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Elastic modes, phase fluctuations, and long-range order in type-II superconductors

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We consider the effect of thermal fluctuations on the phase of the superconducting order parameter of the Abrikosov flux-line-lattice state of a type-II superconductor. While in the electrodynamic gauges usually chosen there are apparently divergent phase fluctuations, we show that the gauge-invariant phase has a finite average value. Thus, conventional superconducting long-range order is present in a three-dimensional flux line lattice.

The discovery of type-II superconducting material with high-critical temperatures has brought renewed interest to the study of the Abrikosov flux-line lattice (FLL). In particular, the relatively short correlation lengths in these materials (and hence the correspondingly large values of the Ginzburg-Landau parameter κ) and the high-critical temperatures enhance the effects of thermal fluctuations leading to, among other things, a large depression of the melting temperature of the lattice below the mean-field H_{c2} phase boundary.[1-6] The study of thermal fluctuations in the Abrikosov state has led to some controversy regarding the lower critical dimension of superconducting long-range order, i.e., the dimensionality below which the long-range order is destroyed at *all* temperatures. Some authors[4,7,8,] have argued that three is the lower critical dimension with superconducting correlations decaying algebraically in that dimension. This conclusion is found on examining the phase of the superconducting order parameter in the fluctuating flux lattice. With the choice of gauge made by these authors, the fluctuations of this phase diverge logarithmically in three dimensions, which leads then to an algebraic decay of the order-parameter correlations. In this paper we evaluate the *gauge-invariant* order parameter, and show explicitly that its phase is finite in three dimensions. The previous results to the contrary are apparently artifacts of the particular choice of gauge.

Our starting point is the Ginzburg-Landau (GL) free energy of a superconductor in the presence of an external magnetic field, written in dimensionless units,[5,9]

$$f=-|\psi|^2+\tfrac{1}{2}|\psi|^4+H^2+\left|\left(\frac{\nabla}{i\kappa}-A\right)\psi\right|^2 . \qquad (1)$$

Here, $\kappa=\lambda/\xi$, where λ and ξ are the penetration depth and correlation length, respectively. The order-parameter ψ is measured in units of the London solution, the magnetic field is measured in units of $\sqrt{2}\,H_c$, where H_c is the critical fields, and lengths are measured in units of λ. The microscopic field $\mathbf{H}$ and vector potential $\mathbf{A}$ are related by $\nabla\times\mathbf{A}=\mathbf{H}$. While the high-$T_c$ materials are anisotropic, for simplicity we restrict our discussion to the isotropic case shown in Eq. (1). It is relatively straightforward to incorporate anisotropy into our subsequent analysis, at least in the case where the magnetic field is along the c axis of the crystal; we only consider that geometry here.

In the absence of fluctuations the GL equations that result from minimizing Eq. (1) with respect to variations in ψ and $\mathbf{A}$ yield the well-known Abrikosov triangular FLL, with flux lines (FL's) parallel to the magnetic field which we take to be along the z axis. It will be convenient to work in the symmetric gauge specified by $\mathbf{A}=(B/2)\times(\hat{\mathbf{z}}\times\mathbf{r})$ where $B=\langle H(\mathbf{r})\rangle_{\text{sp}}$, the brackets $\langle\,\rangle_{\text{sp}}$ indicating a spatial average. The Abrikosov solution for a FLL with N lines is then given by[9]

$$\mathbf{r}_v(z)=\mathbf{R}_v+\mathbf{s}_v(z)\,, \qquad (2)$$

where $\mathbf{R}_v=(X_v,Y_v,z)$ and $\mathbf{s}_v=(s_v^x(z),s_v^y(z),0)$ is the displacement of the point on the FL initially located at $\mathbf{R}_v$. Replacing X_v and Y_v in Eq. (2) by $X_v(z)$ and $Y_v(z)$ and expanding to first order in s yields a solution ψ_l then of the linearized GL equations. However, as first noted by Brandt,[9] this solution leads to a divergent order parameter. Brandt obtained a well-behaved solution of the full nonlinear GL equations by multiplying ψ_l by a slowly varying function which modulates the amplitude and phase of the order parameter but leaves the zeros at the points defined in Eq. (2),

$$\psi=\psi_l(1+\theta/2)^2e^{i\kappa\chi}\,. \qquad (3)$$

The functions θ and χ are then determined variationally by minimizing the free energy. The results found by

Brandt reduce in the continuum limit of the lattice to

$$\theta(\mathbf{k})=\frac{k_\psi^2+k_z^2}{k_\psi^2+k^2}\frac{2b\kappa^2(\nabla\cdot\mathbf{s})}{k_\perp^2} \tag{4}$$

and

$$\chi(\mathbf{k})=-\frac{b\kappa}{k^2}\frac{k_z^2}{k_\perp^2}\hat{\mathbf{z}}\cdot(\nabla\times\mathbf{s})\,; \tag{5}$$

here $k_\psi^2=2(1-b)/\xi^2$, and we have assumed that $\mathbf{s}(\mathbf{r})=\mathbf{s}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$, $\mathbf{k}=(\mathbf{k}_\perp,k_z)$.

The $\mathbf{s}$-dependent portion of the phase of the linearized solution is given by

$$\phi_l=\frac{b\kappa^2}{k_\perp^2}\hat{\mathbf{z}}_0\cdot(\nabla\times\mathbf{s})\,. \tag{6}$$

Then from Eqs. (5) and (6) we find the phase of the wave function Eq. (3) to be

$$\phi=\phi_l+\kappa\chi=\frac{b\kappa^2}{k^2}\hat{\mathbf{z}}\cdot(\nabla\times\mathbf{s})\,. \tag{7}$$

Correlation functions of ϕ can now be calculated readily using the nonlocal harmonic elastic Hamiltonian derived by Brandt.[9] For the triangular FLL this energy takes the form

$$H=\tfrac{1}{2}\sum_k \mathbf{s}_i(-\mathbf{k})\{c_L(\mathbf{k})k_ik_j+\delta_{ij}[c_{66}(\mathbf{k})k_\perp^2+c_{44}(\mathbf{k})k_z^2]\}s_j(\mathbf{k})\,, \tag{8}$$

$(i,j)=(x,y)$,

where the elastic moduli are given by

$$c_{44}(\mathbf{k})=\frac{B^2}{4\pi}\langle\omega_0\rangle\left[\frac{1}{k^2+\langle\omega_0\rangle}+\frac{1}{2b\kappa^2}\right], \tag{9a}$$

$$c_{66}=\frac{B_{c2}^2}{4\pi}\frac{b(1-b)^2}{8\kappa^2}, \tag{9b}$$

$$c_L(\mathbf{k})=\frac{B^2}{4\pi}\left[\frac{1}{k^2+\langle\omega_0\rangle}-\frac{1}{k^2+k_\psi^2}\right]-c_{66}\,, \tag{9c}$$

$$\langle\omega_0\rangle=\langle|\psi_0|^2\rangle=2\kappa^2(1-b)/[(2\kappa^2-1)\beta+1]\,, \tag{10}$$

and $\beta=1.16$ for the triangular FLL.

The thermal average of ψ within the ensemble of elastic fluctuations governed by the quadratic Hamiltonian Eq. (8) is readily calculated. The contribution from the amplitude modulation θ will be ignored as it is innocuous (i.e., finite). We then have

$$\langle\psi\rangle=|\psi|\langle e^{i\phi}\rangle=|\psi|e^{-1/2\langle\phi^2\rangle}\,, \tag{11}$$

where using Eq. (7)

$$\langle\phi^2\rangle=b^2\kappa^4\int_0^\infty\frac{dk_z}{4\pi}\int_0^{\Lambda^2}\frac{dk_\perp^2}{2\pi}\frac{(\hat{\mathbf{z}}\times\mathbf{k})_i(\hat{\mathbf{z}}\times\mathbf{k})_j}{k^4}\times\langle s_i(\mathbf{k})s_j(-\mathbf{k})\rangle\,. \tag{12}$$

Here $\langle\,\rangle$ denotes a thermal average with respect to the Hamiltonian Eq. (8). In Eq. (12) we have assumed, for simplicity, a circular Brillouin zone, in the plane perpendicular to the fields, of radius Λ, where $\Lambda^2=2b/\xi^2$.

The elastic propagator appearing in Eq. (12) is found readily from the Hamiltonian Eq. (8) and is given by

$$G_{ij}(\mathbf{k})\equiv\langle s_i(\mathbf{k})s_j(\mathbf{k})\rangle=k_BT\left[\frac{P_T}{c_{66}k_\perp^2+c_{44}k_z^2}+\frac{P_L}{c_{11}k_\perp^2+c_{44}k_z^2}\right], \tag{13}$$

where $P_T=(\delta_{ij}-k_ik_j/k_\perp^2)$, $P_L=k_ik_j/k_\perp^2$, and $c_{11}=c_L+c_{66}$. The second term on the right-hand side of Eq. (13) will not contribute to the integrand of Eq. (12) and we find

$$\langle\phi^2\rangle=b^2\kappa^4\int_0^\infty\frac{dk_z}{2\pi}\int_0^{\Lambda^2}\frac{dk_\perp^2}{4\pi}\frac{k_\perp^2}{k^4}\frac{k_BT}{c_{66}k_\perp^2+c_{44}k_z^2}\,. \tag{14}$$

Since $c_{44}(\mathbf{k})$ and $c_{66}(\mathbf{k})$ are both finite and $\mathbf{k}\to 0$, it is clear that the integral in Eq. (14) will diverge in the infrared ($\mathbf{k}\to 0$) regime. Power counting indicates that this divergence will be *linear* in the size of the system, suggesting that *four* is the lower critical dimensionality. Previous authors[4,7,8] have evaluated Eq. (14) using the form for $c_{44}(\mathbf{k})$ valid at large $\mathbf{k}$, i.e., $k^2\gtrsim\langle\omega_0\rangle$, in which case $c_{44}\sim(B^2/4\pi)\langle\omega_0\rangle/k^2$. This is incorrect, as the divergence of $\langle\phi^2\rangle$ clearly is governed by the behavior as $\mathbf{k}\to 0$. In any event were we to use this form for c_{44} in Eq. (14), we would find

$$\langle\phi^2\rangle\approx b^2\kappa^4\int_0^\infty\frac{dk_z}{2\pi}\int_0^{\Lambda^2}\frac{dk_\perp^2}{4\pi}\frac{k_\perp^2}{k^2}\times\frac{k_BT}{c_{66}k_\perp^2+(B^2/4\pi)\langle\omega_0\rangle k_z^2}\,. \tag{15}$$

The integral in Eq. (15) is *logarithmically* divergent with the system size, suggesting that three is the lower critical dimensionality for superconducting long-range order.

We now demonstrate that these divergences have *no* physical significance by considering the gauge-invariant phase of the wave function which, in our dimensionless units, is given by

$$\tilde{\phi}(\mathbf{r})=\phi(\mathbf{r})-\kappa\int_0^{\mathbf{r}}\mathbf{A}\cdot dl\,, \tag{16}$$

where we have chosen $\tilde{\phi}(0)=\phi(0)=0$. The vector potential $\mathbf{A}$ is given by Brandt's solution of the nonlinear GL equations. The result is stated in terms of the corresponding magnetic field $\mathbf{h}=\mathbf{H}-B\hat{\mathbf{z}}$ which satisfies the second GL equation

$$\nabla\times\mathbf{h}=-\omega\mathbf{Q}\,. \tag{17}$$

Here $\omega=|\psi|^2$, and $\mathbf{Q}$ is the super velocity given by $\mathbf{Q}=\mathbf{A}-(1/\kappa)\nabla\phi$. The field $\mathbf{h}$, after local averaging[9] (which will suffice for our calculation of long-wavelength properties) has the form

$$\bar{h}=\frac{B\langle\omega_0\rangle}{k^2+\langle\omega_0\rangle}\left[-\hat{\mathbf{z}}\cdot(\nabla\cdot\mathbf{s})+\frac{\partial\mathbf{s}}{\partial z}\right]=\frac{B\langle\omega_0\rangle}{k^2+\langle\omega_0\rangle}\nabla\times(\mathbf{s}\times\hat{\mathbf{z}})\,. \tag{18}$$

Using Eqs. (17) and (7), and the definition of $\mathbf{Q}$, we then find

$$\bar{A}=\frac{B\langle\omega_0\rangle}{k^2(k^2+\langle\omega_0\rangle)}\nabla(\hat{\mathbf{z}}\cdot\mathbf{v}\times\mathbf{s})+\frac{B}{k^2+\langle\omega_0\rangle}(\nabla^2\mathbf{s}\times\hat{\mathbf{z}})+\frac{B}{2}\hat{\mathbf{z}}\times\mathbf{r}\,. \tag{19}$$

From Eq. (16) we find [the last term of Eq. (19) makes no contribution to the integral in Eq. (16)] that

$$\tilde{\phi}=\frac{b\kappa^2}{k^2+\langle\omega_0\rangle}(\hat{\mathbf{z}}\cdot\nabla\times\mathbf{s})-\frac{b\kappa^2}{k^2+\langle\omega_0\rangle}\int_0^{\mathbf{r}}(\nabla^2\mathbf{s}\times\hat{\mathbf{z}})\cdot d\mathbf{l}\,. \tag{20}$$

Unlike ϕ, $\tilde{\phi}$ has *no* singularity at $k=0$. Furthermore, it is clear from power counting that the second moment $\langle\tilde{\phi}^2\rangle$ will be finite in three dimensions because of the removal of this singularity at $k=0$. We see then that the apparent breakdown of long-range order given by Eq. (14) is an artifact of the choice of gauge, whether it is the symmetric one used here or the asymmetric gauge used by previous authors.

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