Renormalized field theory for the static crossover in uniaxial dipolar ferromagnets

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We present a field-theoretic description of the crossover in the critical statics of uniaxial dipolar ferromagnets. Within a generalized minimal subtraction scheme we are able to describe the crossover from Ising behavior with nonclassical exponents to asymptotic uniaxial dipolar behavior, which is characterized by classical exponents with logarithmic corrections. The results are discussed in terms of flow diagrams and effective exponents and are compared with experiments on $LiTbF_4$.

I. INTRODUCTION

The influence of the dipole-dipole interaction on the critical statics of isotropic and uniaxial ferromagnets is quite different. Whereas for isotropic ferromagnets the the dipolar interaction leads only to a slight modification of the critical exponents, it was discovered by Larkin and Khmelnitskii[1] that unaxial dipolar ferromagnets show classical behavior with logarithmic corrections in three dimensions. The asymptotic behavior of this system was also studied by means of renormalization-group (RG) theory,[2,3] which revealed that the one-loop calculation agrees with the asymptotic results of Ref. 1.

The existence of logarithmic corrections near the transition temperature T_c was verified experimentally for a number of uniaxial ferromagnetic substances $GdCl_3$,[4,5] $LiTbF_4$,[6–10] $Dy(C_2H_5SO_4)_3{\cdot}9H_20$,[10,11] and TbF_3.[12] However, these experiments were performed in regions of the reduced temperature, where departures from the asymptotic behavior are expected and are observed indeed. If the susceptibility is analyzed in terms of an effective critical exponent, a pronounced maximum is found.[10]

As in the case of isotropic dipolar ferromagnets,[13–19] it is therefore of interest to investigate the crossover from Ising behavior, which is dominated by the exchange interaction, to the asymptotic uniaxial dipolar behavior, which is characterized by classical exponents with logarithmic corrections. First attempts to explore this problem were made in Refs. 20 and 21. In Ref. 20, matching techniques[22] are used to study the crossover from Ising to classical behavior. Since no attempt is made to include the logarithmic corrections, the relevance of this approach for the crossover in uniaxial dipolar ferromagnets may be questioned. In Ref. 21, the asymptotic results from Refs. 2 and 3 are used in order to fit the flow of the coupling constants to the experimental data. It is found that the flow of the coupling constant obtained from fitting different quantities (susceptibility, specific heat, magnetization) gives compatible results. This, however, does not solve the actual problem since none of these quantities is computed and no theoretical explanation for the flow is given. Moreover, the underlying starting Hamiltonian does not even apply in the isotropic region.

In this paper we study the crossover from a critical behavior, dominated by the exchange interaction, to the asymptotic uniaxial dipolar behavior by a generalized minimal subtraction scheme similar to Refs. 19 and 23. This field-theoretic method allows the calculation of the complete flow of the coupling constants and parameters from the Ising fixed point to the uniaxial dipolar fixed point. Herewith, the main obstacle to a renormalization-group treatment is surmounted, namely, the differing upper critical dimensionalities of the isotropic and uniaxial dipolar fixed points, four and three, respectively. We interpret the experiments mentioned above by a crossover from mean field via Ising critical to asymptotic uniaxial dipolar critical behavior.

The paper is organized as follows: In Sec. II we introduce the model and the generalized renormalization procedure. This will be used in Secs. III and IV to calculate the renormalization constants and the flow of the coupling constants. In Sec. IV we calculate the susceptibility, which will be analyzed in Sec. V in terms of an effective exponent and compared with experiments on $LiTbF_4$. After a discussion of the specific heat in Sec. VI, we give a short summary of the results in Sec. VII.

II. MODEL AND RENORMALIZATION

The Landau-Ginzburg free-energy functional for an n-component uniaxial spin system with an isotropic exchange coupling and long-range dipolar interactions is given by[1–3,16]

$$\mathcal{H}=-\tfrac{1}{2}\int_k \left[r_0+k^2+g_0^2\frac{q^2}{p^2}\right] S_0^\alpha(\mathbf{k})S_0^\alpha(-\mathbf{k}) - \frac{u_0}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4} F^{\alpha\beta\gamma\delta} S_0^\alpha(\mathbf{k}_1)S_0^\beta(\mathbf{k}_2) \times S_0^\gamma(\mathbf{k}_3)S_0^\delta(\mathbf{k}_4)\delta\left[\sum_{i=1}^{4}\mathbf{k}_i\right] . \quad (2.1)$$

Here $S_0^\alpha(\mathbf{q})$ ($\alpha=1,2,\ldots,n$) are the components of the bare spin variables. (The most relevant case is the Ising system $n=1$.) The d-dimensional wave vector $\mathbf{k}=(\mathbf{p},\mathbf{q})$ is decomposed into $\mathbf{q}$, the component along the uniaxial direction, and $\mathbf{p}$, the remaining $(d-1)$ components. In Eq. (2.1) we used the notations $\int_k=\int d^dk/(2\pi)^d$ and

$$F^{\alpha\beta\gamma\delta}=\tfrac{1}{3}(\delta^{\alpha\beta}\delta^{\gamma\delta}+\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma}) .$$

The bare reduced temperature is given by

$$r_0=\frac{T-T_c^0}{T_c^0}$$

and g_0^2 is a measure of the relative strength of the dipolar interaction with respect to the exchange interaction. The Gaussian propagator for the graphical expansion has the form

$$G_0(r_0,g_0,\mathbf{k},\mathbf{q})=\left[r_0+k^2+g_0^2\frac{q^2}{p^2}\right]^{-1} . \qquad (2.2)$$

Contrary to Refs. 3 and 21, the propagator (2.2) contains the full gradient term k^2, and not only the part orthogonal to the uniaxial direction p^2. This allows us to describe the crossover from Ising behavior to uniaxial dipolar behavior. If one would take into account only the p^2 term, the theoretical analysis would be confined to the strong dipolar limit, i.e., to an asymptotic analysis. As can be inferred from the propagator (2.2), the dipolar coupling constant g_0 has the same canonical dimension as the wave vector k. Consequently the RG flow of the dipolar coupling constant tends to infinity analogous to the flow of the dipolar coupling constant in isotropic dipolar ferromagnets[19] and the flow of the anisotropy constant in the context of bicritical points.[23] Therefore, the conventional minimal subtraction scheme, where solely the ϵ poles ($\epsilon=d_c-d$) are subtracted,[24–26] is not applicable in the present case. Nevertheless, the theory can be renormalized by including divergences of the type ln$g$ in the renormalization procedure. This extension of the minimal renormalization scheme was formulated by Amit and Goldschmidt[23] in the context of bicritical points and has been successfully applied to the crossover problem in isotropic dipolar ferromagnets.[19] In the present case there is an additional peculiarity, namely that the propagator vanishes in the limit of infinite dipolar coupling. Therefore, in conjunction with any ϵ pole there is a ln$g$ divergence, i.e., the structure of the divergence D to be subtracted in order to renormalize the theory, is given by

$$D\propto\frac{1}{\epsilon}(1-\epsilon\ln g)$$

in an ϵ expansion around dimension four. In particular, this structure is maintained in all orders of the perturbation series. Thus, one is confronted with a situation where the actual expansion coefficient is not the four-point coupling u_0 but

$$u_0g^{-\epsilon}\approx u_0(1-\epsilon\ln g) .$$

The determination of the renormalization factors is fixed by the requirements that (i) in the limit of vanishing dipolar coupling one gets the Ising behavior and (ii) for finite g_0 the asymptotic behavior is given by classical behavior with logarithmic corrections. After introducing a momentum scale μ the renormalized parameters, coupling constants, and fields are defined by

$$u_0=\mu^\epsilon Z_u S_d^{-1}u , \qquad (2.3a)$$

$$r_0=Z_r r , \qquad (2.3b)$$

$$g_0=Z_g g , \qquad (2.3c)$$

$$S_0^\alpha(\mathbf{q})=(Z_\Phi)^{1/2}S^\alpha(\mathbf{q}) , \qquad (2.3d)$$

where the factor

$$S_d=\frac{2}{(4\pi)^{d/2}\Gamma\left[\dfrac{d}{2}\right]}$$

is introduced for convenience. The renormalization factors Z have to be determined according to the general minimal subtraction scheme given in Refs. 19 and 23 where one has to account for the modifications mentioned above.

We shall consider the Fourier transforms of bare N-point vertex functions defined via the appropriate generating functional.[26] In the remaining part of this section we derive the renormalization-group equation for the renormalized two-point vertex function $\Gamma_R^{(2)\alpha\beta}=\delta^{\alpha\beta}\Gamma_R$, which also serves for an introduction of our notation. The bare vertex functions Γ_B are related to the renormalized vertex functions by

$$\Gamma_R=Z_\Phi\Gamma_B . \qquad (2.4)$$

As usual, the renormalization-group equations are obtained by noting that the bare quantities are independent of the momentum scale μ. Applying the differential operator $\mu(d/d\mu)|_0$ to the bare two-point vertex functions we find

$$\left[\mu\frac{\partial}{\partial\mu}+\zeta_r\left[u,\frac{g}{\mu}\right]r\frac{\partial}{\partial r}+\zeta_g\left[u,\frac{g}{\mu}\right]g\frac{\partial}{\partial g}+\beta\left[u,\frac{g}{\mu}\right]\frac{\partial}{\partial u}+\zeta_\Phi\left[u,\frac{g}{\mu}\right]\right]\Gamma_R(r,g,u,\mu)=0 , \qquad (2.5)$$

where the β and ζ functions are given by

$$\beta_u\left[u,\frac{g}{\mu}\right]=\mu\frac{\partial}{\partial\mu}u|_0 , \qquad (2.6a)$$

$$\zeta_r\left[u,\frac{g}{\mu}\right]=\mu\frac{\partial}{\partial\mu}\ln Z_r^{-1}|_0 , \qquad (2.6b)$$

$$\zeta_g\left[u,\frac{g}{\mu}\right]=\mu\frac{\partial}{\partial\mu}\ln Z_g^{-1}|_0 , \qquad (2.6c)$$

$$\zeta_\Phi\left[u,\frac{g}{\mu}\right]=\mu\frac{\partial}{\partial\mu}\ln Z_\Phi^{-1}|_0 . \qquad (2.6d)$$

The symbol $|_0$ indicates that all derivatives are to be taken at fixed bare parameters r_0, g_0, and u_0. One should realize that, due to the generalized renormalization procedure, the renormalization factors depend on the dipolar coupling, which implies that the β and ζ functions depend on u as well as on g/μ.

Partial differential equations like (2.5) are solved by the method of characteristics which, in the present case, are given by

$$\mu(l)=\mu l \;, \tag{2.7a}$$

$$l\frac{dr(l)}{dl}=r(l)\zeta_r\left[u(l),\frac{g(l)}{\mu(l)}\right]\equiv r(l)\zeta_r(l) \;, \tag{2.7b}$$

$$l\frac{dg(l)}{dl}=g(l)\zeta_g\left[u(l),\frac{g(l)}{\mu(l)}\right]\equiv g(l)\zeta_g(l) \;, \tag{2.7c}$$

$$l\frac{du(l)}{dl}=\beta_\mu\left[u(l),\frac{g(l)}{\mu(l)}\right]\equiv\beta_u(l) \;, \tag{2.7d}$$

with the initial conditions $r(1)=r$, $g(1)=g$, and $u(1)=u$. Combining Eq. (2.5) with Eqs. (2.7), the following ordinary differential equation for the scaled vertex functions

$$\Gamma_R[r(l),g(l),u(l),\mu(l)]\equiv\Gamma_R(l)$$

is obtained:

$$l\frac{d}{dl}\Gamma_R(l)=-\zeta_\Phi(l)\Gamma_R(l) \;. \tag{2.8}$$

This is solved by

$$\Gamma_R(l)=\exp\left[-\int_1^l\frac{d\rho}{\rho}\zeta_\Phi(\rho)\right]\Gamma_R(1) \;, \tag{2.9}$$

where $\Gamma_R^\alpha(1)=\Gamma_R^\alpha(r,g,u,\mu)$ and the flow equations (2.7b and 2.7c) are solved by

$$r(l)=r\exp\left[\int_1^l\frac{d\rho}{\rho}\zeta_r(\rho)\right] \;, \tag{2.10a}$$

$$g(l)=g\exp\left[\int_1^l\frac{d\rho}{\rho}\zeta_g(\rho)\right] \;. \tag{2.10b}$$

By dimensional analysis one finds that Γ_R^α has the dimension μ^2. This implies that the dimensionless vertex function

$$\hat{\Gamma}_R\left[\frac{r}{\mu^2},\frac{g}{\mu},\frac{k}{\mu},u\right]=\frac{1}{\mu^2}\Gamma_R(r,g,k,u,\mu)$$

obeys the RG equation

$$\hat{\Gamma}_R\left[\frac{r}{\mu^2},\frac{g}{\mu},\frac{k}{\mu},u\right]=l^2\exp\left[\int_1^l\frac{d\rho}{\rho}\zeta_\Phi\left[u(\rho),\frac{g(\rho)}{\mu(\rho)}\right]\right]\times\hat{\Gamma}_R\left[\frac{r(l)}{\mu^2(l)},\frac{g(l)}{\mu(l)},\frac{k}{\mu(l)},u(l)\right] \;. \tag{2.11}$$

III. RENORMALIZATION OF THE COUPLING CONSTANT U AND THE FLOW DIAGRAM

In this section we evaluate the renormalization factor of the four-point coupling constant. The resulting flow will elucidate the fixed points and the crossover in between. The renormalization constant Z_u for the coupling constant u is determined by the requirement that the renormalized four-point vertex function

$$\Gamma_R^{(4)\alpha\beta\gamma\delta}(r,g,\{k_i=0\},u,\mu)=F^{\alpha\beta\gamma\delta}\Gamma_R^{(4)}(r,g,\{k_i=0\},u,\mu) \;, \tag{3.1}$$

which is related to the bare vertex function $\Gamma_B^{(4)}$by

$$\Gamma_R^{(4)}(r,g,\{k_i=0\}u,\mu)=Z_\phi^2\Gamma_B^{(4)}(r_o,g_0,\{k_i=0\},u_0) \;, \tag{3.2}$$

is finite according to the renormalization procedure of Sec. II. The one-loop contribution is given by

$$\Gamma_B^{(4)}(r_0,g_0,\{k_i=0\},u_0)=u_0-u_0^2\frac{n+8}{18}[I_4(r_0,g_0)+\cdots] \;, \tag{3.3}$$

where the ellipsis represents two permutations referring to the external legs of the corresponding one-loop diagram. After performing the q integration in

$$I_4(r,g)=\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\int\frac{dq}{2\pi}\frac{1}{(r+p^2+q^2+g^2q^2/p^2)^2} \;, \tag{3.4}$$

one obtains

$$I_4(r,g)=\tfrac{1}{4}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{p}{(r+p^2)^{3/2}(g^2+p^2)^{1/2}} \;. \tag{3.5}$$

In determining the divergent part of this integral, one has to take care that both limits $g\to0$ and $g\to\infty$ are taken into account correctly. Let us first consider the asymptotic limit ($g\to\infty$), in which case the integral (3.5) reduces to

$$I_4(r,\infty)=\frac{1}{4g}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{p}{(r+p^2)^{3/2}} \;. \tag{3.6}$$

This shows two essential points: (i) the integral is logarithmically divergent in $d=3$ dimensions, i.e., the upper critical dimension is $d_c=3$, and (ii) the effective expansion coefficient is no longer the four-point coupling constant u but $\bar{v}=u/g$. This asymptotic limit was studied extensively in the literature.[1-3] It is found that the flow of the coupling constant $\bar{v}$ tends to zero as the inverse of the logarithm of the flow parameter.

For finite nonzero dipolar coupling g, a dimensional analysis shows that the integral (3.5) is proportional to g^{d-4} implying that the effective expansion coefficient is no longer u but $v=ug^{d-4}$. In $d=3$ dimensions this leads to an effective coupling constant $\bar{v}=u/g$ in agreement with the asymptotic analysis.

Using standard techniques, the divergent part of the integral $I_4(r=0,g)$ is found to be

$$[I_4(r=0,g)]_{\rm div}=\frac{1}{\epsilon}S_d\left[a+\left[\frac{g}{\mu}\right]^{\sigma}\right]^{-\epsilon/\sigma} \tag{3.7}$$

in $d=4-\epsilon$ dimensions. The coefficients a and σ are fixed by the requirements that (i) for $g=0$ one obtains the result without dipolar interaction and (ii) in the asymptotic limit ($g\to\infty$) the coupling constant v flows to zero as the inverse of the logarithm of the flow parameter in $d=3$ dimensions. The first requirement gives

$$a=1 \tag{3.8a}$$

and the second leads to

$$\sigma=1\ . \tag{3.8b}$$

In one-loop order there is no field renormalization and hence the renormalized four-point vertex function is given by

$$\Gamma_R^{(4)}(k_i=0)=\mu^{\epsilon}\left[Z_u u S_d^{-1}-\frac{n+8}{18}Z_u^2 S_d^{-2}u^2\times[I_4(Z_r r/\mu^2,Z_g g/\mu)+\cdots]\right]\ . \tag{3.9}$$

where the ellipsis represents two permutations. Therefore, the renormalization constant for the four-point coupling constant is found to be

$$Z_u=1+\frac{n+8}{6}\frac{u}{\epsilon}\left[1+\frac{g}{\mu}\right]^{-\epsilon}\ . \tag{3.10}$$

With $\beta_u=\mu(\partial/\partial\mu)|_0 u$ one obtains for the corresponding β function

$$\beta_u\left[u,\frac{g}{\mu}\right]=-\epsilon u+\frac{n+8}{6}u^2\left[1-\frac{1}{1+\mu/g}\right]\ . \tag{3.11}$$

Knowing the β function, we can now study the flow of the coupling constant $u(l)$ from the flow equation

$$l\frac{du(l)}{dl}=\beta_u\left[u(l),\frac{g(l)}{\mu l}\right]=-\epsilon u(l)+\frac{n+8}{6}u^2(l)\left[1-\frac{1}{1+\mu l/g(l)}\right]\ . \tag{3.12}$$

The renormalization of the dipolar coupling g is determined solely by the renormalization of the fields because there is no contribution to the divergent part of $\Gamma^{(2)}(\mathbf{k})$ of the form q^2/p^2. Therefore, in one-loop order, the flow of the dipolar coupling is given by $g(l)=g$, i.e. the flow of g is solely determined by its canonical dimension.

In the limit $l\to 0$, the flow equation (3.12) reduces to

$$l\frac{du(l)}{dl}=-\epsilon u(l)+\frac{n+8}{6}u^2(l)\frac{\mu l}{g}\ . \tag{3.13}$$

The solution of Eq. (3.13) for $\epsilon=1$ reads

$$u(l)=\frac{6}{n+8}\frac{g/\mu l}{\ln\hat{l}-\ln l}\ , \tag{3.14}$$

where the quantity $\hat{l}$ is determined by the boundary condition $u(l=1)=u$ leading to

$$\ln\hat{l}=\frac{g u_H}{\mu u}\ . \tag{3.15}$$

The quantity

$$u_H=\frac{6\epsilon}{n+8} \tag{3.16}$$

denotes the Heisenberg fixed point, which reduces to the Ising fixed point for $n=1$: $u_H(n=1)=u_I$.

From Eq. (3.14) one infers that the effective coupling constant $v(l)=u(l)/\hat{g}(l)$ tends to zero as the inverse of the logarithm of the flow parameter l well known from earlier asymptotic studies.[2,3] Here $\hat{g}(l)$ is the canonical flow of the dipolar coupling constant g

$$\hat{g}(l)=\frac{g}{\mu l}\ . \tag{3.17}$$

For $l\to 0$ one obtains

$$v(l)=\frac{6}{n+8}\frac{1}{\ln\hat{l}-\ln l}\ . \tag{3.18}$$

These considerations demonstrate that the generalized renormalization procedure gives the correct asymptotic flow of the coupling constant. Let us now study the general solution of the flow equation (3.12). For $\epsilon=1$ we find

$$u(l)=u_H\hat{g}(l)\left[\frac{g u_H}{\mu u}+\ln\left[\frac{1+\hat{g}(l)}{1+g/\mu}\right]\right]^{-1}\ . \tag{3.19}$$

Without dipolar interaction, $g\to 0$, this equation reduces to

$$u^{-1}(l)|_{g\to 0}=u_H^{-1}+l\left[\frac{1}{u}-\frac{1}{u_H}\right]\ . \tag{3.20}$$

Therefore, in the absence of dipolar forces the asymptotics are determined by the Ising (Heisenberg) fixed point, i.e., in the limit $l\to 0$ one obtains, from Eq. (3.20), $u(l)\to u_H$. For $g\neq 0$ one finds in the asymptotic limit $l\to 0$ that $v(l)=u(l)/\hat{g}(l)$ tends to zero as the inverse logarithm of the flow parameter

$$v(l\to 0)\to\frac{1}{\ln(1/l)}\to 0\ . \tag{3.21}$$

These limiting cases suggest to define a coupling constant $w(l)$ by

TABLE I. Fixed points of the flow equations ($n=1$). The Gaussian (G) and Ising (I) fixed points are unstable, whereas the uniaxial dipolar (UD) fixed point is infrared stable.

	g^*	w^*
G	$g_G^*=0$	$w_G^*=0$
I	$g_I^*=0$	$w_I^*=\frac{6}{n+8}\epsilon$
UD	$g_{\rm UD}^*=\infty$	$w_{\rm UD}^*=0$("ln")

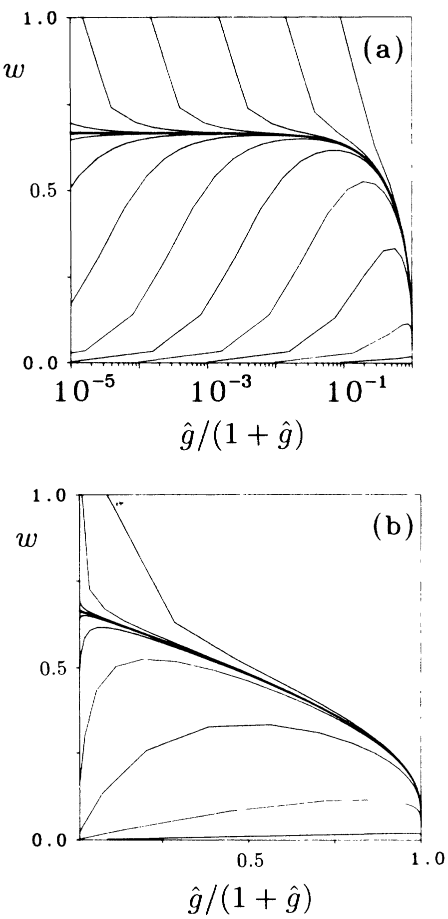

FIG. 1. (a) Flow diagram for the coupling constant $w(l)$ vs $\tilde{g}(l)=\hat{g}(l)/[1+\hat{g}(l)]$ for $\epsilon=1$ and $n=1$. The initial values for the flow are chosen as $w\approx 1$, $w\approx 0$, and $g=10^{-i}$ ($i=1,2,\ldots,10$). (b) Flow diagram for the coupling constant $w(l)$ as in (a) where $\tilde{g}$ is given on a linear scale in the interval $0\le\tilde{g}\le 1$.

$$w(l)\equiv\frac{u(l)}{1+\hat{g}(l)}\ , \tag{3.22}$$

with finite values in both limits. Then, one finds for (i) $g\to 0$: $w(l)\to u(l)$ and for (ii) $g\to\infty$: $w(l)\to v(l)$.

The topology of the $(w,\tilde{g})$ flow diagram is determined by the fixed points summarized in Table I. In order to map the range of $\hat{g}(l)$ onto the interval [0,1], it is convenient to define the quantity

$$\tilde{g}(l)=\hat{g}(l)/[1+\hat{g}(l)]\ .$$

The flow diagram is depicted in Fig. 1, where the momentum scale is chosen as $\mu=1$, the number of components as $n=1$ (uniaxial case), and the space dimension as $d=3$. The flow diagram in Fig. 1 is divided into two regions by a separatrix, which is the renormalization-group trajectory from the Ising (I) fixed point to the uniaxial (UD) fixed point. All renormalization-group trajectories flow into the separatrix asymptotically. For $u\le u_I$, there are, besides the stable uniaxial dipolar fixed point, two unstable fixed points (I and G). Starting from the Gaussian (G) fixed point (i.e., for weak dipolar systems with small initial values for g), the renormalization-group trajectories traverse regions near the I fixed point before they finally end up in the stable UD fixed point after joining the separatrix. This implies that a weak dipolar system shows Ising behavior in approaching the critical temperature before there is a crossover to uniaxial dipolar behavior. The onset of this crossover region depends on the relative strength of the dipolar interaction, but the behavior in the crossover region is universal. For strong dipolar systems (large values of the initial value of the dipolar coupling g), the flow tends directly to the uniaxial dipolar fixed point. The effects of the Ising fixed point are seen merely in a maximum of the flow of the coupling constant.

IV. MASS RENORMALIZATION AND SUSCEPTIBILITY

In this section we study the mass renormalization and compute the scaling function of the susceptibility. As renormalization is done at the critical point, the mass renormalization is determined by the requirement that the renormalized vertex function $\Gamma^{(2,1)}_{R\alpha\beta}$ with one S^2-insertion is finite according to the renormalization procedure defined in Sec. II. In one-loop order there is the following relation between the bare and the renormalized vertex function:

$$\Gamma^{(2,1)}_{R\alpha\beta}(r=0,g,\mathbf{k}_1,\mathbf{k}_2,\mathbf{p},u,\mu)=Z_r\Gamma^{(2,1)}_{B\alpha\beta}(r_0=0,g_0,\mathbf{k}_1,\mathbf{k}_2,\mathbf{p},u_0)\ , \tag{4.1}$$

where the one-loop contribution to the bare vertex function is given by

$$\Gamma^{(2,1)}_{B\alpha\beta}(r=0,g_0,\mathbf{k}_1,\mathbf{k}_2,\mathbf{p})=\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{p})\delta^{\alpha\beta}\left[1-\frac{n+2}{6}u_0I_4(r_0=0,g_0,\mathbf{p})\right]\ . \tag{4.2}$$

The divergent part of this one-loop diagram is identical with Eq. (3.7). Therefore, the renormalization constant for the mass is found to be

$$Z_r=1+u\frac{n+2}{6\epsilon}\left[1+\frac{g}{\mu}\right]^{-\epsilon} \tag{4.3}$$

and the corresponding ζ function is given by

$$\zeta_r\left[u,\frac{g}{\mu}\right]=u\frac{n+2}{6}\left[1-\frac{1}{1+\mu/g}\right]\ . \tag{4.4}$$

Let us now calculate the susceptibility to one-loop order. The unrenormalized (bare) two-point vertex function is found to be

$$\Gamma^{(2)}_{B\alpha\beta}=\delta^{\alpha\beta}\left[r_0+u_0\frac{n+2}{6}I_2(r_0,g_0)\right], \quad (4.5)$$

where the integral I_2 is defined by

$$I_2(r,g)=\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\int\frac{dq}{2\pi}\frac{1}{p^2+q^2+g^2p^2/q^2+r}. \quad (4.6)$$

As in the isotropic case, the dipole-dipole interaction leads to a shift of the transition temperature,[19] which can be taken into account by a counter term or equivalently by subtracting from the integral $I_2(r,g)$ its value at $r=0$

$$I_2(r=0,g)=\tfrac{1}{2}S_d\Gamma(d/2)\Gamma(-1+\epsilon/2)g^{2-\epsilon}, \quad (4.7)$$

which is a measure for the magnitude of the shift of the transition temperature.

According to the above considerations, one finds for the renormalized two-point vertex function after subtracting the counter term

$$\Gamma^{(2)}_{R\alpha\beta}(r,g,\mu)=\mu^2\delta^{\alpha\beta}\left[Z_r\frac{r}{\mu^2}-uS_d^{-1}\frac{n+2}{6}\times[I_2(r/\mu^2,g/\mu)-I_2(0,g/\mu)]\right]. \quad (4.8)$$

Using standard techniques for the evaluation of integrals in the dimensional regularization scheme, the integral $I_2(r,g)$ is found to be

$$I_2(r,g)=-\frac{1}{2\pi}\left[\frac{1}{2}S_d\Gamma\left[\frac{d}{2}\right]\right]\frac{\Gamma(-\frac{1}{2})\Gamma(2-\epsilon/2)}{\Gamma(\frac{3}{2}-\epsilon/2)}\times\Gamma\left[-1+\frac{\epsilon}{2}\right]\times\int_0^1dx\frac{[g^2x+r(1-x)]^{1-\epsilon/2}}{[x(1-x)]^{1/2}}. \quad (4.9)$$

The evaluation of the parameter integral

$$J(r,g)=\int_0^1dx[x(1-x)]^{-1/2}[g^2x+r(1-x)]^{1-\epsilon/2} \quad (4.10)$$

in Eq. (4.9) results in (see the Appendix)

$$\hat{J}(r,g)=J(r,g)-J(r=0,g)=\frac{\pi}{2}r(r+g^2)^{-\epsilon/2}\left[1+\epsilon\ln2+\frac{\epsilon}{2}-\frac{\epsilon}{2}\ln\left[\frac{g^2}{r+g^2}\right]-\epsilon\frac{r+g^2}{r}\ln\left[1+\frac{\sqrt{r}}{g}\right]-\epsilon\frac{r-g^2}{r}\frac{1}{1+g/\sqrt{r}}\right]. \quad (4.11)$$

Inserting this result in Eqs. (4.9) and (4.8), one finds for the renormalized two-point vertex function

$$\Gamma^{(2)}_{R\alpha\beta}=\delta^{\alpha\beta}r\left\{1-u\frac{n+2}{6}\left[\frac{r+g^2}{\mu^2}\right]^{-\epsilon/2}\left[1-\ln\left[\frac{g}{\mu+g}\right]-\frac{r-g^2}{r}\frac{1}{1+g/\sqrt{r}}-\frac{r+g^2}{r}\ln(1+\sqrt{r}/g)\right]\right\}, \quad (4.12)$$

which will be analyzed in terms of an effective exponent in the next section.

V. EFFECTIVE EXPONENT OF THE SUSCEPTIBILITY

One possibility for analyzing the susceptibility is the so-called effective exponent $\gamma_{\rm eff}$ defined by

$$\gamma_{\rm eff}=\frac{d\ln\chi^{-1}(r/\mu^2,g/\mu,u)}{d\ln r}. \quad (5.1)$$

Using the renormalization-group equation [see Eq. (2.11)]

$$\chi^{-1}\left[\frac{r}{\mu^2},\frac{g}{\mu},u\right]=l^2\chi^{-1}\left[\frac{r(l)}{\mu^2l^2},\frac{g}{\mu l},u(l)\right] \quad (5.2)$$

and the flow equation for the mass

$$r(l)=r\exp\left[\int_1^l\frac{d\rho}{\rho}\zeta_r(\rho)\right], \quad (5.3)$$

one finds from Eq. (4.12) to one-loop order

$$\gamma_{\rm eff}=1+\tfrac{1}{2}\zeta_r\left[\frac{g}{\mu l},u(l)\right]+\frac{d\ln\chi^{-1}(1,g/\mu l,u(l))}{d\ln r}, \quad (5.4)$$

where the matching condition

$$\frac{r(l)}{\mu^2 l^2}=1 \tag{5.5}$$

has been chosen. The ζ function for the mass is known from Sec. IV, Eq. (4.4),

$$\zeta_r(l)=\frac{n+2}{6}u(l)\left[1+\frac{g}{\mu l}\right]^{-1} . \tag{5.6}$$

Using the Eqs. (4.12) and (5.3), the logarithmic derivative in Eq. (5.4) is found to be

$$\frac{d\ln\chi^{-1}(1,g/\mu l,u(l))}{d\ln r}=-\tfrac{1}{2}u(l)\frac{n+2}{6}\left[1+\left[\frac{g}{\mu l}\right]^2\right]^{-\epsilon/2}\left\{\frac{\epsilon}{1+(\mu l/g)^2}\left[\frac{g}{\mu l}-\left[\frac{g}{\mu l}\right]^2\ln\left[1+\frac{\mu l}{g}\right]\right]-\left[\frac{g}{\mu l}+\frac{g/\mu l}{1+\mu l/g}-2\left[\frac{g}{\mu l}\right]^2\ln\left[1+\frac{\mu l}{g}\right]\right]\right\} . \tag{5.7}$$

In order to be consistent to first order in u and ϵ, the first term in the curly brackets, proportional to ϵ, has to be neglected. Then one finds, for $d=3$, (i.e., $\epsilon=1$), from Eq. (5.7)

$$\frac{d\ln\chi^{-1}(1,g/\mu l,u(l))}{d\ln r}=u(l)\frac{n+2}{12}\left[1+\left[\frac{\mu l}{g}\right]^2\right]^{-1/2}\left[1+\frac{1}{1+\mu l/g}-2\frac{g}{\mu l}\ln\left[1+\frac{\mu l}{g}\right]\right] . \tag{5.8}$$

Before analyzing the effective exponents in general, let us consider some limiting cases.

(i) In the limit of vanishing dipolar coupling $g\to 0$ one finds

$$\zeta_r(g=0,l)=\frac{n+2}{6}u(g=0,l) . \tag{5.9}$$

Since $u(g=0,l)$ is given by Eq. (3.20), the ζ function asymptotically ($l\to 0$) reduces to

$$\zeta_r^*(g=0)=\frac{n+2}{n+8}\epsilon . \tag{5.10}$$

Furthermore, since the logarithmic derivative of $\chi^{-1}(1,g/\mu l,u(l))$ vanishes linearly in g, one finds, in the case of vanishing dipolar coupling for the exponent ν of the correlation length,

$$\frac{1}{\nu}-2=-\frac{n+2}{n+8}\epsilon . \tag{5.11}$$

(ii) If the dipolar coupling is finite, classical behavior with logarithmic correction results. As is inferred from eq. (5.6) the ζ function becomes

$$\zeta_r(l\to 0)=\frac{n+2}{6}u(l)\frac{\mu l}{g}=\frac{n+2}{6}v(l) . \tag{5.12}$$

According to Eq. (5.8), the logarithmic derivative of $\chi^{-1}(1,g/\mu l,u(l))$ vanishes linearly in l. Combining this with Eq. (4.14), one finds, for the effective exponent of the susceptibility,

$$\gamma_{\text{eff}}(1\to 0)\to 1+\frac{n+2}{2(n+8)}\frac{1}{\ln\hat{l}-\ln l} , \tag{5.13}$$

which implies, for the susceptibility,

$$\chi^{-1}(r)=r|\ln r|^{-(n+2)/(n+8)} \tag{5.14}$$

in agreement with Refs. 2 and 3.

For the further analysis of the effective exponent γ_{eff} one is allowed to set $r=\mu^2 l^2$ in Eqs. (5.6) and (5.8) to one-loop order of Eqs. (5.3) and (5.5). The effective exponent resulting from combining Eqs. (5.4), (5.6), and (5.8) is shown in Fig. 2 for the initial value $u(1)=u_I$ ($n=1$) as a function of the scaling variable $y=r/g^2$ for a series of initial values of the dipolar coupling constant $g=k/10$ with $k=1,2,\ldots,5$ indicated in the graph. The curves start at the initial value $l=1$ of the flow parameter where the effective exponent is given by

$$\gamma_{\text{eff}}(l=1)=1+\tfrac{1}{6}$$

for small values of the dipolar coupling g and $n=1$. With increasing dipolar coupling the value of γ_{eff} at $l=1$ decreases, implying that the crossover from $\gamma_{\text{eff}}=1+\frac{1}{6}$ to the asymptotic dipolar behavior can be regarded as

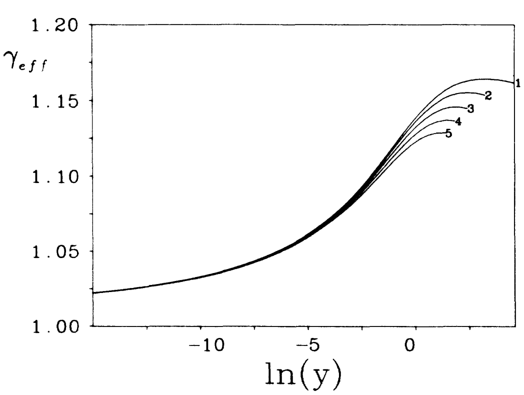

FIG. 2. Effective exponent of the susceptibility vs the scaling variable $\ln(y)=\ln(r/g^2)$ for the initial value $u(1)=u_I$ and a series of initial values of the dipolar coupling constant $g=k/10$ with $k=1,2,\ldots,5$ indicated in the graph.

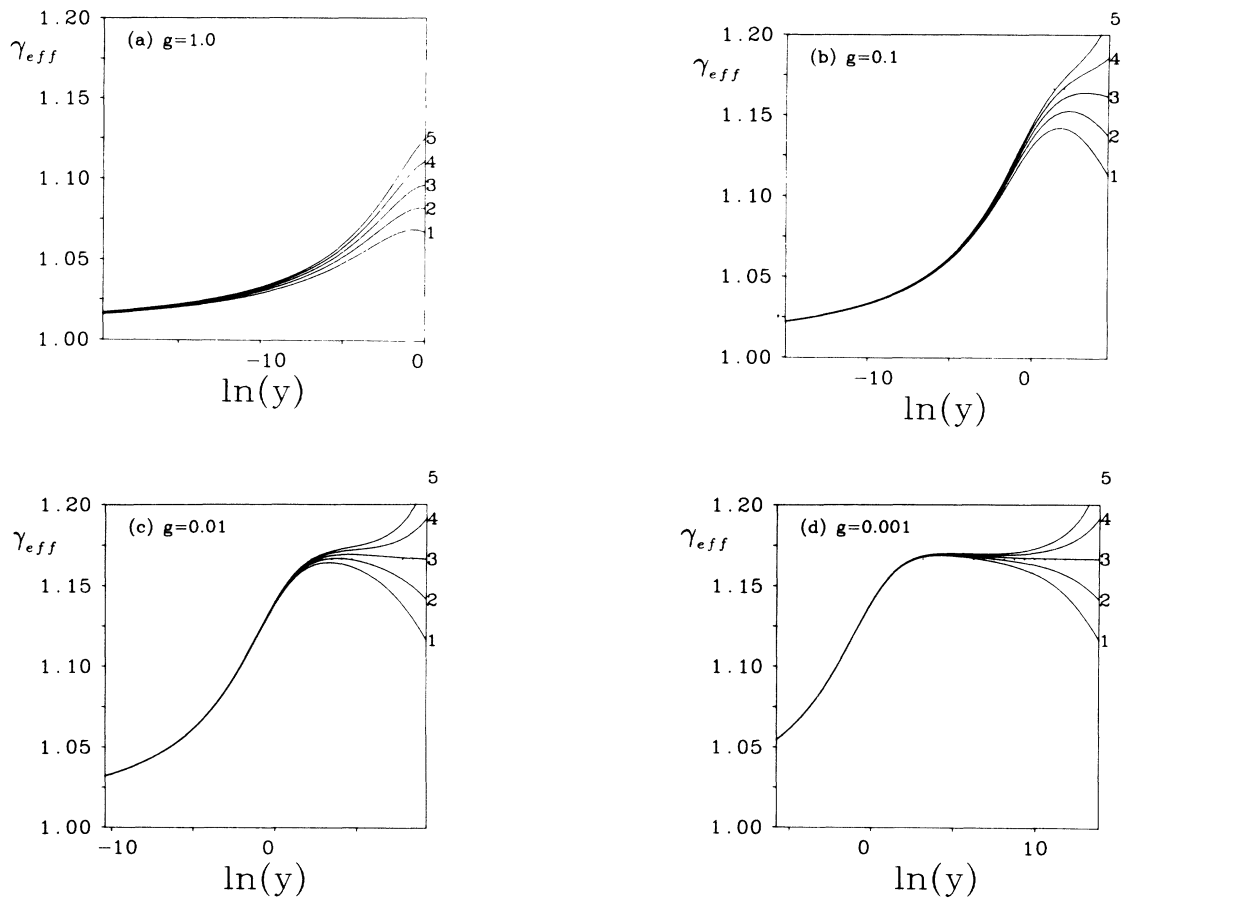

FIG. 3. Effective exponent of the susceptibility for (a) $g=1.0$, (b) $g=0.1$, (c) $g=0.01$, (d) $g=0.001$ vs the scaling variable $\ln(y)=\ln(r/g^2)$ for a series of initial values $u(1)=\frac{2}{3}+(k-3)/10$ with $k=1,2,\ldots 5$ indicated in the graphs.

universal for small dipolar couplings only. However, the reduction of the initial value $\gamma_{\text{eff}}(l=1)$ is essentially due to the fact that the effective coupling constant is not u but $u/(1+g)$. Thus, the initial value of γ_{eff} can be approximated by

$$\gamma_{\text{eff}}(l=1)\approx 1+\frac{1}{6(1+g)} .$$

Let us now study in what sense the crossover function for the effective exponent can be regarded as universal, i.e., how the choice of the initial values $u(1)$ effects the form of the crossover function. For this purpose $\gamma_{\text{eff}}(r,g)$ is displayed in Fig. 3 versus $\ln(y)=\ln(r/g^2)$ for fixed dipolar couplings (a) $g=1.0$, (b) $g=0.1$, (c) $g=0.01$, (d) $g=0.001$, and a series of initial values $u=\frac{2}{3}+(k-3)/10$ with $k=1-5$ indicated in the graphs. For weak dipolar systems [Figs. 3(c) and 3(d)], the crossover from the Ising value $\gamma_I=1+\frac{1}{6}$ to classical behavior with logarithmic correction is a universal feature of uniaxial dipolar ferromagnets. For stronger dipolar systems [Figs. 3(a) and 3(b)], the Ising fixed point is only weakly attractive for temperatures well separated from T_c, but the crossover is no longer universal, i.e., it depends on the initial value of the coupling constant u.

Now we turn to the comparison of our theoretical results with experiments on $LiTbF_4$, which is one of the most thoroughly studied uniaxial ferromagnets. In Fig. 4 the experimental data for γ_{eff} of Ref. 10, ranging over the interval $0.001\leq r\leq 20$ of the reduced temperature, are compared with the present theory where the

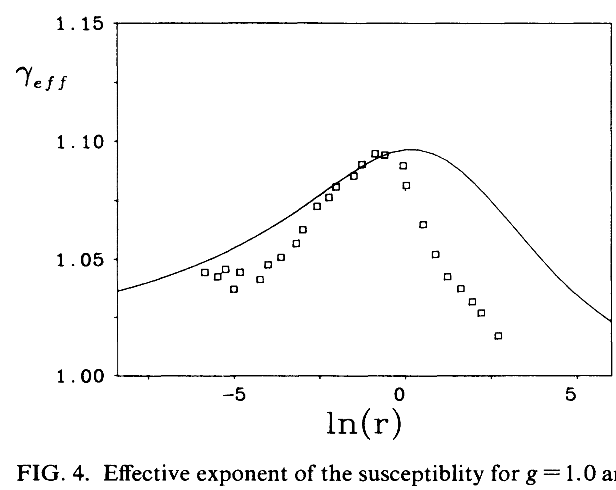

FIG. 4. Effective exponent of the susceptiblity for $g=1.0$ and $u=u_I$ vs $\ln(r)$. The momentum scale is chosen as $\mu=1$. Experimental data on $LiTbF_4$ from Ref. 10.

nonuniversal parameters in Eqs. (5.4), (5.6), and (5.8) have been chosen as $g=1$ and $u=u_H=\frac{1}{3}$. In the range $r\leq 1$ of the reduced temperature there is excellent agreement between theory and experiment. Especially, it is found that the observed crossover corresponds to the flow from the Ising fixed point to the uniaxial dipolar fixed point. For $r\geq 1$ the data tend to the mean-field value $\gamma_{\text{eff}}=1$ corresponding to a crossover to the Gaussian fixed point. This can be described only qualitatively within our critical theory by the limit $l\to\infty$. In as far as this crossover to the mean-field limit is concerned with critical behavior, it could be described by a method given by Lawrie.[24]

VI. SPECIFIC HEAT

In this section we study the crossover behavior of the specific heat, which is given by the cummulant (C)

$$C_B=\langle \tfrac{1}{2}S_0^2(p=0)\tfrac{1}{2}S_0^2(p=0)\rangle_C \ . \tag{6.1}$$

Above T_c this quantity is related directly to the vertex function $\Gamma_B^{(0,2)}$ with two S^2 insertions

$$C_B=\Gamma_B^{(0,2)}(p=0) \ . \tag{6.2}$$

This vertex function has to be renormalized additatively[26]

$$\Gamma_R^{(0,2)}=Z_{\phi^2}^2\Gamma_B^{(0,2)}-(Z_{\phi^2}^2\Gamma_B^{(0,2)})_{\text{sing}} \ , \tag{6.3}$$

where the renormalization constant Z_{ϕ^2} is identical to the mass renormalization[26]

$$Z_{\phi^2}=Z_r \ . \tag{6.4}$$

The RG equation for the specific heat is obtained from Eq. (6.3) by applying the differential operator $\mu(d/d\mu)|_0$ and noting that the bare quantities do not depend on the momentum scale μ

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\left[u,\frac{g}{\mu}\right]\frac{\partial}{\partial u}+\zeta_r\left[u,\frac{g}{\mu}\right]\left[2+r\frac{\partial}{\partial r}\right]+\zeta_g\left[u,\frac{g}{\mu}\right]g\frac{\partial}{\partial g}\right]C_R(r,g,u,\mu)=\mu^{-\epsilon}\hat{B}_{\phi^2}\left[u,\frac{g}{\mu}\right] . \tag{6.5}$$

The inhomogeneity of the RG equation (6.5) has the form

$$\hat{B}_{\phi^2}\left[u,\frac{g}{\mu}\right]=\mu^{\epsilon}Z_r^2\mu\frac{d}{d\mu}Z_r^{-2}[Z_r^2\Gamma_B^{(0,2)}(p=0)]_{\text{sing}} \ . \tag{6.6}$$

The RG equation (6.5) is solved by the method of characteristics. Confining ourselves to the one-loop order, we find for the renormalized specific heat

$$\hat{C}\left[\frac{r}{\mu^2},\frac{g}{\mu},u\right]=\mu^{\epsilon}C_R(r,g,u,\mu) \tag{6.7}$$

with

$$\hat{C}\left[\frac{r}{\mu^2},\frac{g}{\mu},u\right]=\exp\left[\int_1^{\rho}\frac{dl}{l}[2\zeta_r(l)-\epsilon]\right]\hat{C}\left[\frac{r(\rho)}{\mu^2\rho^2},\frac{g}{\mu\rho},u(\rho)\right]-\int_1^{\rho}\frac{dl}{l}\hat{B}_{\phi^2}(l)\exp\left[\int_1^{l}\frac{dl'}{l'}[2\zeta_r(l')-\epsilon]\right] . \tag{6.8}$$

Since there is no zero-loop contribution to the specific heat, the vertex function $\Gamma_B^{(0,2)}$ is given to one-loop order by

$$\Gamma_B^{(2,0)}(p,0)=-\tfrac{1}{2}\int_k G^{\alpha\beta}(r_0,g_0,\mathbf{k})G^{\alpha\beta}(r_0,g_0,\mathbf{k}) \ . \tag{6.9}$$

Since the integral in Eq. (6.9) already appeared in connection with the renormalization of the four-point coupling u, the additive renormalization is found to be

$$\hat{B}_{\phi^2}\left[\frac{g}{\mu}\right]=\frac{n}{2}S_d\frac{1}{1+g/\mu} \ . \tag{6.10}$$

For the renormalized specific heat one finds

$$\hat{C}\left[\frac{r}{\mu^2},\frac{g}{\mu},u\right]=\frac{n}{2}S_d\left[\frac{r+g^2}{\mu^2}\right]^{-\epsilon/2}\left[\frac{1}{2}+\ln\left[1+\frac{\mu}{g}\right]-\ln\left[1+\frac{\sqrt{r}}{g}\right]-\frac{1}{1+g/\sqrt{r}}\right] . \tag{6.11}$$

With the matching condition

$$\frac{r(l)}{\mu^2l^2}=1 \tag{6.12}$$

and use of the RG equation (6.8), we get, for the dimensionless renormalized specific heat, the expression

$$\hat{C}\left[\frac{r}{\mu^2},\frac{g}{\mu},u\right]=\exp\left[\int_1^{\rho}\frac{dl}{l}[2\zeta_r(l)-\epsilon]\right]\hat{C}\left[1,\frac{g}{\mu\rho},u(\rho)\right]-\int_1^{\rho}\frac{dl}{l}\hat{B}_{\phi^2}(l)\exp\left[\int_1^{l}\frac{dl'}{l'}[2\zeta_r(l')-\epsilon]\right] , \tag{6.13}$$

where the flow parameter ρ and the reduced temperature r are, according to the matching condition (6.12), related by

$$r=\mu^2\rho^2\exp\left[-\int_1^{\rho}\frac{dl}{l}\zeta_r(l)\right] . \tag{6.14}$$

$\zeta_r(l)$ is given by Eq. (5.6) and the flow of the four-point coupling $u(l)$ is determined by Eq. (3.19) for $\epsilon=1$. The additive renormalization in terms of the flow parameter reads

$$\hat{B}_{\phi^2}(l)=\frac{n}{2}\frac{1}{1+g/\mu l} . \tag{6.15}$$

Combining Eq. (6.11) with (6.12), the term $\hat{C}(1,g/\mu l,u(l))$ in Eq. (6.13) can be written as

$$\hat{C}\left[1,\frac{g}{\mu l},u(l)\right]=\frac{n}{2}\left[1+\left[\frac{g}{\mu l}\right]^2\right]^{-\epsilon/2}\times\left[\frac{1}{2}-\frac{1}{1+g/\mu l}\right] . \tag{6.16}$$

Here and in the following, the common prefactor S_d of $\hat{B}_{\phi^2}$ and $\hat{C}(1)$ has been omitted.

Let us study two important limiting cases. (i) With nonzero dipolar coupling g, one finds, in the asymptotic limit ($l\to 0$) for $d=3$ dimensions,

$$\zeta_r(l)\to\frac{n+2}{6}u(l)\frac{\mu l}{g} ,$$

$$u(l)\to\frac{6}{n+8}\frac{g/\mu l}{\ln\hat{l}-\ln l} ,$$

$$\hat{B}_{\phi^2}(l)\to\frac{n}{2}\frac{\mu l}{g} ,$$

$$\hat{C}\left[1,\frac{g}{\mu l},u(l)\right]\to\frac{n}{4}\frac{\mu l}{g} ,$$

and consequently the following asymptotic expression for the specific heat:

$$\hat{C}\left[\frac{r}{\mu^2},\frac{g}{\mu},u\right]=\frac{n(n+8)}{2(4-n)}\frac{u_H}{u}\times\left\{\left[1+\frac{\mu u}{g u_H}\ln\left[\frac{1}{\rho}\right]\right]^{(4-n)/(n+8)}-1\right\} . \tag{6.17}$$

One should note that our asymptotic result (6.17) is valid for any value of the strength of the dipolar coupling in the limit $l\to 0$. It reduces to the logarithmic behavior

$$\hat{C}\left[\frac{r}{\mu^2},\frac{g}{\mu},u\right]=\frac{n(n+8)}{2(4-n)}\frac{u_H}{u}\times\left\{\left[\frac{\mu u}{g u_H}\ln\left[\frac{1}{\rho}\right]\right]^{(4-n)/(n+8)}-1\right\} \tag{6.18}$$

found by purely asymptotic analysis in Refs. 2 and 3 only for very small values of the flow parameter ρ or equivalently for large values of the dipolar coupling constant g, i.e., in the strong dipolar limit. Equation (6.17), however, has the same structure as Eq. (42) of Ref. 1, which was obtained from a resummation of the leading logarithms in each order of perturbation theory. This confirms that the present field-theoretic approach is valid for both the strong and weak dipolar limits. As we will see later by comparison with experiments, Eq. (6.17) is closer to the data than the ultimate asymptotic logarithmic behavior (6.18). (ii) For vanishing dipolar coupling $g=0$, one finds in the limit $l\to 0$

$$\zeta_r(l)\to\frac{n+2}{n+8}\epsilon ,$$

$$\hat{B}_{\phi^2}(l)\to\frac{n}{2} ,$$

$$\hat{C}\left[1,\frac{g}{\mu l},u(l)\right]\to-\frac{n}{4} ,$$

and consequently

$$\hat{C}\left[\frac{r}{\mu^2},u\right]=A-Br^{-\alpha} , \tag{6.19}$$

where A and B are constants and the critical exponent of the specific heat is found to be

$$\alpha=\frac{2\zeta_r^*-\epsilon}{2-\zeta_r^*} \tag{6.20}$$

with $\zeta_r^*=[(n+2)/(n+8)]\epsilon$.

Now we turn to the numerical analysis of the specific heat, Eq. (6.13). In Figs. 5(a) and 5(b), the specific heat versus the scaling variable r/g^2 is displayed for the parameters $u=u_I$, $n=1$, $\mu=1$, $\epsilon=1$, and a series of dipolar couplings (a) $g=k/3$ and (b) $g=10^{-k}$ with $k=1,2,3$ indicated in the graphs. The solid lines represent the numerical solution of Eq. (6.13) and the dashed curves the asymptotic law Eq. (6.17). For systems with a weak dipolar coupling [see Fig. 5(b)], the behavior of the specific heat in the experimental accessible region looks more like a power-law behavior with a positive α [see Eq. (6.19)] than a logarithmic divergence. Closer to T_c there is a crossover from positive to negative α and ultimately to logarithmic behavior. For stronger dipolar systems [see Fig. 5(a)], the specific heat is nicely represented by Eq. (6.17) and this approximation is better the larger the dipolar coupling. The ultimate asymptotic behavior (6.18), even for strong dipolar systems, fails to describe the behavior of the specific heat in the experimental accessible region. The reason is that, due to the slow increase of the term $\ln(\rho)$, the asymptotic behavior is reached for small flow parameters only. In particular, this implies that the purely logarithmic asymptotic behavior is outside the experimental range. In order to substantiate this point, the asymptotic expression (6.18) for the specific heat is also displayed in Fig. 5(a) as the dot-dashed curves.

In Sec. V we have compared our theoretical results with measurements of the susceptibility in $LiTbF_4$, where we have chosen the parameters according to

$u = u_I$, $n = 1$, $g = 1$, $\mu = 1$, and $d = 3$. Now we compare with specific-heat experiments in $LiTbF_4$ using the same set of parameters as in Sec. V. The resulting curve is exhibited in Fig. 6 versus the reduced temperature r (solid curve). For comparison we have also plotted Eq. (42) of Ref. 1 (dot-dashed curve)

$$C^+ = (A/b^z)\{[1 + b\ln(a/r)]^z - 1\} , \qquad (6.21)$$

where z equals $\frac{1}{3}$ for $n = 1$ and the parameters in Eq. (6.21) have been determined by a least-squares fit in the temperature range $10^{-3} \le r \le 10^{-2}$. According to Ref. 5, one finds $A/R = 0.4394$, $b = 2.425$, and $a = 0.2084$, where $R = 8.3144\ \mathrm{J\,mol^{-1}\,K^{-1}}$. Equation (6.21) gives quite a good description in the temperature range given above, while it falls below the data for reduced temperature $r \ge 10^{-2}$. In contrast, the present theory, Eq. (6.13), gives a satisfactory description in the whole temperature range $10^{-3.5} \le r \le 10^{-1}$. Furthermore, the main advantage of the theoretical results of this section is that (besides of a nonuniversal scale for the amplitude of the specific heat) there are no adjustable parameters. More precisely spoken, if one makes a choice of the set of nonuniversal parameters by comparing, for instance, the theory with susceptibility measurements, this set of parameters has to be maintained for all further quantities of the same substance. This we have done in comparing our theory with measurements of the specific heat.

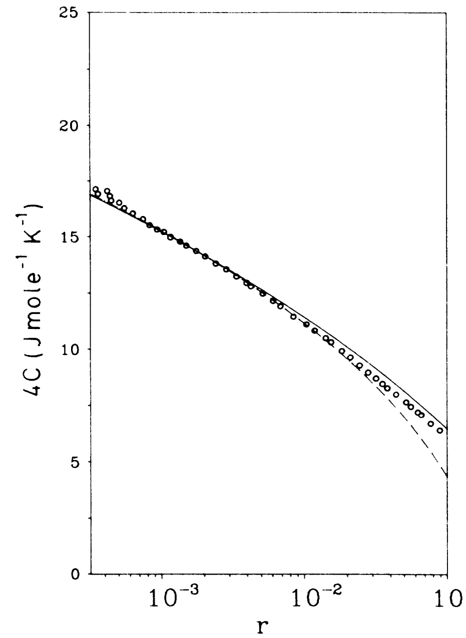

FIG. 6. Specific heat of the uniaxial dipolar ferromagnet vs the rescued temperature r. The solid curve represents Eq. (6.13) with the parameters $g = 1$ and $u = u_I$. The dot-dashed curve represents Eq. (6.21) as described in the text. The data points are taken from Ref. 5.

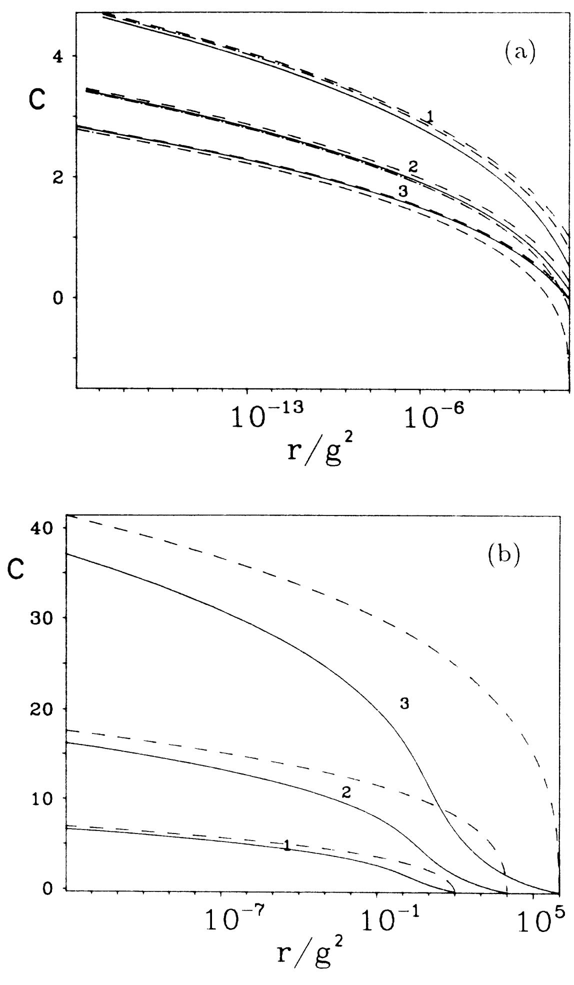

FIG. 5. (a) Specific heat of the uniaxial dipolar ferromagnet vs the scaling variable r/g^2. The solid lines represent Eq. (6.13) with the parameters $u = u_I$ and $g = k/3$, where $k = 1,2,3$ is indicated in the graph. The dashed curves represent Eq. (6.17) and the dot-dashed curves Eq. (6.18). (b) Specific heat of the uniaxial dipolar ferromagnet vs the scaling variable r/g^2. The solid lines represent Eq. (6.13) with the parameters $u = u_I$ and $g = 10^{-k}$, where $k = 1,2,3$ is indicated in the graph. The dashed curves represent Eq. (6.17).

VII. SUMMARY

In this paper we have developed a field theory for the crossover in uniaxial ferromagnets mediated by the dipolar interaction. In particular, we succeeded in describing the crossover from Ising to classical behavior with logarithmic corrections, which is accompanied by a change in the upper critical dimension. The theory is applicable for both weak and strong dipolar systems. Within this theory we were able to give a quantitative interpretation of susceptibility and specific-heat measurements on $LiTbF_4$, where solely the strength of the dipolar coupling g and the initial value of the four-point coupling u entered as adjustable parameters. These parameters are, however, nonuniversal quantities depending on the substance under consideration. The important point is that this set of parameters is adjusted by comparing the theory with measurements of one particular correlation function, for instance, the susceptibility, and has to be maintained for all other physical quantities.

Let us compare the present crossover to the situation in isotropic dipolar ferromagnets. There one finds a minimum in the effective critical exponent γ_{eff} of the order-parameter susceptibility. This minimum appears in the crossover region between isotropic and dipolar critical behavior, and has to do with the split up of the order-parameter components in two transverse and one uncritical longitudinal component. The maximum of Fig. 4 is of quite different an origin. It is located in the isotropic critical region and appears because the isotropic exponent is higher than the asymptotic uniaxial and the classical at the border of the critical region.

A particular virtue of the present approach is the capability to describe the crossover between fixed points of different upper critical dimensionalities. The method introduced here in the context of uniaxial dipolar magnets has a variety of potential applications.

ACKNOWLEDGMENTS

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APPENDIX: HARTREE INTEGRAL

In Sec. IV one has to calculate the parameter integral

$$J(r,g)=\int_0^1 dx[x(1-x)]^{-1/2}[g^2x+r(1-x)]^{1-\epsilon/2} . \tag{A1}$$

In order to give the correct result in both limits $r\to 0$ and $g\to 0$, it is necessary to extract the factor $(r+g^2)^{1-\epsilon/2}$ from the integral without making an ϵ expansion of this term. Then one finds

$$J(r,g)=(r+g^2)^{1-\epsilon/2}\int_0^1 dx[x(1-x)]^{-1/2}\left[\frac{g^2}{r+g^2}x+\frac{r}{r+g^2}(1-x)\right]\left[1-\frac{\epsilon}{2}\ln\left[\frac{g^2}{r+g^2}x+\frac{r}{r+g^2}(1-x)\right]\right] . \tag{A2}$$

The remaining integrals can be evaluated analytically with the result

$$J(r,g)=\frac{\pi}{2}(r+g^2)^{1-\epsilon/2}\left[1+\frac{\epsilon}{2}\frac{r-g^2}{r+g^2}+\epsilon\ln 2-\frac{\epsilon}{2}\ln\left[\frac{g^2}{r+g^2}\right]-\frac{\epsilon}{\pi}\frac{g^2}{r+g^2}I^{(2)}(r/g^2)-\frac{\epsilon}{\pi}\frac{r}{r+g^2}I^{(1)}(r/g^2)\right] , \tag{A3}$$

where

$$I^{(1)}(r/g^2)=\int_0^1 dx\, x^{-1/2}(1-x)^{1/2}\ln\left[1+\frac{r}{g^2}\frac{1-x}{x}\right]=\pi\left[\ln\left[1+\frac{\sqrt{r}}{g}\right]+\frac{1}{1+g/\sqrt{r}}\right] , \tag{A4a}$$

$$I^{(2)}(r/g^2)=\int_0^1 dx\, x^{1/2}(1-x)^{-1/2}\ln\left[1+\frac{r}{g^2}\frac{1-x}{x}\right]=\pi\left[\ln\left[1+\frac{\sqrt{r}}{g}\right]-\frac{1}{1+g/\sqrt{r}}\right] . \tag{A4b}$$

Hence, by additionally subtracting the counter term $J_{\text{CT}}=J(r=0,g)$, one gets for $\hat{J}(r,g)=J(r,g)-J(r=0,g)$

$$\hat{J}(r,g)=\frac{\pi}{2}r(r+g^2)^{-\epsilon/2}\left[1+\frac{\epsilon}{2}+\epsilon\ln 2-\frac{\epsilon}{2}\ln\left[\frac{g^2}{r+g^2}\right]-\epsilon\frac{r-g^2}{r}\frac{1}{1+g/\sqrt{r}}-\epsilon\frac{r+g^2}{r}\ln(1+\sqrt{r}/g)\right] . \tag{A5}$$

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[1]A. I. Larkin and D. E. Khmelnitskii, Zh. Eksp. Teor. Fiz. **56**, 2087 [Sov. Phys.—JETP **29**, 1123 (1969)].

[2]A. Aharony, Phys. Rev. B **8**, 3363 (1973); Phys. Lett. **44A**, 313 (1973).

[3]E. Brezin and J. Zinn-Justin, Phys. Rev. B **13**, 251 (1976).

[4]J. Kötzler and W. Scheithe, Solid State Commun. **12**, 643 (1973); J. Kötzler and G. Eiselt, Phys. Lett. **58**, 69 (1976).

[5]G. Ahlers, A. Kornblit, and H. J. Guggenheim, Phys. Rev. Lett. **34**, 1227 (1975).

[6]J. Als-Nielsen, Phys. Rev. Lett. **37**, 1161 (1976).

[7]J. A. Griffin, J. D. Lister, and A. Linz, Phys. Rev. Lett. **38**, 251 (1977).

[8]R. Frowein, J. Kötzler, and W. Assmus, Phys. Rev. Lett. **42**, 739 (1979).

[9]P. Beauvillain, C. Chappert, and I. Laursen, J. Phys. C **13**, 1481 (1980).

[10]R. Frowein, J. Kötzler, B. Schaub, and H. G. Schuster, Phys. Rev. B **25**, 4905 (1982).

[11]R. Frowein and J. Kötzler, Z. Phys. B **25**, 279 (1976).

[12]J. Brinkmann, R. Courths, and H. J. Guggenheim, Phys. Rev. Lett. **19**, 1286 (1978).

[13]A. Aharony and M. E. Fisher, Phys. Rev. B **8**, 3323 (1973); Phys. Rev. Lett. **30**, 559 (1973).

[14]A. Aharony, Phys. Rev. B **8**, 3342 (1973); **8**, 3358 (1973).

[15]A. Aharony and A. D. Bruce, Phys. Rev. B **10**, 2973 (1974); **10**, 2078 (1974).

[16]T. Nattermann and J. Trimper, J. Phys. C **9**, 2589 (1976).
[17]A. D. Bruce, J. M. Kosterlitz, and D. R. Nelson, J. Phys. C **9**, 825 (1976).
[18]A. D. Bruce, J. Phys. C **10**, 419 (1977).
[19]E. Frey and F. Schwabl, J. Phys. (Paris) Colloq. **49**, C8-1569 (1988); (unpublished).
[20]D. Bailin and A. Love, J. Phys. C **10**, 3633 (1977).
[21]R. Folk and G. Moser, Phys. Lett. A **120**, 39 (1987).
[22]D. R. Nelson and J. Rudnick, Phys. Rev. Lett. **35**, 178 (1975); D. R. Nelson and E. Domany, Phys. Rev. B **13**, 236 (1976); J. Rudnick and D. R. Nelson, *ibid.* **13**, 2208 (1976).
[23]D. J. Amit and Y. Y. Goldschmidt, Ann. Phys. (N.Y.) **114**, 356 (1978).
[24]I. D. Lawrie, J. Phys. A **9**, 961 (1976).
[25]G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972).
[26]D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena,* 2nd ed. (World Scientific, Singapore, 1984).