shear modulus.

As one motivation for the present work, we mention briefly the following. Recent decoration experiments carried out on Bi-Sr-Ca-Cu-O (Ref. 11) show unexpected features in the vortex structure (vortex chains embedded in a nearly regular FLL), when the magnetic field is tilted away from the $\hat{\mathbf{c}}$ axis. In order to investigate the *stability* of various such vortex arrangements, it is important to gain a complete understanding of the elastic properties of the FLL in arbitrarily oriented fields. We will derive these properties from the general anisotropic interaction between arbitrarily curved vortices. As a further application the resulting elastic energy also allows determination of the anisotropy of *thermal fluctuations* and the phase boundary between flux-line lattice and vortex liquid as the field direction is varied. Finally, the anisotropic nonlocal moduli for general field direction will also be useful in *pinning theories* since the statistical summation of pinning forces depends crucially on the elastic response of the FLL to the pins.

II. ANISOTROPIC LONDON THEORY

The starting point of our analysis is the free energy for a system of London vortices in an anisotropic superconductor given in Ref. 12,

$$F=\frac{\Phi_0^2}{8\pi}\sum_{i,j}\int\int dl_i^\alpha dl_j^\beta V_{\alpha\beta}(\mathbf{r}_i-\mathbf{r}_j)\ . \tag{2}$$

In (2) the summations run over all vortices in the system, including the terms $i=j$, which represent the vortex self-energies. Only in these self-energy terms is an inner cutoff of the potential required to account for the finite vortex core, but not in terms involving *different* vortices. Equation (2) applies to bulk superconductors and discards the energy of the stray field generated outside the sample by the vortex ends, but image vortices may in principle be included. For the inclusion of surface effects into the elastic energy see Ref. 13. Equation (2) implies that *all* line elements dl_i^α interact with each other by the *three-dimensional* anisotropic London potential

$$V_{\alpha\beta}(\mathbf{r})=\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{r}}\tilde{V}_{\alpha\beta}(\mathbf{k})\ , \tag{3}$$

where for uniaxial symmetry we have

$$\tilde{V}_{\alpha\beta}(\mathbf{k})=\frac{1}{1+\Lambda_1k^2}\left[\delta_{\alpha\beta}-\frac{q_\alpha q_\beta\Lambda_2}{1+\Lambda_1k^2+\Lambda_2q^2}\right] \tag{4}$$

with $(\alpha,\beta)\in(x,y,z)$, $\mathbf{q}=\mathbf{k}\times\hat{\mathbf{c}}$, $\Lambda_1=\lambda_{ab}^2$, and $\Lambda_2=\lambda_c^2-\lambda_{ab}^2$ in the notation of Ref. 14. λ_{ab} is the in-plane magnetic penetration depth and λ_c the penetration depth for currents flowing perpendicular to the *ab* plane. The relative simplicity of the present London approach to obtaining the nonlocal elastic properties of the FLL, as compared to Ginzburg-Landau theory,[6,7(c),7(d)] depends crucially on the applicability of (2), which assumes that the vortices can actually be defined at all points in space. More sophisticated treatments may be envisaged by using the Lawrence-Doniach model of layered superconductors.[15,16] Equation (2) demonstrates that in anisotropic superconductors the vortex-vortex interaction is *tensorial* in nature, whereas in an isotropic superconductor it is *vectorial* [see Eq. (12) below]. This fact possibly puts approximations where one only considers the *scalar* repulsion between straight, parallel vortices even more into question than in the isotropic case.

The tensorial real-space interaction $V_{\alpha\beta}(\mathbf{r})$ between vortex-line elements (3) explicitly reads (for details of the calculation see Appendix A)

$$V_{\alpha\beta}(\mathbf{r})=V_1(r)\delta_{\alpha\beta}+V_2^{\alpha\beta}(\mathbf{r})\ , \tag{5}$$

where the first, isotropic term is

$$V_1(r)=\frac{1}{4\pi\lambda_{ab}^2r}\exp\left[-\frac{r}{\lambda_{ab}}\right]\ . \tag{6}$$

Without loss of generality at this point, we may choose a coordinate system in which $\hat{\mathbf{z}}\|\hat{\mathbf{c}}$, thus obtaining for the anisotropic part $V_2^{\alpha\beta}(\mathbf{r})$

$$V_2^{zz}(\mathbf{r})=0\ , \tag{7}$$

$$V_2^{\alpha\beta}(\mathbf{r})=\frac{1}{4\pi\lambda_{ab}\rho^2}\left[[G_1(\mathbf{r})-G_2(\mathbf{r})]\delta_{\alpha\beta}+G_2(\mathbf{r})\frac{x_\alpha x_\beta}{\rho^2}\right]\ , \tag{8}$$

where now $(\alpha,\beta)\in(x,y)$, and the two functions $G_1(\mathbf{r})$ and $G_2(\mathbf{r})$ are given by

$$G_1(\mathbf{r})=\exp\left[-\frac{r}{\lambda_{ab}}\right]-\exp\left[-\frac{(\rho^2+\epsilon^2z^2)^{1/2}}{\lambda_c}\right] \tag{9}$$

and

$$G_2(\mathbf{r})=\left[2+\frac{\rho^2}{\lambda_{ab}r}\right]\exp\left[-\frac{r}{\lambda_{ab}}\right]-\left[2+\frac{\rho^2}{\lambda_c(\rho^2+\epsilon^2z^2)^{1/2}}\right]\times\exp\left[-\frac{(\rho^2+\epsilon^2z^2)^{1/2}}{\lambda_c}\right]\ . \tag{10}$$

In Eqs. (8)–(10), we have defined the quantities $\rho^2=x^2+y^2$ and $\epsilon=\lambda_c/\lambda_{ab}$.

These expressions contain a considerable amount of information. Not only do they give the interaction of two arbitrarily oriented vortex segments, they also allow the calculation of the magnetic field distribution of a *general configuration of arbitrarily curved vortices* via the expression[12]

$$B_\alpha(\mathbf{r})=\Phi_0 \sum_i \int dl_i^\beta V_{\alpha\beta}(\mathbf{r}-\mathbf{r}_i) \ . \tag{11}$$

In the isotropic limit $\lambda_c/\lambda_{ab}\to 1$, $V_2^{\alpha\beta}(\mathbf{r})$ vanishes, and (2) reduces to the well-known result[12,17]

$$F=\frac{\Phi_0^2}{8\pi}\sum_{i,j}\int\int d\mathbf{r}_i\cdot d\mathbf{r}_j V_1(|\mathbf{r}_i-\mathbf{r}_j|) \ . \tag{12}$$

The interaction between vortices is thus quite different from an ordinary scalar repulsion. Although it is true that straight, parallel vortices repel each other (with a logarithmic potential at distances much smaller than the penetration depth), it is clear from (11) that vortices tilted by more than 90° with respect to each other *attract* each other. We suggest that in vortex systems where the tilt modulus is small and transverse meandering of the flux lines is significant,[1,2,18–20] such considerations should be taken into account in the statistical mechanics of, for instance, weakly pinned entangled flux-line liquids. In fact, the vectorial nature of the integral measure $d\mathbf{r}_i\cdot d\mathbf{r}_j$ in (11) will, under quite general circumstances, lead to an *instability* of a two-vortex configuration provided that one allows for *local* bending of the vortices in the vicinity of their closest approach.[21] This instability will not occur if the vortices are regarded as *rigid*.[22] Furthermore, preliminary results in the isotropic case (11) indicate that an instability similar to that reported in Ref. 21 may occur for entangled vortex configurations in a *lattice*.[23]

In anisotropic superconductors, the angle at which the interaction between stiff, parallel vortices vanishes is $>90°$ when they are tilted symmetrically away from the $\hat{\mathbf{c}}$ axis. This is because the currents tend to flow in the *ab* plane. One might thus argue that the anisotropy makes vortex cutting harder. However, this effect is compensated (or possibly overcompensated) by the tendency of vortices to align parallel to the *ab* plane; such vortices have a smaller line tension (self-energy $\propto 1/\lambda_{ab}\lambda_c$) than vortices oriented along $\hat{\mathbf{c}}$ (self-energy $\propto 1/\lambda_{ab}^2$). Therefore it costs little energy when the vortices tilt *locally* to form an angle $>90°$ such that they can cut easily.

III. ELASTIC ENERGY

To account for distortions and rotations of the vortices we let $\mathbf{r}_i=\mathbf{R}_i+\mathbf{u}_i$, $\mathbf{R}_i$ being the equilibrium position of the *i*th vortex in the lattice and $\mathbf{u}_i$ a two-dimensional (2D) displacement vector about this position, chosen perpendicular to the direction of the average magnetic field. [Note that the displacement component parallel to the displaced line, u_z, has no physical meaning, cf. also Eq. (17) below.] Without loss of generality, we choose $\hat{\mathbf{z}}$ along the equilibrium direction of the flux line which always coincides with the orientation of the average field $\mathbf{B}$, whereas in general the applied $\mathbf{B}_a$ may deviate from this direction, namely, when $\mathbf{B}_a$ is not $\|\hat{\mathbf{c}}$ or $\perp\hat{\mathbf{c}}$ and the magnetization is not discarded. Expanding $V_{\alpha\beta}(\mathbf{r}_i-\mathbf{r}_j)$ and the line elements up to quadratic order in the displacements and collecting all terms, we find the excess energy ΔF due to vortex displacements

$$\Delta F=\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}u_\alpha(-\mathbf{k})\tilde{\Phi}_{\alpha\beta}(\mathbf{k})u_\beta(\mathbf{k}) \ . \tag{13}$$

Here, summation over repeated indices is understood and the integration over $\mathbf{k}_\perp$ is restricted to the first Brillouin zone, while $k_z\in(-\infty,\infty)$. The elastic matrix $\tilde{\Phi}_{\alpha\beta}(\mathbf{k})$ is given by (see Appendix B for details)

$$\tilde{\Phi}_{\alpha\beta}(\mathbf{k})=\frac{B^2}{4\pi}\sum_{\mathbf{Q}}[k_z^2\tilde{V}_{\alpha\beta}(\mathbf{k}+\mathbf{Q}) +(\mathbf{k}+\mathbf{Q})_\alpha(\mathbf{k}+\mathbf{Q})_\beta\tilde{V}_{zz}(\mathbf{k}+\mathbf{Q}) -Q_\alpha Q_\beta\tilde{V}_{zz}(\mathbf{Q})] \ , \tag{14}$$

where $(\alpha\beta)\in(x,y)$, $\tilde{V}_{\alpha\beta}$ is given by (4), and the sum in (14) runs over all reciprocal lattice vectors $\mathbf{Q}$ at a given inclination of $\mathbf{B}$ with respect to $\hat{\mathbf{c}}$. The basis vectors of the undistorted equilibrium lattice when $\mathbf{B}$ is tilted by an arbitrary angle Θ away from the $\hat{\mathbf{c}}$ axis are given by[8–10] $\mathbf{a}_1=C\hat{\mathbf{x}}$ and $\mathbf{a}_2=C(\gamma\hat{\mathbf{x}}+\sqrt{3}\hat{\mathbf{y}}/\gamma)/2$, where $\gamma^4=(M/M_z)\sin^2\Theta+\cos^2\Theta$ and $C^2=2\Phi_0/\sqrt{3}B$. The corresponding reciprocal lattice vectors for arbitrary field inclination are given by $\mathbf{Q}_{mn}=n\mathbf{Q}_1+m\mathbf{Q}_2$ with m,n integers and

$$\mathbf{Q}_1=\frac{2\pi}{C}\left[\frac{\hat{\mathbf{x}}}{\gamma}-\frac{\gamma\hat{\mathbf{y}}}{\sqrt{3}}\right],\quad \mathbf{Q}_2=\frac{2\pi}{C}\frac{2\gamma}{\sqrt{3}}\hat{\mathbf{y}} \ . \tag{15}$$

Equation (14), with (15) inserted, is the central result of this paper. It is a generalization to anisotropic superconductors of the results of Ref. 7(b) and gives a complete description of the elastic properties of a FLL in a uniaxially anisotropic superconductor when $b\ll 1$ and $\kappa\gg 1$, for *any* orientation of the applied field relative to the underlying crystal. Its validity is not limited to a continuum description, which is obtained by expanding the elastic matrix $\Phi_{\alpha\beta}(\mathbf{k})$ for $|\mathbf{k}|\ll k_{\mathrm{BZ}}$, where k_{BZ} is the radius of the circularized Brillouin zone of the FLL. Note that, due to the summation over $\mathbf{Q}$ in (14), and to the appearance of the argument $\mathbf{k}$ only in the combination $\mathbf{k}+\mathbf{Q}$, the elastic matrix is a *periodic function* in $\mathbf{k}$ space. Also, it is seen that $\tilde{\Phi}_{\alpha\beta}(\mathbf{k}=0)=0$, as it should be since a uniform displacement of all flux lines does not cost energy in the pin-free case.

In the limit where only the term $\mathbf{Q}=0$ in (14) is kept, $\tilde{\Phi}_{\alpha\beta}(\mathbf{k})$ reduces to

$$\tilde{\Phi}_{\alpha\beta}(\mathbf{k})=\frac{B^2}{4\pi}[k_z^2\tilde{V}_{\alpha\beta}(\mathbf{k})+k_\alpha k_\beta\tilde{V}_{zz}(\mathbf{k})] \ . \tag{16}$$

From this lattice-structure-independent term, we can obtain the various nonlocal elastic tilt and bulk moduli but not, for instance, any of the shear moduli. Moreover, in the very small-$\mathbf{k}$ limit, where $V_{\alpha\beta}(\mathbf{k})=\text{const}$ and $\tilde{\Phi}_{\alpha\beta}(\mathbf{k})\propto k^2$ (the local limit), the elastic energy is completely *material independent,* since the prefactor is determined entirely by the magnitude of the magnetic field only, which for negligible reversible magnetization coincides with the applied field.

The limit defined by (16) may be denoted as *flux-line liquid limit.*[18,19] The continuum limit to (14) is obtained from (16) when the energy of shear, squash, and rotation modes[8] (rotation of the FLL with respect to the underlying crystal) is added to (16). The energy cost of rotating the FLL with respect to the crystal is a novel feature of anisotropic superconductors and occurs when the magnetic field is applied in an off-symmetry direction (i.e., away from $\hat{\mathbf{c}}$); this was first discussed in Ref. 8. In the present formalism, these structure-dependent contributions to the elastic energy have their origin in the $\mathbf{Q}\neq 0$ terms in (14). The corresponding moduli will in general depend on material properties such as mass anisotropy and κ, as well as on the direction of the applied field; they are proportional to λ^{-2} as is easily seen from (14) and (4) noting that $Q^2\lambda^2 \gg 1$.

In the limit of very small $B \ll B_{c1}$ where $Q^2\lambda \gg 1$, all moduli not connected with the tilt become exponentially small, e.g.,[24] $c_{11}=3c_{66} \propto \exp(-a/\lambda)$ in the isotropic case. This nearest-neighbor limit, however, is only of academic interest since its c_{66} value is extremely small such that the always present pinning distorts the FLL *plastically.* A more useful fit to a numerically obtained c_{66} at somewhat larger B is[7(a)] $c_{66} \approx (B^2/32\pi\kappa^2)\exp(-1/3b\kappa^2)$, see also the review.[25] Note that, since these structure-dependent modes contribute to the elastic energy only via the terms $\mathbf{Q}\neq 0$, the shear and rotation moduli are *local,* i.e., nondispersive for $|\mathbf{k}|$ not too close to k_{BZ}, say, $|\mathbf{k}| < k_{\mathrm{BZ}}/2$.[7(b)]

Within the flux-line liquid limit defined by (16), the form of the energy density associated with the vortex displacements is given by[8]

$$f=\tfrac{1}{2}(4\lambda_{zxzx}u_{xz}^2+4\lambda_{zyzy}u_{yz}^2+2\lambda_{xxyy}u_{xx}u_{yy} +\lambda_{xxxx}u_{xx}^2+\lambda_{yyyy}u_{yy}^2) \ . \tag{17}$$

Here, $u_{ij}=(\partial_j u_i+\partial_i u_j)/2$. The first two terms in (17) are tilt modes, the three last ones compressional modes. Note that the form of the free energy Eq. (17) is consistent with the fact that the displacement vector $\mathbf{u}$ has no z component, i.e., terms like $\lambda_{zzzz}u_{zz}^2$ do not occur. In general, there are two tilt, and three compressional moduli, all of which in the general case could differ from each other. In Ref. 8, the previously mentioned shear, squash, and rotation moduli are computed within a *local* approach and as we have argued, these moduli indeed can generally be considered well approximated by their local values; their weak dispersion is of *geometric* nature, i.e., they depend on ka but no $k\lambda$. The results of Ref. 8 may thus be combined with (17), to form a *complete description* of the FLL in the continuum limit, provided the coefficients λ_{ijlm} are known; this is the topic of the next section.

IV. NONLOCAL ELASTIC MODULI

The nonlocal moduli are obtained by comparing Eqs. (13) and (17) for the elastic energy and using Eq. (16) for the elastic matrix. We consider the two geometries $\mathbf{B}\|\hat{\mathbf{c}}$ and $\mathbf{B}\perp\hat{\mathbf{c}}$.

(1) $\mathbf{B}\|\hat{\mathbf{c}}$. In this case, the tilt moduli are given by

$$\lambda_{zxzx}(\mathbf{k})=\frac{B^2}{4\pi}\frac{1}{1+\lambda_c^2(k_x^2+k_y^2)+\lambda_{ab}^2k_z^2} \ , \tag{18}$$

$$\lambda_{zyzy}(\mathbf{k})=\lambda_{zxzx}(\mathbf{k})\equiv c_{44}(\mathbf{k}) \ . \tag{19}$$

Furthermore, we find for the bulk moduli

$$\lambda_{xxxx}(\mathbf{k})=\frac{B^2}{4\pi}\frac{1+\lambda_c^2k^2}{(1+\lambda_{ab}^2k^2)[1+\lambda_c^2(k_x^2+k_y^2)+\lambda_{ab}^2k_z^2]} \ , \tag{20}$$

$$\lambda_{yyyy}(\mathbf{k})=\lambda_{xxyy}(\mathbf{k})=\lambda_{xxxx}(\mathbf{k})\equiv c_{11}(\mathbf{k}) \ . \tag{21}$$

These results have already been derived from the GL theory in Ref. 6, and were later rederived from a hydrodynamic treatment of flux-line liquids.[18] Note that the present form of the modulus c_{11} is somewhat simpler than the one given in Ref. 6. This is due to the London limit where terms $\sim O(\kappa^{-2})$ are neglected compared to 1. A detailed discussion of this has been given for the isotropic case in Ref. 7(b).

(2) $\mathbf{B}\perp\hat{\mathbf{c}}\perp\hat{\mathbf{y}}$. Now we find for the tilt moduli

$$\lambda_{zyzy}(\mathbf{k})=\frac{B^2}{4\pi}\frac{1}{1+\lambda_c^2(k_y^2+k_z^2)+\lambda_{ab}^2k_x^2} \ , \tag{22}$$

$$\lambda_{zxzx}(\mathbf{k})=\frac{B^2}{4\pi}\frac{1+\lambda_c^2k^2}{(1+\lambda_{ab}^2k^2)[1+\lambda_c^2(k_y^2+k_z^2)+\lambda_{ab}^2k_x^2]} \ , \tag{23}$$

where $\lambda_{zxzx}(\mathbf{k})$ and $\lambda_{zyzy}(\mathbf{k})$ represent out-of-plane and in-plane tilt moduli, respectively. The planes refer to the basal planes. The *nonlocal* tilt moduli now differ from each other due to lack of rotational symmetry around the $\hat{\mathbf{z}}$ axis. Similarly, we find for the bulk moduli

$$\lambda_{xxxx}(\mathbf{k})=\frac{B^2}{4\pi}\frac{1}{1+\lambda_c^2(k_y^2+k_z^2)+\lambda_{ab}^2k_x^2} \ , \tag{24}$$

$$\lambda_{yyyy}(\mathbf{k})=\frac{B^2}{4\pi}\frac{1+\lambda_{ab}^2k^2+2(\lambda_c^2-\lambda_{ab}^2)k_z^2}{(1+\lambda_{ab}^2k^2)[1+\lambda_c^2(k_y^2+k_z^2)+\lambda_{ab}^2k_x^2]} \ , \tag{25}$$

$$\lambda_{xxyy}(\mathbf{k})=\frac{B^2}{4\pi}\frac{1}{1+\lambda_{ab}^2k^2} \ . \tag{26}$$

Notice that in the local limit all moduli reduce to the value $B^2/4\pi$, as in the isotropic case. The reason for this has been given above. Note also that the anisotropic nonlocal moduli depend on the orientation of $\mathbf{k}$, not only $|\mathbf{k}|$.

V. REMARKS ON LARGE κ AND LARGE κ

In the case of an *isotropic* superconductor $\lambda_c/\lambda_{ab}=1$, within London theory (valid for $\kappa^2 \gg 1$ and $b \le 0.2$), and for $|\mathbf{k}| \ll k_{\mathrm{BZ}}$, even all *nonlocal* tilt and bulk moduli collapse to one and the same quantity,

$$c_{44}(\mathbf{k})=c_{11}(\mathbf{k})=\frac{B^2}{4\pi}\frac{1}{1+\lambda^2k^2}\ , \tag{27}$$

which is not orientation dependent. However, an additional *geometric* anisotropy and nonlocality associated with the FLL structure itself appears in the moduli at **k** values approaching the boundary of the first Brillouin zone.[7(b)] This *geometric* dispersion occurs in the isotropic as well as the anisotropic cases and is fully contained in the elastic matrix Eq. (14), but not in its limiting form, Eq. (16). Moreover, the general expression (14) contains also the case of *almost isolated* flux lines ($B \ll B_{c1}$), where the moduli (18)–(27) do not apply.

A remark is appropriate as for the small difference between $c_{44}(k)$ and $c_{11}(k)$. As is well known,[26] thermodynamic considerations yield a bulk modulus (for isotropic compression) $c_{11}(0)-c_{66}=B^2/4\pi$ and a slightly larger tilt modulus $c_{44}(0)=BB_a/4\pi$. The difference is small since $B_a-B=\text{const}/\kappa^2 \ll B$ as soon as $B \geq 2B_{c1}$. In most publications so far, the slightly more general expression

$$c_{44}(k)=\frac{B^2}{4\pi}\left[\frac{1}{1+\lambda^2k^2}+\cdots\right]\ , \tag{28}$$

where the ellipsis represents correction terms, was used, with k-independent correction term (valid as $\mathbf{k}\to 0$)

$$\cdots=(B_a-B)/B\approx(1-b)/k_{\rm BZ}^2\lambda^2=(1-b)/2b\kappa^2. \tag{29}$$

However, as pointed out in Ref. 27, and as is implicit in Sec. 4.4 of Ref. 7(b), for large values $k^2\geq k_s^2=k_{\rm BZ}^2/\ln(1/\xi k_s)$, the true ($k$-dependent) correction term even *exceeds* the main term since the isolated-vortex result

$$c_{44}(k)\approx\frac{B\Phi_0}{(4\pi\lambda)^2}\ln(1/\xi k_z) \tag{30}$$

has to be recovered in (28). Thus for $k>k_s$ (29) should be replaced by

$$\cdots\approx\frac{\Phi_0}{4\pi\lambda^2B}\ln\left[\frac{1}{\xi k_z}\right]=\frac{B_{c1}}{B}\frac{\ln(1/\xi k_z)}{\ln(\lambda/\xi)}\ . \tag{31}$$

Note that in the isolated-vortex result, only the variable k_z enters in the dispersion, since the perpendicular components of **k** no longer have physical meaning. In the large (even dominating) part of the Brillouin zone $k>k_s$ where (31) dominates over $1/(1+\lambda^2k^2)$, tilted vortices behave as if they were *isolated*, i.e., their elastic response is determined by the (logarithmically dispersive) line tension of an isolated flux line. This fact slightly reduces the thermal fluctuations of the FLL as compared to the results of Refs. 5 and 6. Note that the component k_z may be larger than $k_{\rm BZ}$. The correct cutoff from GL theory is $k_z^2\leq 1/\xi^2=k_{\rm BZ}^2/2b \gg k_{\rm BZ}^2$.

It has recently been argued that the limit $\kappa\to\infty$ may be taken such that vortices in type-II superconductors behave like vortices in superfluid ^{4}He.[28] However, for typical applications, the limit $\kappa=\lambda/\xi\to\infty$ is unphysical. If this limit means $\xi\to 0$ for $\lambda=\text{const}$, then $c_{44}(k)$ (30) *diverges.* As a consequence of this infinite vortex-core stiffness, both thermal fluctuations and pinning-caused distortions of the FLL would be *suppressed.* On the other hand, considering this limit further we see that since $B_{c1}=\Phi_0\ln\kappa/4\pi\lambda^2$ the lower critical field also diverges, so *no vortices even exist.*

Another reason for the unphysical nature of the limit $\kappa\to\infty$ becomes clear when this is interpreted as $\lambda\to\infty$ for $\xi=\text{const}$. This limit not only would make $c_{44}(k)=0$ and $B_{c1}=0$; it would also require specification of whether the limit $\lambda\gg L$ or $\lambda\ll L$ is considered, where L is the smallest extension of the specimen, e.g., the thickness of the slab. If $\lambda\gg L$ is meant, it is *inconsistent* to consider bulk problems, since all vortices are "close to the surface," i.e., image forces have to be considered. Furthermore, the problem of an infinite slab would become 2D and stray-field terms (from the field outside the specimen) would have to be accounted for in the elastic energy of the FLL.[14] If $\lambda\ll L$ is meant, then the limit $\kappa\to\infty$ does not imply that $c_{44}(k)\propto 1/k^2$ as expected in Ref. 28. Rather one has the usual nonlocal result (28), which for $k\to 0$ reduces to the thermodynamic result $c_{44}(0)=BB_a/4\pi$.[26] However, some properties of stacks of point vortices in superconducting layers with no Josephson coupling (pancake vortices[29]) may be obtained by letting $\lambda_c\to\infty$ in the *anisotropic* London theory of Secs. II–IV.

All statements in this paragraph follow from the general (periodic) elastic matrix given by (14) [and by the expressions given in Ref. 7(b) for isotropic superconductors], from which the isolated-vortex limit is easily obtained as well as the improved tilt moduli (28)–(31). Our results were derived from a description of the FLL that *a priori* avoided the continuum approximation. The subsequent continuum limit could be taken in a straightforward manner, confirming the results obtained in Ref. 6 for the case $\mathbf{B}\|\hat{\mathbf{c}}$. This seems to settle a point of criticism which was raised recently,[28] where the results of Refs. 6, 7(c), and 7(d) were attributed to an incorrect approach to the continuum limit. Our present results, derived by the methods of Ref. 7(b) strongly suggest that this is not the case. This finding, in turn, lends further support to the result of Ref. 30, that phase fluctuations of the order parameter do not destroy superconductivity in 3D in the mixed state of a type-II superconductor. For completeness let us remark finally that the simplicity and transparency of the above London approach disproves the (to us unclear) arguments by Matsushita,[31] who claims that the tilt modulus of the FLL is not dispersive.

VI. SUMMARY

We have presented results for the nonlocal elastic properties of the flux-line lattice in anisotropic superconductors in an arbitrarily tilted magnetic field. In addition to the exact anisotropic real-space potential between arbitrarily curved London vortices and the nonlocal elastic matrix for general field direction, we have given explicit expressions for the bulk and tilt moduli for the two cases $\mathbf{B}\|\hat{\mathbf{c}}$ and $\mathbf{B}\perp\hat{\mathbf{c}}$. The general solution for the magnetic field $\mathbf{B}(\mathbf{r})$ is also given [Eq. (11)]. The general intervortex potential, Eq. (5), is relevant for discussing the stability of entangled flux-line liquids, at least in the weak-pinning regime, and in principle allows the calculation of the energy barrier for flux cutting. The general result for the elastic matrix, Eq. (14), is exact for reduced induction $b \ll 1$ and GL parameter $\kappa \gg 1$, since the continuum description of the FLL is avoided.

By inverting the elastic matrix and integrating over the Brillouin zone, one can obtain directly the variation of thermal fluctuations of the FLL as **B** is tilted away from the $\hat{\mathbf{c}}$ axis. Such a calculation, which may be relevant for FLL melting, as estimated by a Lindemann criterion, is the topic of a forthcoming paper. The obtained anisotropic elastic matrix is further required for the summation of random pinning forces and for the calculation of activation energies for thermally activated depinning.

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APPENDIX A

We compute the anisotropic part of Eq. (3) by considering the auxiliary integral

$$I_0=\frac{\Lambda_2}{(2\pi)^3}\int d^3k \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(1+\Lambda_1 k_\perp^2+\Lambda_1 k_z^2)[1+(\Lambda_1+\Lambda_2)k_\perp^2+\Lambda_1 k_z^2]} . \tag{A1}$$

On performing the azimuthal and $k_\perp$ integration, we get

$$I_0=\frac{2}{(2\pi)^2}\int_0^\infty \cos(k_z z)\frac{K_0(z_1)-K_0(z_2)}{1+\Lambda_1 k_z^2} \equiv I_0^1-I_0^2 . \tag{A2}$$

Here, $z_i=\alpha_i(1+\Lambda_1 k_z^2)^{1/2}$ $\alpha_i=c_i(x^2+y^2)^{1/2}$, $c_1^{-1}=(\Lambda_1+\Lambda_2)^{1/2}$, $c_2^{-1}=(\Lambda_1)^{1/2}$, and K_0 is a modified Bessel function of zeroth order. The anisotropic part of the London potential, $V_2^{\alpha\beta}(\mathbf{r})$ in Eq. (5), is given by

$$V_2^{\alpha\beta}(\mathbf{r})=\left[\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\delta_{\alpha\beta}-\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\right]I_0 . \tag{A3}$$

Defining the quantity Γ_i by

$$\Gamma_i \equiv \frac{\partial I_0^i}{\partial \alpha_i} \tag{A4}$$

we obtain

$$\frac{\partial^2 I_0}{\partial x_\alpha \partial x_\beta}=\frac{\partial \Gamma_1}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial x_\alpha}\frac{\partial \alpha_1}{\partial x_\beta}+\Gamma_1\frac{\partial^2\alpha_1}{\partial x_\alpha \partial x_\beta} -\frac{\partial \Gamma_2}{\partial \alpha_2}\frac{\partial \alpha_2}{\partial x_\alpha}\frac{\partial \alpha_2}{\partial x_\beta}-\Gamma_2\frac{\partial^2\alpha_2}{\partial x_\alpha \partial x_\beta} . \tag{A5}$$

After a lengthy calculation we get

$$\Gamma_i=\frac{\partial I_0^i}{\partial \alpha_i}=-\left[\frac{\pi}{2}\right]^{1/2}\frac{2}{4\pi^2}\frac{1}{\alpha_i}\left[\frac{\tilde{z}_i}{\Lambda_1}\right]^{1/2}K_{1/2}(\tilde{z}_i) , \tag{A6}$$

where $K_{1/2}$ is a modified Bessel function of order $\frac{1}{2}$, given by $K_{1/2}(x)=\sqrt{\pi/2x}\ \exp(-x)$, and $\tilde{z}i=(\alpha_i^2\Lambda_1+z^2)^{1/2}/\Lambda_1^{1/2}$.

Performing the necessary derivatives of Γ_i and α_i and collecting terms, we get Eqs. (8)–(10).

APPENDIX B

The expression for $\tilde{\Phi}_{\alpha\beta}(\mathbf{k})$ is derived along the lines described in Ref. 7(b). Expanding the free energy, Eq. (2), and retaining all terms up to quadratic order in the displacement vectors, we get for the excess free energy due to fluctuations

$$\Delta F=\frac{\Phi_0^2}{8\pi}\sum_{i,j}\int\int dz_i dz_j \left[\frac{du_i^\alpha}{dz_i}\frac{du_j^\alpha}{dz_j}V^{\alpha\beta}(\mathbf{R}_i-\mathbf{R}_j) +\frac{1}{2!}(\mathbf{u}_i-\mathbf{u}_j)^\alpha(\mathbf{u}_i-\mathbf{u}_j)^\beta \times \nabla_\alpha\nabla_\beta V^{zz}(\mathbf{r})|_{\mathbf{r}=\mathbf{R}_i-\mathbf{R}_j}\right] . \tag{B1}$$

The last term arises because the only line-element components that do not contain displacement fields are dl_i^z. The coefficient of $u_i^\alpha u_j^\beta$ is denoted $\Phi_{\alpha\beta}(\mathbf{R}_i-\mathbf{R}_j)$. To extract its functional form, it suffices to let $\mathbf{R}_j=0$. We get, after partial integration,

$$\Phi_{\alpha\beta}(\mathbf{R}_\mu)=-\frac{\Phi_0^2}{4\pi}\left[\frac{\partial^2}{\partial z^2}V^{\alpha\beta}(\mathbf{r}) +(1-\delta_{\mu,0})\nabla_\alpha\nabla_\beta V^{zz}(\mathbf{r})\right]_{\mathbf{r}=\mathbf{R}_\mu} \tag{B2}$$

which we represent, using Eq. (3), as

$$\Phi_{\alpha\beta}(\mathbf{R}_\mu)=\frac{\Phi_0^2 n}{4\pi}\int\frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{r}}[q_z^2\tilde{V}^{\alpha\beta}(\mathbf{q}) +(1-\delta_{\mu,0})q_\alpha q_\beta \tilde{V}^{zz}(\mathbf{q})] . \tag{B3}$$

The lattice Fourier transform of this is given by

$$\tilde{\Phi}^{\alpha\beta}(\mathbf{k})=\sum_\mu \int dz\ e^{-i\mathbf{k}\cdot\mathbf{R}_\mu}\Phi^{\alpha\beta}(\mathbf{R}_\mu) . \tag{B4}$$

Using the result

$$\sum_\mu \int dz\ e^{i(\mathbf{q}-\mathbf{k})\cdot\mathbf{R}_\mu}=(2\pi)^3 n\,\delta(q_z-k_z)\sum_{\mathbf{Q}}\delta(\mathbf{q}_\perp-\mathbf{k}_\perp-\mathbf{Q}), \tag{B5}$$

and performing the $\mathbf{q}$ integration, we get Eq. (14). Here $n=B/\Phi_0$ is the areal density of vortices.

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