Casimir effects for charged particles

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The theory of forces due to quantum electrodynamic field fluctuations are of great consequence in condensed-matter physics and chemistry, and are routinely applied to polarizable media to explain long-ranged van der Waals forces. Less well studied are the Casimir forces on particles with a net charge. Some examples of the charged-particle case are discussed here. In particular, methods are explored for treating the infrared photon "divergences" which arise from the ease with which an accelerated charged particle emits radiation.

I. INTRODUCTION

The theory of long-ranged van der Waals forces in condensed-matter systems rests on an interesting feature of quantum electrodynamics in which the condensed-matter renormalization of electromagnetic-field fluctuations (in one region of space) produce "effective potentials" (in other regions of space) via photon propagation. For the case of polarizable media, these effects are well studied.[1] Less well investigated is the nature of Casimir forces on particles with a net charge. In some situations, Casimir forces on charged particles, when calculated to lowest orders in quantum electrodynamic perturbation theory, contain infrared divergences which require somewhat subtle soft-photon renormalizations. These will here be explored.

The nature of Casimir forces on charged particles can be argued in a simple manner as follows: (i) A particle (having charge e and mass m) moving classically in an electromagnetic field,

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \ , \tag{1}$$

has a four-momentum p obeying

$$[p^\mu-(e/c)A^\mu][p_\mu-(e/c)A_\mu]=-(mc)^2 \ . \tag{2}$$

(ii) In the charged-particle "center-of-mass" frame

$$p^\mu=(\mathbf{0},m^*c^2) \ , \tag{3}$$

and in the temporal gauge

$$A^\mu=(\mathbf{A},0) \ , \tag{4}$$

Eqs. (2), (3), and (4) read, for the effective mass,

$$(m^*)^2=m^2+(e/c^2)^2|\mathbf{A}|^2 \ . \tag{5}$$

Equation (5) can be derived nonperturbatively (for example) from the Dirac equation for an electron in a plane-wave radiation field by using somewhat more elaborate[2] mathematics. (iii) In lowest-order perturbation theory, the mass shift due to electromagnetic fluctuations is shown from Eq. (5) to be

$$\frac{\delta m}{m}=\frac{1}{2}\left[\frac{e}{mc^2}\right]^2\langle|\mathbf{A}|^2\rangle \ . \tag{6}$$

(iv) Of interest here is not the (divergent) vacuum fluctuation contribution to the mass, which is included in the "definition" of the "observed physical mass," but the change in mass (presumably finite) induced by condensed-matter renormalization of the electromagnetic fluctuations themselves. Using the rest-mass–energy equivalence, the mass shift in Eq. (6) is then written as a "potential" for the Casimir force on a charged particle,

$$U=\tfrac{1}{2}(e^2/mc^2)[\langle|\mathbf{A}|^2\rangle-\langle|\mathbf{A}|^2\rangle_0] \ , \tag{7}$$

where the first term on the right-hand side of Eq. (7) represents the electromagnetic fluctuations renormalized by a condensed-matter environment, and the second term is that contribution present if the charged particle were in the vacuum. Equation (7) is the starting point for computing the Casimir forces on a charged particle.

II. PHOTON PROPAGATORS

In the temporal gauge, the retarded photon propagator[3] is defined as

$$D_{ij}(\mathbf{x},\mathbf{y},\zeta)=(i/\hbar c)\int_0^\infty dt\, e^{i\zeta t}\langle[A_i(\mathbf{x},t),\ A_j(\mathbf{y},0)]\rangle \ . \tag{8}$$

In terms of the Matsubara frequencies,

$$\omega_n=(2\pi k_B T/\hbar)n,\quad n=0,\pm1,\pm2,\pm3,\ldots \ . \tag{9}$$

The electromagnetic fluctuations obey

$$\langle|\mathbf{A}(\mathbf{r})|^2\rangle=(k_B Tc)\sum_{n=-\infty}^{\infty}\mathrm{tr}\mathbf{D}(\mathbf{r},\mathbf{r},i|\omega_n|) \ , \tag{10}$$

where the trace in Eq. (9) is over vector indices in the propagator. Equations (7) and (10) yield the Casimir potential on a charged particle in terms of the photon propagator; i.e.,

$$U(\mathbf{r})=(k_B T/2)(e^2/mc)\sum_{n=-\infty}^{\infty}\mathrm{tr}[\mathbf{D}(\mathbf{r},\mathbf{r},i|\omega_n|)-\mathbf{D}_0(\mathbf{r},\mathbf{r},i|\omega_n|)] , \quad (11)$$

where $\mathbf{D}_0$ is the vacuum photon propagator.

III. INTERACTION WITH PERFECT CONDUCTORS

Consider a charged particle at a distance z from a perfect metal as shown in Fig. 1. For this problem the force may be regarded as a photon exchange process with an image charge.[4] The static one-photon exchange process gives rise to the usual electrostatic potential energy

$$U_0(z)=-e^2/4z . \quad (12)$$

In addition to the electrostatic potential of Eq. (12), we have a fluctuation exchange force, i.e., the Casimir force as given by Eq. (11). It is (see the Appendix for the geometry under discussion)

$$U_1(z)=-(k_B T/2mc^2)(e^2/z)\sum_{n=-\infty}^{\infty}\exp(-2|\omega_n|z/c) . \quad (13)$$

Adding the contributions in Eqs. (12) and (13) yields the resulting total potential

$$U(z)=-(e^2/4z)[1+2(k_B T/mc^2)\times\coth(2\pi k_B Tz/\hbar c)+\cdots] , \quad (14)$$

which, in the low-temperature limit ($T\to 0$, simplifies to

$$U(z)=-(e^2/4z)[1+(\hbar/\pi mc)(1/z)+\cdots] . \quad (15)$$

Equation (15) describes the lowest-order quantum electrodynamic correction to the usual electrostatic image force problem for a perfect conductor.

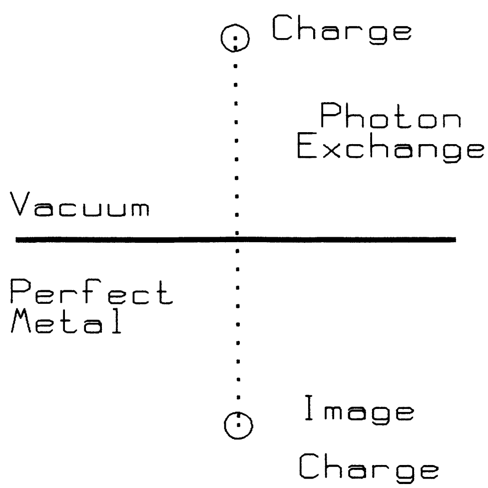

FIG. 1. Electron is attracted to a perfect metal by a one-photon-exchange process with its oppositely charged image. This includes the usual electrostatic term as well as the Casimir term.

Finally, the zero-temperature limit of Eq. (11) is *in general* obtained by replacing the sum over Matsubara frequencies by an integral (as $T\to 0$); i.e., Eq. (11) becomes

$$U(\mathbf{r})=(\hbar e^2/2\pi mc)\int_0^\infty d\omega\,\mathrm{tr}[\mathbf{D}(\mathbf{r},\mathbf{r},i\omega)-\mathbf{D}_0(\mathbf{r},\mathbf{r},i\omega)] . \quad (16)$$

IV. INTERACTION WITH POLARIZABLE ATOMS

Consider a charged particle a distance r from a spherical object with polarizability $\alpha(\zeta)$, as shown in Fig. 2. The electric field E that the charged particle applies to the atom has a magnitude

$$E=e/r^2 . \quad (17)$$

This static electric field lowers the atoms energy by an amount[5]

$$U_0(r)=-\tfrac{1}{2}\alpha(0)E^2=-e^2\alpha(0)/2r^4 , \quad (18)$$

yielding a potential $U_0(r)$ as in Eq. (18). In addition to this static effect, there is a two-photon exchange potential which can be calculated from Eq. (16) as

$$U_1(r)=-(\hbar e^2/2\pi mc^2)\int_0^\infty d\omega\,\omega^2\alpha(i\omega)\times\mathrm{tr}[\mathbf{D}_0(\mathbf{r},\mathbf{0},i\omega)\mathbf{D}_0(\mathbf{0},\mathbf{r},i\omega)] . \quad (19)$$

The trace over vector indices in the vacuum propagator product on the right-hand side of Eq. (19) is direct and reads

$$U_1(r)=-(\hbar e^2/\pi mc^4 r^2)\int_0^\infty d\omega\,\alpha(i\omega)\omega^2 e^{-2\omega r/c}\times g(\omega r/c) , \quad (20a)$$

$$g(x)=1+2/x+5/x^2+6/x^3+3/x^4 . \quad (20b)$$

Formally, the integral in Eq. (20a) diverges badly in the infrared limit $\omega\to 0$. The reason is related to the usual "infrared catastrophe" of quantum electrodynamics, i.e., that an accelerating electron radiates more and more photons at lower and lower frequencies.[6] Thus low-order perturbative expansions in powers of $(e^2/\hbar c)$ are incapable of describing the very large number of very-low-frequency photons. The infrared catastrophe in the divergent equation (20a) is due to the fact that here (strictly) only two-photon exchange has been taken into account in the integral.

The situation can be rectified by an infrared-cutoff factor of the usual "power-law type" in the elastic scattering amplitude at the charged-particle vertex in Fig. 2, i.e., by the replacement

$$g(x)\to g(x,\beta)=x^\beta g(x) . \quad (21)$$

The perturbative limit $\beta\to 0$ should be taken only after the frequency integral is performed. In detail, Eqs. (20) and (21) combine into the formal limit

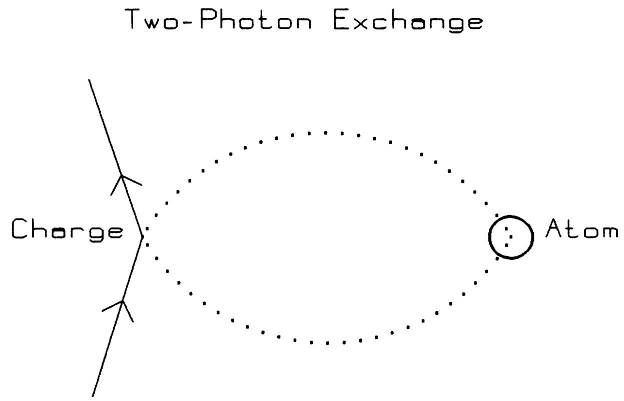

FIG. 2. Electron is attracted to a polarizable spherical uncharged atom by a two-photon exchange process.

$$U_1(r) = -\lim_{\beta\to 0}\left[\frac{\hbar e^2}{\pi m c^4 r^2}\int_0^\infty d\omega\,\omega^2\alpha(i\omega) \times e^{-2\omega r/c} g(\omega r/c,\beta)\right] . \tag{22}$$

With the Γ function defined (as is usual in infrared-cutoff integrals)

$$\Gamma(z) = \int_0^\infty \left[\frac{dx}{x}\right] x^z e^{-x} , \tag{23a}$$

$$\Gamma(1+z) = z\Gamma(z) . \tag{23b}$$

Equation (22) reads (for large r)

$$U_1(r) = -[\hbar e^2\alpha(0)/\pi m c r^5]\lim_{\beta\to 0}\chi(\beta) , \tag{24a}$$

$$\chi(\beta) = (1/2^{3+\beta})[\,\Gamma(3+\beta)+4\Gamma(2+\beta)+20\Gamma(1+\beta) +48\Gamma(\beta)+48\Gamma(\beta-1)] , \tag{24b}$$

which turns out to be perfectly finite (and repulsive); i.e.,

$$U_1(r) = (11/4\pi)[\hbar e^2\alpha(0)/m c r^5] . \tag{25}$$

Combining the two contributions in Eqs. (18) and (25) yields the total potential between a charged particle and a polarizable spherical atom (as $r \to \infty$):

$$U(r) = -[e^2\alpha(0)/2r^4][1-(11/2\pi)(\hbar/mcr)+\cdots] . \tag{26}$$

That the Casimir (two-photon) correction to the static attraction is *repulsive* (due to the soft-photon infrared renormalization) is *not* completely intuitive. However, it is by no means new that power-law regulators can reverse the sign of a diverging sum (or integral) in the course of producing a finite result. This is discussed further in the concluding section.

V. CONCLUSIONS

The new feature of calculating Casimir forces on charged particles, as opposed to only considering uncharged polarizable objects, is the possibility of having to regulate "infrared divergences" in low-order photon exchange potentials. More often, in the computation of Casimir effects, high-frequency divergences have to be regulated. In all cases, the fact that the sign of divergent sums can change during the process of renormalization is not very intuitive and looks somewhat strange when written out in the short direct form. An example follows.[7]

To see what is involved, one may consider the Casimir effect for vibrations on a mechanical string (say, a guitar, violin, or piano string) of length L with fixed ends. The mode frequencies of such a string have the form

$$\omega_k = (\pi v/L)k, \quad k = 1,2,3,\ldots , \tag{27}$$

where v is the velocity of waves along the string. Including the two possible polarizations, one has, for the quantum zero-point oscillation energy of the string,

$$E_0 = 2\sum_{k=1}^{\infty}(\hbar\omega_k/2) , \tag{28}$$

yielding the obviously infinite sum

$$E_0 = (\pi\hbar v/L)\sum_{k=1}^{\infty} k . \tag{29}$$

The power-law regulator involves the Riemann ζ function

$$\zeta(z) = \sum_{k=1}^{\infty} 1/k^z , \tag{30}$$

from which Eq. (29) is computed using

$$E_0 = (\pi\hbar v/L)\lim_{z\to -1}\zeta(z) . \tag{31}$$

Equation (29) is perfectly finite:

$$E_0 = -(\pi/12)(\hbar v/L) . \tag{32}$$

However, comparing Eqs. (29) and (32) gives rise to some wonder that Casimir effects are indeed observed in laboratories (for the quantum electrodynamic case). This leaves little doubt in the authors' minds that mechanical (say, string) Casimir effects are also in principle observable in the laboratory (for, say, a violin string, if not for other kinds of "strings"), despite the fact that ζ-function (or power-law) regulators are at first glance rather strange. (We could have buried this "strangeness" in a long-winded complicated derivation, but prefer to leave it bare where it may be explicitly noted.) In going from Eq. (22) to Eq. (25), a similar sign change results from the same sort of regularization as for the string; but it is the low-frequency divergence that is here of importance.

This unusual feature of Casimir forces on charged particles, i.e., that infrared divergences have to be regulated, is central to this work. Other examples (than those here discussed) are worthy of further study.

APPENDIX

We here collect some well-known equations for photon propagators used in the above work. The vacuum temporal gauge propagator is given by

$$\mathbf{D}_0(\mathbf{x},\mathbf{y},\zeta)=[1+(c/\zeta)^2\nabla\nabla](e^{i\zeta R/c}/cR)\ , \tag{A1}$$

$$R=|\mathbf{x}-\mathbf{y}|\ , \tag{A2}$$

and obeys the trace condition

$$\mathrm{tr}\mathbf{D}_0(\mathbf{x},\mathbf{y},\zeta)=(2/cR)e^{i\zeta R/c}\ . \tag{A3}$$

For the geometry of a plane separating a vacuum from a perfect metal, with $\mathbf{y}$ as a vacuum source point, and $\mathbf{y}_I$ as the image of that source point in the metal, the propagator is that of the vacuum in addition to that of the image, i.e.,

$$\mathrm{tr}\mathbf{D}(\mathbf{x},\mathbf{y},\zeta)=\mathrm{tr}\mathbf{D}_0(\mathbf{x},\mathbf{y},\zeta)-\mathrm{tr}\mathbf{D}_0(\mathbf{x},\mathbf{y}_I,\zeta)\ . \tag{A4}$$

Equations (A3) and (A4) have been employed for deriving Eq. (13) from Eq. (11).

Another interesting effect concerns the temperature variation of the charged-particle mass in a blackbody radiation field. From Eq. (6) one finds

$$\frac{\Delta m(T)}{m}=\frac{1}{2}\left[\frac{e}{mc^2}\right]^2[\langle|\mathbf{A}|^2\rangle_T-\langle|\mathbf{A}|^2\rangle_0]\ . \tag{A5}$$

From Eqs. (A1) and (10) it follows (with split spatial points $R=|\mathbf{x}-\mathbf{y}|$) that

$$\langle\mathbf{A}(\mathbf{x})\cdot\mathbf{A}(\mathbf{y})\rangle_T=(ck_BT)\sum_{n=-\infty}^{\infty}\mathrm{tr}\mathbf{D}(\mathbf{x},\mathbf{y},i|\omega_n|)\ , \tag{A6}$$

$$\langle\mathbf{A}(\mathbf{x})\cdot\mathbf{A}(\mathbf{y})\rangle_T=(2k_BT/R)\coth(\pi k_BTR/\hbar c)\ . \tag{A7}$$

As the point splitting distance grows small ($R\to 0$),

$$\langle\mathbf{A}(\mathbf{x})\cdot\mathbf{A}(\mathbf{y})\rangle_T=(2/\pi)(\hbar c/R^2)+(2\pi/3\hbar c)(k_BT)^2+\cdots\ , \tag{A8}$$

so that Eqs. (A5) and (A8) yield the temperature variation of the mass of a charged particle in a blackbody radiation field;[8] i.e.,

$$\frac{\Delta m(T)}{m}=\frac{\pi}{3}\frac{e^2}{\hbar c}\left[\frac{k_BT}{mc^2}\right]^2\ , \tag{A9}$$

to lowest order in the coupling strength $(e^2/\hbar c)$.

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