Schur functions, chiral bosons, and the quantum-Hall-effect edge states

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I demonstrate how the many-body wave function may be used to describe the bosonization of the edge excitations of a droplet of $\nu=1$ quantum-Hall liquid. In particular, I exhibit an isomorphism between the charge-neutral edge-state excitations of the droplet and the space of universal symmetric polynomials. There are two natural bases for this space; the first, the Schur functions, correspond to the fermion picture; the second, generated by the power sums, yields the Bose picture and the Kac-Moody algebra. I also show explicitly how the loop group LU(1) acts to create the coherent-state deformations of the droplet shape used in path-integral bosonization and in the quantization of chiral bosons.

I. INTRODUCTION

The two-dimensional electron gas (2DEG) in a strong magnetic field is a Fermi system that can be mapped into an equivalent Bose system in two distinct, but related, ways. The bosonization of the bulk degrees of freedom by means of a Chern-Simons statistics field[1] leads to the approximate mean-field theory of the fractional quantum-Hall effect (FQHE) which was further developed by Zhang, Hansen, and Kivelson[2] and by Read.[3] The second bosonization concentrates on the edge states, the only low-energy states when the bulk of the two-dimensional electron gas exhibits the quantum-Hall effect (QHE).[4] These states have been shown by Wen[5] to span representations of Kac-Moody current algebras.[6] The fundamental representations of these algebras are unique up to unitary equivalence but their generators may be written in terms of either Bose or Fermi operators and this dual description extends to the dynamics: the fermion operators serve to promote individual fermions to excited states while the chiral boson fields create coherent-edge waves or deformations in the shape of the droplet of 2DEG.[7]

The bosonic description of the edge degrees of freedom is a chiral variant of one-dimensional bosonization, a by now classic idea which goes back to Jordan[8] and Tomonaga[9] and whose modern formulation is due to Luther and Peschel[10] and to Coleman and Mandelstam.[11] The formalism invoked is rather abstract: Commuting variables are introduced via the language of field theory using Schwinger terms, vertex operators, operator product expansions, and renormalization. At no point are the wave functions of the individual electrons mentioned. In the QHE the many-body states are most familiarly exhibited as wave functions[12] and it is from the wave-function picture that much of our physical understanding of the effect is derived.

An alternative approach to bosonization used coherent states and coherent-state path integrals.[13] This formalism is the geometric counterpart of the algebraic representation theory of the Kac-Moody algebras, and focuses on the groups obtained by exponentiating the Lie algebras.[14] Since the geometry of infinite-dimensional Lie groups can seem complicated, it is valuable to have an explicit example of the resulting many-body coherent states. The Hall edge states provide a physical realization of these constructions and this paper is intended to provide an exegesis of the ideas behind them.

In the second section I will describe the physical basis for the Fermi-Bose mapping in the case of filling-fraction $\nu=1$, an observation due to Haldane,[15] and show how the Schur functions map the states of the QHE onto the ring of symmetric polynomials. The third section reviews some of the algebraic properties of this ring. The fourth section will show how it becomes a Hilbert space, and how two natural bases form the Bose and Fermi pictures of the states. The fifth section will use the constructions from the earlier sections to produce the explicit coherent-state wave functions generated by the loop group. I will show that they describe deformations of the edges of the QHE sample. In the discussion section I will briefly discuss the FQHE states.

I should make it clear that none of the mathematics in this paper is original, only the physical application is believed to be new. In particular the key constructions rely heavily on Secs. 10.3 and 10.4 of Ref. 14.

II. SLATER DETERMINANTS AND SCHUR FUNCTIONS

In the symmetric gauge, and with a choice of length scale, the lowest Landau-level single-particle wave functions can be identified with the space of analytic functions, $f(z)\ z=x+iy$, subject to the restriction that

$$\psi(z)=f(z)e^{-(1/2)|z|^2} \tag{2.1}$$

be square integrable. A normalized basis for this space is given by

$$\psi_n(z)=\frac{1}{\sqrt{\pi n!}}z^n e^{-(1/2)|z|^2}\ . \tag{2.2}$$

The simplest N-body state, a homogeneous circular droplet of electron liquid with filling-fraction $\nu=1$, has an antisymmetric Slater determinant of the basis func-

tions as its wave function

$$\Psi_0(z_1,z_2,\ldots,z_N)=\begin{vmatrix} z_1^{N-1} & z_1^{N-2} & \cdots & 1 \\ z_2^{N-1} & z_2^{N-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ z_N^{N-1} & z_N^{N-2} & \cdots & 1 \end{vmatrix} \times\exp\left[-\tfrac{1}{2}\sum_i |z_i|^2\right] . \tag{2.3}$$

Expectation values in this state are well known[16] to be calculable as statistical averages in a two-dimensional one-component Coulomb gas with the Gaussian factor providing the neutralizing background charge.

In the absence of a perturbing potential all N-body states with energy below the cyclotron energy gap, $\omega_c=eB/m$, are degenerate with this one. They are created by moving some electrons to states outside the droplet. A typical wave function is

$$\Psi_{\{\lambda\}}(z_1,z_2,\ldots,z_N)=\begin{vmatrix} z_1^{\lambda_1+N-1} & z_1^{\lambda_2+N-2} & \cdots & z_1^{\lambda_N} \\ z_2^{\lambda_1+N-1} & z_2^{\lambda_2+N-2} & \cdots & z_2^{\lambda_N} \\ \vdots & \vdots & \ddots & \vdots \\ z_N^{\lambda_1+N-1} & z_N^{\lambda_2+N-2} & \cdots & z_N^{\lambda_N} \end{vmatrix} \times\exp\left[-\tfrac{1}{2}\sum_i |z_i|^2\right] \tag{2.4}$$

with $\lambda_1\geq\lambda_2\geq\lambda_3$, etc.

The wave function $\Psi_{\{\lambda\}}$ still describes an N-electron state but has $\lambda_1+\lambda_2+\cdots+\lambda_n=M$ extra powers of z. Each λ_i means that an electron has been moved from its position in the "sea" and raised from, say, the z^m state to the $z^{m+\lambda_i}$ state. If the droplet were confined in a potential well, the states near the edge would see a potential gradient and, since the single-particle z^m states are localized near a circle whose radius depends on m, they will have energy $E_m=Cm$. The new many-body state has energy $E=CM$. The Hall droplet becomes a physical realization of a "chiral" Dirac sea where all the states near the Fermi surface, identified with the physical edge of the droplet, move in the direction of the Hall current induced by the gradient.[7]

The set of distinct λ_i with $\sum_i\lambda_i=M$ are in one-to-one correspondence with the partitions of M, which I will label by the collective symbol $\{\lambda\}$,

$$\{\lambda\}\equiv\{\lambda_1,\lambda_2,\ldots,\lambda_N\} . \tag{2.5}$$

The number $p(m)$ of distinct partitions of m is given by the number theory partition function

$$Z=\frac{1}{\prod_{n>0}(1-x^n)}=\sum_m p(m)x^m . \tag{2.6}$$

Put $x=e^{-\beta C}$

$$Z=\frac{1}{\prod_{n>0}(1-e^{-\beta Cn})}=\sum_m p(m)e^{-\beta Cm} \tag{2.7}$$

and the similarity to a statistical-mechanical partition function already suggests a description of the charge-neutral excitations in terms of bosonic oscillators with energy Cn.

To explore what this means for the wave functions we will concentrate on the determinant part of the wave function. Define

$$D_{\{\lambda\}}(z)=\begin{vmatrix} z_1^{\lambda_1+N-1} & z_1^{\lambda_2+N-2} & \cdots & z_1^{\lambda_N} \\ z_2^{\lambda_1+N-1} & z_2^{\lambda_2+N-2} & \cdots & z_2^{\lambda_N} \\ \vdots & \vdots & \ddots & \vdots \\ z_N^{\lambda_1+N-1} & z_N^{\lambda_2+N-2} & \cdots & z_N^{\lambda_N} \end{vmatrix} \tag{2.8}$$

or, in compact but, I hope, self-explanatory notation

$$D_{\{\lambda\}}(z)=\det|z_s^{\lambda_t+n-t}| . \tag{2.9}$$

The empty partition corresponds to the Vandermonde determinant

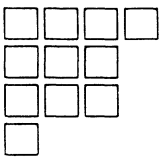
$$D(z)=\begin{vmatrix} z_1^{N-1} & z_1^{N-2} & \cdots & 1 \\ z_2^{N-1} & z_2^{N-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ z_N^{N-1} & z_N^{N-2} & \cdots & 1 \end{vmatrix}=\prod_{i<j}(z_i-z_j) \tag{2.10}$$

which is a factor of all the $D_{\{\lambda\}}(z)$—so the quotient

$$\Phi_{\{\lambda\}}(z)=D_{\{\lambda\}}(z)/D(z) \tag{2.11}$$

is a *symmetric* polynomial in the z_i. Clearly all charge-neutral "excited" states $\Psi_{\{\lambda\}}$ are obtained by multiplying the "ground state" Ψ_0 by one of these symmetric polynomials[15] and the quantum Hilbert space of neutral excitations is isomorphic to the linear space spanned by them.

The polynomial $\Phi_{\{\lambda\}}(z)$ is the *Schur function* associated with the partition $\{\lambda\}$ of M. These functions are familiar in physics as the characters of the groups $GL(n)$ or $U(n)$.[17] The partitions are usually displayed as *Young tableaux*—diagrams with λ_i boxes in the ith row. For example,



represents the partition $\{4,3,3,1\}$. Schur functions are also important as they form a linear basis for the space of symmetric polynomials and it is in this role, rather than that of group characters, that we will use them.

III. SYMMETRIC FUNCTIONS

In this section I will review those parts of the theory of symmetric functions that we need in the sequel. The material is standard and can be found in many books, e.g., Ref. 18.

Given a set of symbols α_i, $i=1,N$ (for example, complex variables like our z_i) form the ring $\mathcal{S}(\alpha)$ of polyno-

mials in the α_i which are invariant under arbitrary permutations of the α_i.

Elements of this ring include the *elementary symmetric functions*, a_n, $n=1,N$ defined by

$$\prod_i (1-\alpha_i x)=1-a_1 x+a_2 x^2+\cdots\pm a_N x^N , \tag{3.1}$$

and the *homogeneous product sums* h_i, $i=1,\infty$, defined by

$$\frac{1}{\prod_i (1-\alpha_i x)}=1+h_1 x+h_2 x^2+h_3 x^3+\cdots . \tag{3.2}$$

Inverting the power-series definitions, we can express the a_n as polynomials in the h_n and vice versa. Remarkably the coefficients are integers in both directions. Sums and products of either the a_n or the h_n generate the ring $\mathcal{S}(\alpha)$.

The *power sums* S_k, $k=1,\infty$ are defined as

$$S_k(\alpha)=\sum_i \alpha_i^k \tag{3.3}$$

and using the relation

$$\exp\left[\sum_k \frac{1}{k}x^k S^k(\alpha)\right]=1+h_1 x+h_2 x^2+h_3 x^3+\cdots \tag{3.4}$$

we can express the S_n as polynomials in the h_n (with integer coefficients), and the h_n as polynomials (with rational coefficients) in the S_k, so sums of products of the S_n also generate $\mathcal{S}(\alpha)$.

The Schur functions can be written in terms of the h_n as

$$\Phi_{\{\lambda\}}(\alpha)=\det|h_{\lambda_s-s+t}| . \tag{3.5}$$

For example,

$$\Phi_{\{p\}}=h_p \tag{3.6}$$

and

$$\Phi_{\{p,q,r\}}=\begin{vmatrix} h_p & h_{p+1} & h_{p+2} \\ h_{q-1} & h_q & h_{q+1} \\ h_{r-2} & h_{r-1} & h_r \end{vmatrix} . \tag{3.7}$$

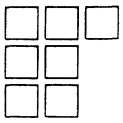
We can easily evaluate

$$\Phi_{\{1^N\}}(z)=z_1 z_2\cdots z_N=a_N , \tag{3.8}$$

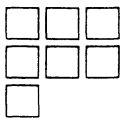
and this is an example of the expression for $\Phi_{\{\lambda\}}$ in terms of the a_n: We use the *conjugate partition,* the Young tableau with the rows and columns interchanged, in the same manner of determinant as the h expression, e.g.,

$$\Phi_{\{322\}}=\begin{vmatrix} h_3 & h_4 & h_5 \\ h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \end{vmatrix}=\begin{vmatrix} a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 \\ 0 & 1 & a_1 \end{vmatrix} , \tag{3.9}$$

where $\{3,2,2\}$



is the partition conjugate to $\{3,3,1\}$



None of the relations between the h_n, a_n, S_n, and $\Phi_{\{\lambda\}}$ depend explicitly on N, but when N is finite the a_i for $i>N$ are zero, as are the Schur functions with more than N rows in the partition $\{\lambda\}$. Similarly only the first N of the h_n and S_n's are functionally independant. It is convenient to regard N as infinite and then we speak of the ring of *universal* symmetric functions.

The Schur functions form a *linear basis* for the ring $\mathcal{S}(\alpha)$: Any element of $\mathcal{S}(\alpha)$ is a linear combination of $\Phi_{\{\lambda\}}$'s. To see this, observe that any symmetric function can be converted to an antisymmetric one by multiplying by $D(\alpha)$, and that antisymmetrizing each monomial in the resulting expression converts it to a $D_{\{\lambda\}}(\alpha)$. Finally dividing out the catalytic $D_{\{\lambda\}}$ returns the symmetric function as a sum of $\Phi_{\{\lambda\}}$.

There is a key identity connecting the Φ's and the $\mathcal{S}_k$'s: Take two sets of indeterminates, α_i and β_t, then

$$\exp\left[\sum_k \frac{1}{k}S_k(\alpha)S_k(\beta)\right]=\sum_{\{\lambda\}}\Phi_{\{\lambda\}}(\alpha)\Phi_{\{\lambda\}}(\beta) . \tag{3.10}$$

It is straightforward to prove (3.10) from Cauchy's determinant identity[19]

$$\det\left|\frac{1}{1-\alpha_s\beta_t}\right|=\frac{D(\alpha)D(\beta)}{\prod_{ij}(1-\alpha_i\beta_j)} . \tag{3.11}$$

Once we have the relation (3.10) we can take all but one, say $\beta_1=x$, of the β_i to be zero. We see that (3.4) is a special case of (3.10).

IV. $\mathcal{S}(z)$ AS A HILBERT SPACE

As mentioned in the Introduction, a general reference for this section is Secs. 10.3 and 10.4 of Ref. 14.

The symmetric algebra $\mathcal{S}(z)$ is both a vector space spanned by the $\Phi_{\{\lambda\}}$ and a ring generated by the $S_n(z)$. We now define an inner product on $\mathcal{S}(z)$ which serves to make it into a Hilbert space. I wish the generators $S_k(z)$ to be independent so we take the number N of the z_i to infinity and this means taking the thermodynamic limit of the 2DEG.

We define the inner product of two polynomial functions of the S_k to be

$$\langle f(S)|g(S)\rangle=\int\prod_k\left[\frac{d^2S_k}{\pi k}\right]f^*(S)g(S)\times\exp\left[-\sum_k\frac{1}{k}|S_k|^2\right]. \tag{4.1}$$

In particular the product of the S_n's themselves is

$$\langle S_k|S_{k'}\rangle=k\delta_{k,k'}. \tag{4.2}$$

A general element in the ring is a sum of products of the S_n such as

$$S_{(l)}(z)=S_1^{l_1}S_2^{l_2}\cdots S_n^{l_n}, \tag{4.3}$$

and their inner products are

$$\langle S_{(l)}|S_{(l')}\rangle=(1^{l_1}l_1!2^{l_2}l_2!\cdots n^{l_n}l_n!)\delta_{(l)(l')}. \tag{4.4}$$

The inner product is essentially that of a Bargman-Fock space where creation operators are represented by multiplication by S_k and annihilation operators by their adjoint, $k\partial_{S_k}$. Our Hilbert space is therefore isomorphic to the space created from a cyclic vector $|0\rangle$ by application of bosonic $a_k^\dagger$'s whose commutation relations are

$$[a_k,a_{k'}^\dagger]=k\delta_{k,k'},\quad k>0. \tag{4.5}$$

If we define $a_{-k}=a_k^\dagger$ we can write

$$[a_k,a_{k'}]=k\delta_{k+k',0}. \tag{4.6}$$

They are the commutation relations of a level-one Abelian Kac-Moody algebra.

The remarkable property of this inner product is that the $\Phi_{\{\lambda\}}$, which we already know to form a linear basis for the Hilbert space $\mathcal{S}(z)$, are *orthonormal* with respect to it. To prove this use the reproducing kernel identity

$$\int\left[\prod_k\frac{d^2S_k(z)}{\pi k}\right]F(S_k(z))\exp\left[-\sum_k\frac{1}{k}|S_k(z)|^2\right]\times\exp\left[\sum_k\frac{1}{k}S_k(\bar z)S_k(z')\right]=F(S_k(z')) \tag{4.7}$$

and (3.9)

$$\exp\left[\sum_k\frac{1}{k}S_k(\bar z)S_k(z')\right]=\sum_{\{\lambda\}}\Phi_{\{\lambda\}}(\bar z)\Phi_{\{\lambda\}}(z') \tag{3.10'}$$

with the choice $F(S_k(z))=\Phi_{\{\mu\}}(z)$.

Since the $\Phi_{\{\lambda\}}(z)$ are linearly independent [they are independent on the torus of U(n) where the $z_i=e^{i\theta_i}$, so *a fortiori* independent on the larger space of all z_i] we must have

$$\langle\Phi_{\{\lambda\}}|\Phi_{\{\lambda'\}}\rangle=\int\left[\prod_k\frac{d^2S_k}{\pi k}\right]\Phi^*_{\{\lambda\}}\Phi_{\{\lambda'\}}\times\exp\left[-\sum_k\frac{1}{k}|S_k|^2\right]=\delta_{\{\lambda\}\{\lambda'\}}. \tag{4.8}$$

An alternative demonstration is via Frobenius's famous reciprocity formula connecting the characters of the permutation group with the characters of GL_n.[20] Frobenius's formula asserts that

$$S_{(l)}=\sum_{\{\lambda\}}\chi_{(l)}^{\{\lambda\}}\Phi_{\{\lambda\}}(z), \tag{4.9}$$

where the $\chi_{(l)}^{\{\lambda\}}$ are the characters of the representation $\{\lambda\}$ of a permutation group. The conjugacy classes of the group are labeled by (l). As group characters the $\chi_{(l)}^{\{\lambda\}}$ obey the orthogonality conditions

$$\frac{1}{g}\sum_{(l)}g_{(l)}\chi_{(l)}^{\{\lambda\}}\chi_{(l)}^{\{\lambda'\}}=\delta_{\{\lambda\}\{\lambda'\}}, \tag{4.10}$$

where $(l_1+2l_2+3l_3+\cdots)!=g$ is the order of the permutation group and

$$g_{(l)}=\frac{g}{1^{l_1}2^{l_2}\cdots l_1!l_2!\cdots l_n!} \tag{4.11}$$

is the number of permutation group elements in the conjugacy class. Equation (4.9) can now be inverted to give $\Phi_{\{\lambda\}}$ in terms of the S_k,

$$\Phi_{\{\lambda\}}(z)=\frac{1}{g}\sum_{(l)}g_{(l)}\chi_{(l)}^{\{\lambda\}}S_{(l)} \tag{4.12}$$

and then (4.4) and (4.10) yield (4.6). This demonstration is not as independent as it seems—the conventional proof of (4.9) depends on (3.9)—but the formulas (4.9) and (4.12) relating the two bases of the Hilbert space are worth exhibiting.

We must now ask the crucial question: Is the inner product we have defined on $\mathcal{S}(z)$ the same as the one given by quantum mechanics? The answer is, in general, *No*: the states (2.4) obtained by multiplying a normalized Ψ_0, Eq. (2.3), by $\Phi_{\{\lambda\}}$ are orthogonal with respect to both products—but when regarded as elements of $\mathcal{S}(z)$ they are normalized to unity, while they are *not* [because we need to include the $\sqrt{n}!$'s from (2.2)] normalized wave functions themselves. This is not, however, a problem in the thermodynamic limit. When the curvature of the droplet edge can be ignored on the length scales of interest, states near the edge reduce to those found in the Landau gauge. For these physically accessible states, the normalization is independent of n and the two products do coincide.

The fruit of our labors is two separate descriptions of the space of edge excitations of the Hall droplet: one in terms of the bosonic a_n's, and one in terms of the elementary fermionic excitations corresponding to the Schur functions. The next step is the identification of the a_n operators in the Kac-Moody algebra (4.6) with the Fourier components of the surface currents used in Ref. 7

or, equivalently, with the mode expansion of the chiral boson describing the edge waves. This we will do in the next section.

V. COHERENT STATES

In the path-integral route to bosonization of chiral fermions[13] one uses generalized coherent states[21,22] which are obtained by the action of a loop group on the Dirac sea. The loop-group LG acts on each first-quantized, single-particle state in the sea by multiplying the wave function with a common position-dependent element of the group G. For example, the loop-group LU(1) has elements $g(x)=e^{i\varphi(x)}$ and acts by

$$e^{ikx}\mapsto e^{i\varphi(x)}e^{ikx} . \tag{5.1}$$

If one thinks of the states in the sea as spanning a hyperplane (or, more precisely, a closed subspace) in the Hilbert space, this common group action rotates the hyperplane bodily into a new orientation. The resulting many-body wave function is the Slater determinant made from any set of wave functions spanning the new subspace. There will be an *isotropy subgroup* of elements which leave the subspace fixed, merely performing rotations within the hyperplane and within its orthogonal complement. The set of coherent states corresponds to the quotient space of the full group by the isotropy subgroup.

In the second-quantized language the group generators, $j(x)$, form a current algebra with commutation relations

$$[j(x),j(x')]=\frac{-i}{2\pi}\partial_x\delta(x-x') , \tag{5.2}$$

or their non-Abelian generalization

$$[j_a(x),j_b(x')]=\mathrm{if}^c_{ab}j_c(x)\delta(x-x') - \frac{i}{2\pi}g_{ab}\partial_x\delta(x-x') . \tag{5.3}$$

In this language the coherent states are made by applying exponentials of the generators to the Dirac sea

$$|\varphi(x)\rangle=\exp\left[i\int dx\,\varphi(x)j(x)\right]|0\rangle . \tag{5.4}$$

Since $\langle 0|j(y)|0\rangle=0$ and

$$\exp\left[-i\int\varphi(x')j(x')dx'\right]j(x) \times\exp\left[+i\int\varphi(x')j(x')dx'\right]$$

$$=j(x)-\frac{1}{2\pi}\int dx'\varphi(x')\partial_{x'}\delta(x'-x) \tag{5.5}$$

$$=j(x)+\frac{1}{2\pi}\partial_x\varphi(x) \tag{5.6}$$

we see that the $|\varphi\rangle$ has charge

$$\langle\theta|j(x)|\varphi\rangle=+\frac{1}{2\pi}\partial_x\varphi(x) . \tag{5.7}$$

Assuming that x has period 2π we can Fourier decompose $j(x)$

$$j(x)=\sum_n e^{-inx}j_n \tag{5.8}$$

and read off from (5.1) that

$$[j_n,j_{n'}]=-n\delta_{n+n',0} . \tag{5.9}$$

The j_n, $n>0$ are therefore to be interpreted as creation operators, while the j_n, with negative n annihilate $|0\rangle$ and generate the isotropy group. Quotienting out the isotropy group, we can rewrite (5.4) as

$$|\varphi(x)\rangle\propto\exp\left[i\sum_{n>0}\varphi_n j_n\right]|0\rangle . \tag{5.10}$$

In the case of the Hall droplet it is convenient to adjust the magnetic field so the surface of the droplet coincides with the circle $z=e^{i\theta}$ and then an element of the loop-group LU(1) has a Laurent-Fourier expansion

$$g(z)=\exp\left[\sum_{n=-\infty}^{\infty}i\varphi_n z^n\right] . \tag{5.11}$$

The isotropy group (in this context called B^+) is composed of elements which are boundary values of functions analytic in $z>1$

$$g(z)=\exp\left[\sum_{n=-\infty}^{0}i\varphi_n z^n\right] \tag{5.12}$$

as they will always give determinants with some identical columns after multiplying into Ψ_0. The effective part of $g(z)$ is multiplication by elements of the subgroup N^- consisting of boundary values of functions analytic in $z<1$:

$$g(z)=\exp\left[\sum_{n=1}^{\infty}i\varphi_n z^n\right] . \tag{5.13}$$

Each state z^k in the Slater determinant is changed

$$z^k\mapsto\left[\exp\left[\sum_n i\varphi_n z^n\right]\right]z^k \tag{5.14}$$

and this has the effect of multiplying the whole determinant by the factor

$$\exp\left[\sum_n i\varphi_n S_n(z)\right] . \tag{5.15}$$

Comparing with (5.3) confirms that the operation of multiplying a wave function by S_k is the same as acting on it by the j_k surface current generator of the Kac-Moody algebra. We can identity $a_k^\dagger\equiv j_k$, $k>0$.

To see how these operations deform the charge distribution at the edge of the droplet we can use the Coulomb gas analogy for the wave function. Introduce a distribution of point "charges" of magnitude q_i, at the points $Z_i=e^{i\theta_i}$. Since we are only considering neutral excitations assume $\sum_i q_i=0$. The kth Fourier component of the external charge distribution is then $(1/2\pi)\sum_i q_i\bar{Z}_i^k$.

Taking

$$i\varphi_k=\frac{1}{k}\sum_i q_i\bar{Z}_i^k, \quad k\neq 0 \tag{5.16}$$

multiplies the wave function by

$$\exp\left[\sum_k \frac{1}{k}\left[\sum_i q_i \bar{Z}_i^k\right] S_k(z)\right] = \frac{1}{\prod_{i,j}(1-\bar{Z}_i z_j)^{q_i}} \tag{5.17}$$

and, in the statistical average, the internal z_i charges will be attracted or repelled by the external ones. They will screen them over the scale set by the magnetic length (which is infinitesimal as we have taken the thermodynamic limit while keeping droplet radius fixed at unity) so the deformed state $|\varphi\rangle$ has charge at the edge. Because the division by $1/k$ in the Fourier sum is equivalent to integrating the charge distribution, the charge distribution is given by differentiating the configuration space $\varphi(\theta)$

$$\langle j(\theta)\rangle = \frac{1}{2\pi}\partial_\theta \varphi(\theta)\ , \tag{5.18}$$

thus reproducing the second-quantized operator result.

VI. DISCUSSION

The wave-function picture of chiral bosonization, which seems especially suited to the QHE, provides an alternative to both the field theory and to the coherent-state path-integral formalism. Like all bosonization schemes it provides what is essentially a kinematical, or bookkeeping, tool. But, while the results are derived initially for noninteracting fermions, they may be extended to interacting theories by finding bosonic equivalents of the fermionic interaction terms. The utility of the dual Bose-Fermi pictures lies in the approximation schemes they suggest—effects that are nonperturbative in one language become perturbative in the other.

One can extend these tricks and bosonize the edge states in the FQHE. For the $\nu=1/(2m+1)$ stage in the FQHE hierarchy the wave functions are the Laughlin states. These are simple powers of the $\nu=1$ Slater determinants and the only significant result of this is that the Berry phase, which appears in the coherent-state path integral and determines the level k of the Kac-Moody algebra, is multiplied by $2m+1$. for other hierarchy states the construction seems to be more complicated. Wen[23] has suggested a method which seems very similar in spirit to the Goddard, Kent, and Olive construction[24] of introducing new degrees of freedom and gauging out unwanted ones—indeed this is what is being done for the $\nu=2m+1$ states: We decompose the electron into $2m+1$ fermionic partons and use these to make one U(1) current of level $k=2m+1$ and one set of level $k=1$ SU($2m+1$) currents. The ground state is then the product of $2m+1$ independent Slater determinants, one for each kind of parton. The unwanted SU($2m+1$) degrees of freedom are then frozen out by forcing the each of the $2m+1$ partons to be at the same location as $2m$ others. The individual determinants then coincide and give the Laughlin state.

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