Nonlinear analysis of the modulational instability of a tachyonic wave train in the damped dc-driven sine-Gordon model

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It is known that a near-linear solution (a train of densely packed kinks) of the underdamped sine-Gordon equation with a dc driving term has a region of modulational instability in the tachyonic regime, in which the train's velocity exceeds the limit velocity of the sine-Gordon model. In the present work a nonlinear analysis of this instability is developed for the slightly overcritical case. A system of coupled evolution equations for two complex amplitudes of the growing disturbances is derived, and it is demonstrated that the instability gives rise to a pair of coupled small-amplitude waves of modulation traveling along the underlying wave train. A physical implementation is proposed in terms of an I-V characteristic of a long Josephson junction described by this model. It is shown that, at the point where the instability sets in, the I-V characteristic suffers a break. A corresponding jump of the differential resistance is found.

I. INTRODUCTION

The present work is devoted to the perturbed sine-Gordon (SG) system,

$$\phi_{tt}-\phi_{xx}+\sin\phi=-\alpha\phi_t-F \ , \tag{1}$$

which is one of the fundamental dynamical models of "one-dimensional" condensed-matter physics (see the recent review[1]). In particular, Eq. (1) finds a very important physical application as the model of the dc-biased damped long Josephson junction (LJJ), under the assumption that the bias current is distributed uniformly along the junction.[2]

As is well known, the unperturbed SG equation ($\alpha=F=0$) has exact stable stationary solutions in the form of solitary kinks and periodic trains of unipolar kinks (in the LJJ theory, a kink corresponds to a fluxon, i.e., to a topological soliton carrying a quantum of the magnetic flux). In the limiting case, when the kinks in the periodic train are strongly overlapped, the solution takes the following form:[3]

$$\phi=\phi_0(x,t)$$
$$\equiv pz+p^{-2}(V^2-1)^{-1}\sin(pz)+\cdots, \quad z=x-Vt \ , \tag{2}$$

where p is an arbitrary parameter that determines the period of the train, $L=2\pi/[(1-V^2)^{1/2}p]$, and V is the train's velocity. The kinks are strongly overlapped if $L\ll 2\pi$, i.e., it is assumed that

$$p^2|1-V^2|\gg 1 \ . \tag{3}$$

Throughout this paper, it will be assumed that the parameters in Eqs. (1) and (2) take the values $\alpha\ll 1$, $F\gg 1$, and $p\gg 1$.

If the terms on the right-hand side of Eq. (1) are taken into account as perturbations, in the lowest approximation they do not alter the form of the wave train given by Eq. (2), but impose the following relation on the density and velocity of the train:[3]

$$V=F/\alpha p \ . \tag{4}$$

In the unperturbed SG model, the solution (2) is always stable in the normal case $V<1$, and it is unstable in the "tachyonic" case $V>1$. However, the dissipative term in the perturbed equation (1) may be stabilizing. It has been demonstrated in Ref. 3 that the solution to the perturbed equation given by Eqs. (2) and (4) remains stable in the normal case, and for the tachyonic case the dissipation-induced stability condition is

$$F>F_{\rm cr}\equiv V/(V^2-1)^{1/2} \ . \tag{5}$$

In the range $F<F_{\rm cr}$, the wave train (2) is subject to the modulational instability, which has an oscillatory character. However, this instability has been investigated only in the linear approximation in Ref. 3. The objective of the present work is to investigate a nonlinear regime that sets in via the instability, and to analyze its physical manifestations. This will be done for the case when the "tachyonic" velocity is close to its limiting value:

$$0<v\equiv V-1\ll 1 \ , \tag{6}$$

i.e., the fundamental assumption (3) takes the form

$$p^2v\gg 1 \ . \tag{7}$$

To develop an analytical treatment, it will be necessary to assume that the inequality (7) is strengthened to the form

$$p^4v^3\gg 1 \tag{8}$$

[the inequality (7) is a corollary of those (8) and (6)].

The rest of the paper is organized as follows. In Sec. II, the nonlinear analysis of the modulational instability near the threshold is developed, assuming that

$$0<(F_{\rm cr}-F)/F_{\rm cr}\ll 1 \tag{9}$$

[see Eq. (5)]. A system of coupled nonlinear evolution equations for two complex amplitudes of the modulational disturbance is derived. These equations give rise to a stable stationary solution that may be interpreted as a coupled pair of waves of modulation traveling on the background of the underlying wave train.

In Sec. III, physical manifestations of this nonlinear regime are analyzed in terms of LJJ's. The main experimentally observable dynamical characteristic of the damped dc-biased LJJ is its *I-V* characteristic, i.e., the dependence of the mean voltage,

$$U=-\langle \phi_t\rangle \ , \tag{10}$$

with $\langle \cdots \rangle$ standing for spatial averaging, upon the bias current density F. In the region where the solution given by Eqs. (2) to (4) is stable, the *I-V* characteristic is, as a matter of fact, defined by Eq. (4) as, according to Eq. (2), $\langle \phi_t\rangle=-pV$. I consider a change of this unperturbed *I-V* characteristic generated by the instability. The main result is a discontinuity of the differential resistance

$$R\equiv\frac{dU}{dF} \tag{11}$$

at $F=F_{\rm cr}$:

$$R=\alpha^{-1} \quad \text{at } F-F_{\rm cr}\to 0+ \ , \tag{12a}$$

$$R=2\alpha^{-1} \quad \text{at } F-F_{\rm cr}\to 0- \ . \tag{12b}$$

Thus Eqs. (12) yield a prediction that can be verified in an experiment. At last, some concluding remarks are summarized in Sec. IV.

II. THE NONLINEAR ANALYSIS OF THE MODULATIONAL INSTABILITY

A perturbed solution to Eq. (1) will be looked for in the form

$$\begin{aligned}\phi(z,t)=\phi_0(z)+\{&a_1(t)\exp[\tfrac{1}{2}ip(V+1)z]\\ &+b_1(t)\exp[\tfrac{1}{2}ip(V-1)z]\\ &+b_2(t)\exp[ip(V-1)z]+\text{c.c.}\}+a_0(t)\ ,\end{aligned} \tag{13}$$

where ϕ_0 is the unperturbed solution given by Eqs. (2)–(4), the amplitudes a_1, b_1, and b_2 are complex, while a_0 is real. The term $\sim\exp[ip(V+1)z]$ has been dropped in Eq. (13), as it would give a negligible contribution to the solution to be obtained [see Eq. (22) below]. The perturbed solution (13) is taken in the form which provides a maximum growth rate γ of the modulational disturbances in the linear approximation. Reformulating the results of Ref. 3, one can find that

$$\text{Im}\gamma=\pm pv \ , \tag{14}$$

$$\text{Re}\gamma=-\alpha/2\pm(2p\sqrt{2v}\,)^{-1} \tag{15}$$

[the signs $\pm$ in Eqs. (14) and (15) are mutually independent]. It is easy to see from Eq. (15) that, with regard to Eq. (4), the inequality (5) guarantees that $\text{Re}\gamma$ is negative, which is just the stability condition. Inserting Eq. (13) into Eq. (1) and going beyond the framework of the linear approximation (in the case of the instability), one should expand the nonlinear term $\sin\phi$ in powers of a_1 and b_1 up to the third order, assuming that

$$a_0,b_2\sim a_1b_1 \ , \tag{16}$$

see Eqs. (22) and (25) below. At the second order of the expansion, one obtains, in accordance with Eq. (16), the following equations:

$$\ddot{a}+\alpha\dot{a}_0=-(i/2)(a_1^*b_1-a_1b_1^*) \ , \tag{17}$$

$$\ddot{b}+\alpha\dot{b}_2-2ipvb_2-i\alpha pvb_2=(i/2)a_1b_1 \ , \tag{18}$$

where the overdot stands for d/dt, and the asterisk designates the complex conjugation.

As it follows from Eqs. (14) and (15) and the underlying assumption (8), the frequency $\text{Im}\gamma$ is much larger than the instability growth rate $\text{Re}\gamma$, provided $\text{Re}\gamma>0$. Thus it is natural to represent the amplitudes a_1 and b_1 in the form

$$a_1(t)=A_1(t)\exp[i(pv+\tfrac{1}{2}pv^2)t] \ , \tag{19}$$

$$b_1(t)=B_1(t)\exp[i(pv+\tfrac{1}{2}pv^2)t] \ , \tag{20}$$

in order to separate the rapidly oscillating exponents governed by $\text{Im}\gamma$ and the slowly varying preexponents A_1 and B_1 governed by $\text{Re}\gamma$. Analogously, the amplitude b_2 will be represented in the form

$$b_2(t)=B_2(t)\exp[2i(pv+\tfrac{1}{2}pv^2)t] \ , \tag{21}$$

with the slowly varying preexponent B_2. Inserting Eqs. (19)–(21) into Eq. (18), it is easy to see that, in the lowest approximation, the amplitude B_2 is adiabatically enslaved by the ones A_1 and B_1:

$$B_2=(2\alpha pv)^{-1}A_1B_1 \ . \tag{22}$$

At last, using the relation (22), one can close the evolution equations for A_1 and B_1 at the third order of the expansion. After some algebra, these equations can be brought into the following eventual form:

$$\begin{aligned}\dot{A}_1+(i/4)p^{-1}B_1+\tfrac{1}{2}\alpha A_1-(8\alpha p^2v)^{-1}|B_1|^2A_1&\\ -(i/8)p^{-1}|B_1|^2B_1=0\ ,&\end{aligned} \tag{23}$$

$$\begin{aligned}\dot{B}_1-(i/2)(pv)^{-1}A_1+\tfrac{1}{2}\alpha B_1+(4\alpha p^2v^2)^{-1}|A_1|^2B_1&\\ +(i/2)(pv)^{-1}|B_1|^2A_1+(i/4)(pv)^{-1}B_1^2A_1^*=0\ .&\end{aligned} \tag{24}$$

In principle, Eqs. (23) and (24) as well include terms, respectively, $\sim\dot{a}_0A_1$ and $\sim\dot{a}_0B_1$, that account for coupling of the first harmonics of the disturbance to the zeroth harmonic [see Eq. (13)]. However, these terms may be omitted, as they give a negligible contribution to the solution obtained below. As to the amplitude a_0 of the zeroth harmonic, it is governed by Eq. (17) which, on inserting Eqs. (20) and (21), takes the eventual form

$$\ddot{a}_0+\alpha\dot{a}_0=-(i/2)(A_1^*B_1-A_1B_1^*) \ . \quad (25)$$

It is straightforward to see that the linear parts of Eqs. (23) and (24) recover the familiar expressions (14) and (15) for the imaginary and real parts of the instability growth rate. To follow the transition from the subcritical (stable) to the supercritical (unstable) region, it is convenient to employ the quantity

$$f\equiv F-\alpha p \quad (26)$$

as the control parameter, so that Eq. (4) takes the form

$$v=f/\alpha p \ . \quad (27)$$

As we are interested in the range defined by Eq. (6), $0<v\ll 1$, it will be assumed that

$$0<f\ll \alpha p \ . \quad (28)$$

In terms of the control parameter f, the stability condition (5) takes the form

$$f>f_{\rm cr}\equiv(2\alpha p)^{-1} \ . \quad (29)$$

The inequalities (28) and (29) are compatible, provided that $\alpha p\gg 1$. At the same time, the compatibility of Eq. (29) with the underlying assumption (8) gives rise to the inequality $\alpha^3 p\ll 1$. Thus the condition

$$1\ll \alpha p\ll \alpha^{-2} \quad (30)$$

will be assumed to hold.

Now, it can be readily seen that in the slightly overcritical region,

$$0<(f_{\rm cr}-f)/f_{\rm cr}\ll 1 \ , \quad (31)$$

the system of Eqs. (23) and (24) has, besides the trivial unstable solution $A_1=B_1=0$, the nontrivial one:

$$B_1=2i\alpha p A_1 \ ,$$
$$|A_1|^2=(2\alpha p)^{-2}(f_{\rm cr}-f)/f_{\rm cr} \ , \quad (32)$$
$$|B_1|^2=(f_{\rm cr}-f)/f_{\rm cr} \ .$$

The moduli of the complex amplitudes A_1 and B_1, as well as their relative phases, are fixed by Eqs. (32), while the phase of, say, A_1 remains arbitrary. Inserting Eq. (32) into Eq. (22), one finds the amplitude of the second harmonic for the solution considered:

$$B_2=2i(\alpha p)^2A_1^2, \quad |B_2|=\tfrac{1}{2}(f_{\rm cr}-f)/f_{\rm cr} \ . \quad (33)$$

The bifurcation that gives rise to the solution (32) at $f=f_{\rm cr}$ is a standard forklike bifurcation, and it is obvious that the nontrivial solutions generated by the bifurcation are stable at sufficiently small values of $(f_{\rm cr}-f)/f_{\rm cr}$. If this parameter is not small, the solution loses its sense: According to Eqs. (32) and (33), in this case the amplitudes B_1 and B_2 are no longer small, while the derivation of Eqs. (23) and (24) was based on the expansion in powers of A_1 and B_1.

Finally, it is relevant to interpret the solution given by Eqs. (32) and (33) in terms of the corresponding wave form in the physical space [see Eq. (13)]. According to Eqs. (19)–(21), the instability gives rise to two coupled waves of disturbance traveling on the background of the underlying wave train $\phi_0(z)$ [see Eq. (2)], viz., the disturbance wave with the smaller amplitude A_1 and the larger wave number $k_a\equiv-\frac{1}{2}p(V+1)\approx -p$, and the one with the larger amplitudes B_1 and B_2 and the smaller wave number $k_b\equiv -pv$. These waves have the common frequency ω given by Eq. (14) and the phase velocities [in the (z,t) coordinate frame] $V_a=\omega/k_a=-v$ and $V_b=\omega/k_b=-2$. The signs of the phase velocities are fixed by the sign of the underlying wave train's velocity V: From the very beginning it was assumed that V was close to $+1$. If V is close to -1, ω will be the same, while the signs of the wave numbers and phase velocities will be opposite.

III. INTERPRETATION IN TERMS OF THE I-V CHARACTERISTIC OF A LONG JOSEPHSON JUNCTION

In experiments with LJJ's, a transition from one branch of a solution of the corresponding SG model to another manifests itself as a change of the I-V (current-voltage) characteristic. The I-V characteristic is determined by the dc magnetic field B applied at the edges of the linear LJJ. In the dimensionless notation,[2] $B=\langle\phi_x\rangle$, cf. Eq. (10). Thus to provide the operation of the LJJ in the regime corresponding to the solution (13), it is necessary to apply a sufficiently strong magnetic field, $\langle(\phi_0)_x\rangle\equiv p$, and the dc bias current with the density F close to the critical value defined by Eq. (5).

Inserting Eqs. (13) and (2) into the definition of the mean dc voltage across the junction [Eq. (10)], one arrives at the following expression for the voltage in the overcritical region $f<f_{\rm cr}$:

$$u\equiv U-p=pv-\dot{a}_0 \ . \quad (34)$$

Next, inserting Eq. (32) into Eq. (25), it is easy to find the value of $\dot{a}_0$ corresponding to the stationary solution (32):

$$\dot{a}_0=(2\alpha p)^{-1}(f_{\rm cr}-f)/f_{\rm cr} \ . \quad (35)$$

Substituting Eqs. (27) and (35) into Eq. (34), one concludes that the full I-V characteristic has a break at the point $f=f_{\rm cr}$:

$$u=f/\alpha \quad (36a)$$

at $f>f_{\rm cr}$, and

$$u=f/\alpha-(2\alpha^2p)^{-1}(f_{\rm cr}-f)/f_{\rm cr} \quad (36b)$$

at $f<f_{\rm cr}$.

To describe the peculiarity of this I-V characteristic in more accurate terms, it is natural to consider the differential resistance defined by Eq. (11):

$$R\equiv\frac{dU}{dF}=\frac{du}{df} \ . \quad (37)$$

Substituting Eqs. (36) into Eq. (37), one concludes that at the point $f=f_{\rm cr}$, the differential resistance suffers the discontinuity described by Eqs. (12).

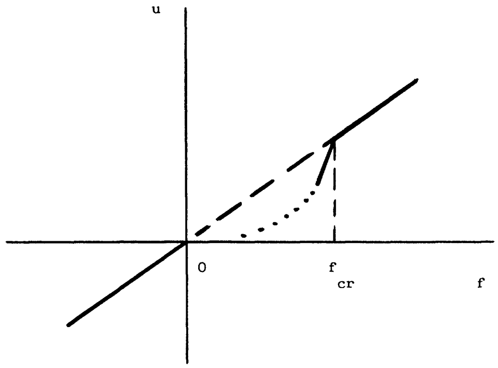

FIG. 1. The schematic *I-V* characteristic at small f. The straight line (solid at $f<0$ and at $f>f_{cr}$, and dashed at $0<f<f_{cr}$) is the unperturbed characteristic given by Eq. (36a). The solid segment in the region $0<(f_{cr}-f)/f_{cr}\ll 1$ corresponds to Eq. (36b), and the dotted segment symbolizes the unknown part of the *I-V* characteristic (which may be hysteretic).

The full *I-V* characteristic for small f ($|f|\ll \alpha p$) is shown schematically in Fig. 1. The characteristic is described by Eq. (36a) both at $f<0$ (the normal regime) and at $f>f_{cr}$, where the "tachyonic" wave train given by Eqs. (2) and (4) is stable. At $0<(f_{cr}-f)/f_{cr}\ll 1$, the *I-V* characteristic is described by Eq. (36b), and at $f>0$, $(f_{cr}-f)/f_{cr}\sim 1$, its form is unknown (in particular, it might be hysteretic).

At last, it is worthy to note that the same model based on Eq. (1) describes the underdamped dc-driven charge-density-wave (CDW) system in the case when the CDW's wave length is commensurable with the spacing of the underlying ionic lattice (see, e.g., Ref. 4). The results obtained in this work can be directly applied to this model of the charge-density-wave conductivity, with the difference that the parameter F in Eq. (1) has the physical meaning of the dc voltage applied to the system, and the quantity U [see Eq. (10)] is the dc current carried by the CDW.

IV. CONCLUSION

The analytical investigation reported in this paper was stimulated by numerical simulations of Ustinov,[5] which demonstrated that, in the region where the "tachyonic" wave train is unstable, the *I-V* characteristic of the model of the LJJ based on Eq. (1) departs from its usual branch given by Eq. (2). The break of the *I-V* characteristic at $f=f_{cr}$, predicted analytically in the present work, seems in accord with the results of the simulations. In the region $f>0$, $(f_{cr}-f)/f_{cr}\sim 1$, where the analytical approach based on the expansion of Eq. (1) in powers of the disturbance amplitudes is irrelevant, the preliminary simulations demonstrated chaotic dynamics of the disturbed wave train. It is quite feasible that the bifurcation considered in this work is but the first link in a chain of bifurcations leading to dynamical chaos. This issue deserves further investigation.

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