Effective theories of the fractional quantum Hall effect: Hierarchy construction

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The effective theories for the hierarchical fractional quantum Hall effect (FQHE) are proposed. We obtain the quantum numbers of the quasiparticles and the structure of the edge excitations for the general hierarchical FQHE state. It is shown that at the filling fractions $\nu = k/(2km \pm 1)$ the Jain states and the hierarchical FQHE states give rise to the same quasiparticles and edge excitations and have the same effective theories (in the dual form). This suggests that these FQHE states are equivalent despite having been obtained from two different schemes.

I. INTRODUCTION

The fractional quantum Hall effect (FQHE) at general filling fractions is explained by the hierarchy schemes. These schemes allow one to construct various incompressible states of electrons in the magnetic field. Today several different construction schemes for the FQHE are known. However, it is not clear yet whether they lead to the same FQH state or not for a given filling fraction.

There are mainly two different ways to construct the FQH states. The first one is the hierarchy construction. This construction was first proposed in Ref. 1 and later developed in Refs. 2 and 3. A closely related approach that uses the particle-hole duality was developed in Refs. 4–6. A different scheme of the construction was proposed recently by Jain in Refs. 7–9.

The first approach uses the assumption that as we change the filling fraction the added quasiparticles in the FQH states can "condense" and form a new incompressible state. This new state supports different kinds of quasiparticles. The new quasiparticles also can condense and form the higher-level FQH state. In the second approach due to Jain, one deals directly with the fermion "condensates." In this approach each electron is represented as a sum of several fictitious fermions. The fermions of each type are assumed to be in their own incompressible state [which is an integral quantum Hall (IQH) state]. Some arguments were given in Ref. 9 to demonstrate the stability of this kind of the electron states.

However, it is not clear whether these two schemes lead to the same FWH states or not for a given filling fraction. It was argued in Ref. 10–12 that the filling fraction is not enough to characterize the FQH state. The FQH states are classified by their topological orders Ref. 13. The latter can be characterized by the allowed quantum numbers of the quasiparticles (i.e., their charges and statistics) in the given FQH state.

A powerful way to investigate the topological order in the FQH states is to construct the effective theories of the FQH states. The effective theory for the Laughlin state with the filling fraction $1/p$ was first proposed in Refs. 14–16. It was proven to be a useful tool for understanding the qualitative properties of these states.

In our previous paper (Ref. 12) the effective theories of the FQHE states proposed by Jain were constructed. We were able to find the quantum numbers of the quasiparticles and the structure of the edge states.

The purpose of the present paper is to construct the effective theories for the hierarchy FQH states. The microscopic wave functions for these states were constructed in Refs. 1–6. Using the effective theory we can calculate the allowed quantum numbers of the quasiparticles in the general hierarchy states. These results enable us to compare the hierarchy FQHE states with the Jain states. We find that for the case of the filling fraction $\nu = k/(mk \pm 1)$ (m is an even integer) the quantum numbers of the quasiparticles in these states coincide. This suggests that these hierarchy FQHE states are equivalent to the corresponding Jain states. We also discuss the particle-hole duality in the framework of the effective theory and the structure of edge excitations.

The plan of the paper is the following. In Sec. II we construct the effective theory for the second-level hierarchy states with a filling fraction $\nu = p_2/(p_1 p_2 \pm 1)$. (Here p_2 is an even integer, p_1 is an odd integer). The quantum numbers of the quasiparticles in these states are obtained. These quantum numbers are compared with the quantum numbers of the quasiparticles in the Jain state. We demonstrate the connection between the effective theory and microscopic wave functions. It is argued that the microscopic construction due to Halperin (Refs. 2 and 3) gives rise to the same quantum numbers of the quasiparticles as our effective theory does. The particle-hole duality is also discussed. In Sec. III we construct the effective theory and calculate the quantum numbers of the quasiparticles for the general hierarchy states at an arbitrary level of the hierarchy construction. We also compare the Jain states and the hierarchical states for the case of the filling fractions $\nu = k/(kp \pm 1)$. In Sec. IV we discuss the dual theory and find the structure of the edge states for the hierarchy construction. The main results are summarized in Sec. V.

II. THE CONSTRUCTION OF THE HIERARCHICAL FQH STATES

In this section we study the second level of the hierarchy construction in the framework of the effective theory. Recall that the microscopic wave functions for the second-level hierarchy states were constructed in terms of the quasiholes in the Laughlin state:[2,3]

$$\psi(z_1,\ldots,z_n)=\prod_{k<1}^{N_h}(\bar{z}_k-\bar{z}_l)^{1/p_1}\prod_{k<l}^{N_h}(\bar{z}_k-\bar{z}_l)^{p_2}\prod_{k=1}^{k=N_h}\exp(-q_h|z_k|^2/4l_0^2)\ , \tag{1}$$

where $z_k=x_k+iy_k$ are the complex coordinates of the quasiholes, p_2 is an even integer, and $l_0=\sqrt{Be/hc}$ is the magnetic length. It is clear that in this construction we have bound an even number k of the flux quanta to each quasihole of the Laughlin state. The microscopic wave function (1) describes the hierarchy state that arises from the "condensation" of the quasiholes (bounded with p_2 units of flux). Similarly it is possible to write down the wave function of the hierarchy state obtained from the "condensation" of the quasiparticles in the Laughlin state. Now let us construct the effective theory for the above hierarchy states. Our starting point will be the effective theory for the Laughlin state with a filling fraction $1/p$.

The Ginsburg-Landau Lagrangian for the Laughlin state can be written as

$$\mathcal{L}=\Phi_e^\dagger i(\partial_0+ia_0-ieA_0)\Phi_e+\frac{1}{2M}\Phi_e^\dagger(\partial_i+ia_i-ieA_i)^2\Phi_e+\frac{1}{4\pi}p\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda+\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu a_\lambda-V(\Phi_e)\ . \tag{2}$$

The order parameter Φ_e describes the electrons that form the incompressible state. It is easy to see that integrating out the auxiliary gauge field c_μ we return to the standard Ginsburg-Landau action of Refs. 15 and 16.

The equations of motion for the Lagrangian (2) have the vortexlike solutions, with the center at $r=0$ and asymptotic behavior for $r\to\infty$:

$$\Phi_e(r,\phi)=\sqrt{n_e}\exp(\pm i\phi)\ ,\qquad a_\phi(r,\phi)=\pm\frac{\hat{\phi}}{er},\quad a_0=0\ . \tag{3}$$

The vortices correspond to the excitations with the nonzero winding numbers in the Φ_e and a fields. They are the quasiparticles in the Laughlin state.

There are two ways to take into account these solutions in the quantum theory. First, we can consider the contributions to the partition function from the different topological sectors in the Φ_e and a fields in which the Φ_e field has different winding numbers. The second possibility is to restrict ourselves only to the trivial topological sectors of the fields Φ_e and a_μ. The vortex can be unwinded with use of a singular gauge transformation. In this case, in order to take into account the vortex contribution, we must introduce a new quantum field Φ_q to keep track of the singularities. This field describes the quasiparticles in the Laughlin state. We must now find the effective Lagrangian for the field Φ_q in the limit when the vortices can be considered as the pointlike particles. This problem was considered in Refs. 17–19 and the following effective Lagrangian was obtained:

$$\mathcal{L}=\Phi_e^\dagger i(\partial_0+ia_0-ieA_0)\Phi_e+\frac{1}{2M}\Phi_e^\dagger(\partial_i+ia_i-ieA_i)^2\Phi_e+p\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda+\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu a_\lambda+i\Phi_q^\dagger(\partial_0+ic_0)\Phi_q+\frac{1}{2M}\Phi_q^\dagger(\partial_i+ic)^2\Phi_q-V(\Phi_e)\ . \tag{4}$$

In order to check the self-consistency of the two different but equivalent descriptions, (2) and (4), of the Laughlin state we would like to show that the field Φ_q has the same quantum numbers (the charge and statistics) as the vortex (3). The equations of motion that follow from the Lagrangian (4) imply that

$$Q_a=\int(n_e-\bar{n}_e)d^2x=-\frac{\Phi_c}{2\pi}\ , \tag{5}$$

$$Q_c=\frac{1}{2\pi}=-\frac{(p\Phi_c+\Phi_a)}{2\pi}\ , \tag{6}$$

$$\Phi_a=\int(a-\bar{a})d^2x=0\ . \tag{7}$$

Equation (7) is the consequence of our assumption that field a is unwinded. The integer l is the number of the quasiparticles, n_e is the electron density, and $\Phi_c=\int(c-\bar{c})d^2x$ is the flux of the auxiliary field c. Note that Φ_c is not an integer since the quasiparticles do not condense in the state under consideration.

Using Eqs. (5)–(7) it is straightforward to check that the quasiparticles described by the field Φ_q have the quantum numbers

$$Q_{\rm el}=\frac{el}{p},\quad \theta=\frac{l^2\pi}{p}\ . \tag{8}$$

These are just the quantum numbers of the quasiparticles in the Laughlin state. Hence it is consistent to use (4) to describe the Laughlin state.

Note that for a vortex with winding number l, the equations of motion that follow from the Lagrangian (2) imply

$$\Phi_a=2\pi l\ ,\qquad Q_a=\frac{-\Phi_c}{2\pi},\qquad p\Phi_c=-\Phi_a\ . \tag{9}$$

Solving these equations we obtain the quantum numbers (8) for the vortices. Hence the winding number of the vortices in the effective theory (2) is just the number of quasiparticles in the effective theory (4) and the Φ_q particles in (4) are just the vortices (3) in the theory (2). The two effective theories are equivalent and both describe the same Laughlin state. It is possible to prove that the partition functions calculated using the Lagrangians (2) and (4) coincide in the limit when the vortices can be represented as pointlike particles (see, e.g., Refs. 19, 17, and 18 for the detailed calculation). The Lagrangian (4) will be the starting point for our future discussion of the hierarchy state.

From the microscopic construction described above it is clear that in orer to construct a hierarchy state we must bind an even number of flux quanta to each quasiparticle. Note the close connection between this construction and the Jain construction that was used to construct the FQH states with the filling fractions $\nu=k/(mk+1)$ (here m is an even integer.) The main difference between these two approaches is that in the Jain case we construct composite fermions, binding an even number of flux quanta to each electron. In the hierarchy construction we construct "composite" quasiparticles, binding an even number of flux quanta to each quasiparticle. This is the origin of the second term in the wave function (1). We refer to Refs. 7 and 20 for a more detailed discussion of this procedure. In order to construct the hierarchy FQHE state we assume that these "composite" objects (that have now the anyon statistics) form an incompressible state.

The binding of the even number of flux quanta to the quasiparticles can be achieved by introducing a new gauge field b_μ. This leads to the following effective Ginsburg-Landau Lagrangian for the hierachy state

$$\begin{aligned}\mathcal{L}=&\, i\Phi_e^\dagger(\partial_0+ia_0-ieA_0)\Phi_e\\&+\frac{1}{2M}\Phi_e^\dagger(\partial_i+ia_i-ieA_i)^2\Phi_e+\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu c_\lambda\\&+i\Phi_q^\dagger(\partial_0+ic_0+ib_0)\Phi_q+\frac{1}{2M_q}\Phi_q^\dagger(\partial_i+ic_i+ib_i)^2\Phi_q\\&+\frac{1}{4\pi}p_1\epsilon^{\mu\nu\lambda}c_\nu\partial_\mu c_\lambda-\frac{1}{4\pi}\frac{1}{p_2}\epsilon^{\mu\nu\lambda}b_\nu\partial_\mu b_\lambda-V(\Phi_e)\,.\end{aligned} \tag{10}$$

In the mean-field theory the hierarchy FQH state is described by the condensates of Φ_e and Φ_q, i.e., $\langle\Phi_e\rangle=$const and $\langle\Phi_q\rangle=$const. Using the equations of motion

$$\begin{aligned}&b+c=a-B=0\,,\\&n_e=\nu B=-\frac{c}{2\pi}\,,\\&n_q=-\frac{a+p_1c}{2\pi}=\frac{1}{p_2}b\,,\end{aligned} \tag{11}$$

we obtain the filling fraction

$$\nu=\frac{p_2}{p_1p_2-1}=\frac{1}{p_1-1/p_2}\,. \tag{12}$$

The condensate densities are given by the equations

$$n_e=\nu B,\ n_q=\frac{\nu}{p_2}B\,. \tag{13}$$

Here $a=\epsilon_{ij}\partial_i a_j$, $b=\epsilon_{ij}\partial_i b_j$, and $c=\epsilon_{ij}\partial_i c_j$ are the auxiliary gauge-field strengths and B is the magnetic-field strength.

This Lagrangian (10) was first obtained in Ref. 19. These authors rederive the results of Halperin[2,3] for the hierarchy states using the effective theory. However, they did not discuss the quantum numbers of general quasiparticles and topological orders in FQH states.

Let us consider the quantum numbers of the quasiparticles in this model. The quasiparticles are vortices in the Φ_e and Φ_q fields. Using equations of motion it is straightforward to obtain

$$Q^a=-\frac{\Phi^c}{2\pi}\,, \tag{14}$$

$$Q^b=\frac{1}{p_2}\Phi^b\,, \tag{15}$$

$$Q^c=-\frac{p_1\Phi^c+\Phi^a}{2\pi}\,, \tag{16}$$

$$Q^b=Q^c,\ \Phi^b+\Phi^c=l_2 2\pi,\ \Phi_a=l_1 2\pi\,. \tag{17}$$

Here

$$\begin{aligned}&\Phi^b=\int(b-\bar{b})d^2x,\ \Phi^c=\int(c-\bar{c})d^2x\,,\\&\Phi^a=\int(a-\bar{a})d^2x\,,\end{aligned} \tag{18}$$

$$Q^b=Q^c=\int(n_q-\bar{n}_q)d^2x,\ Q^a=\int(n_e-\bar{n}_e)d^2x\,, \tag{19}$$

and, for example, $\bar{a}$ means the value of the field a in the ground state. The integers l_1 and l_2 are the winding numbers of the vortices in Φ_q and Φ_e fields. We denote the electron and the quasiparticle densities by n_e and n_q, respectively. Note that in this description of the hierarchical state, only the "condensed" quasiparticles in the Laughlin state are described by the order parameter Φ_q. The integer l_1 is the winding number of vortices in the Φ_e field. These vortices describe the quasiparticles that are not in the "condensate." Their density in the ground state is zero.

Solving Eqs. (14)–(17) we obtain the electric charges

$$Q_{\rm el}=\frac{l_2+p_2l_1}{p_1p_2-1} \tag{20}$$

and the statistics

$$\theta=Q^a\Phi^a+Q^b\Phi^b+Q^c\Phi_c=\frac{p_2l_1^2+p_1l_2^2+2l_1l_2}{p_1p_2-1}\pi \tag{21}$$

of the quasiparticles in the effective theory. The quasiparticles are labeled by two integers l_1 and l_2.

Note that the integers p_1 and p_2 can be both positive and negative. The hierarchy FQH states with $p_2>0$ are obtained by the quasielectron condensation. While the hierarchy FQH states with $p_2<0$ are obtained by the quasihole condensation. From Eq. (21) we see that if there is quasielectron condensation, all quasielectrons have the same sign of statistics θ. If the quasiholes condense, then the quasiparticles may have different signs of statistics.

We have two "fundamental" quasiparticles. By fundamental quasiparticles we mean here the quasiparticles with quantum numbers $l_2=1$, $l_1=0$ or $l_1=1$, $l_2=0$. The quasiparticle with $l_2=1$, $l_1=0$ has the charge and statistics

$$Q_{el}=\frac{1}{p_1p_2-1}e,\ \theta=\frac{p_1}{p_1p_2-1}\pi\ . \tag{22}$$

This kind of quasiparticle was first studied by Halperin.[2,3] Our result (22) agrees with Halperin's results. The second fundamental quasiparticle with $l_2=0$, $l_1=1$ has the quantum numbers

$$Q_{el}=\frac{p_2}{p_1p_2-1}e,\ \theta=\frac{p_2}{p_1p_2-1}\pi\ . \tag{23}$$

This quasiparticle is just the one induced by adiabatic turning on the unit magnetic flux. The result (23) can also be derived from the microscopic calculation. Note that for $p_2\to\infty$ (this limit corresponds to the decoupling of the field b, i.e., we return to the Laughlin state) the quantum numbers of this quasiparticle become the quantum numbers of the quasiparticle in the original Laughlin state.

The effective theory (10) also allows one to consider the particle-hole duality. For example, the state conjugated to the $\nu=1/(k+1)$ Laughlin state (k is an even integer) can be constructed from the $\nu=1$ integer quantum Hall state by binding to its quasiholes k units of flux. Such states have filling fractions $\nu=1/(1+1/k)$ and contain two fundamental quasiparticles. The first one has quantum numbers $Q_{el}=-[1/(k+1)]e$, $\theta=-[1/(k+1)]\pi$, i.e., the same quantum numbers as the quasiparticle in the Laughlin state. The second one has electric charge $Q_{el}=[k/(k+1)]e$ and statistics $\theta=[k/(k+1)]\pi$.

Let us now compare the $\nu=\frac{2}{5}$ states constructed in this chapter with the ones obtained by Jain (see Refs. 7, 8, 9, and 12). In the Jain state the quasiparticles are labeled by two integers l_1 and l_2 and have quantum numbers[12]

$$Q_{el}=\frac{l_1+l_2}{5}e,\ \theta=[l_1^2+l_2^2-\tfrac{2}{5}(l_1+l_2)^2]\pi\ . \tag{24}$$

The quasiparticles in the hierarchy state with a filling fraction $\nu=\frac{2}{5}=1/(3-\frac{1}{2})$ are also labeled by two integers n_1 and n_2. They have the quantum numbers

$$Q_{el}=\frac{n_1+2n_2}{5}e,\ \theta=\frac{2n_1^2+3n_2^2+2n_1n_2}{5}\pi\ . \tag{25}$$

It is easy to see that (24) and (25) coincide if we identify $l_1=n_1+n_2$ and $l_2=n_2$. Thus at filling fraction $\frac{2}{5}$ the quantum numbers of the quasiparticles in both the Jain state and the hierarchical states are the same. This suggests that the two states have the same topological order and are equivalent. At the filling fraction $\nu=\frac{2}{5}$ both the hierarchical state and the Jain state describe the same fractional quantum Hall state, despite these two states having different electronic wave functions.

Above the charges and statistics of the quasiparticles in the hierarchy states were calculated from the effective theory. Now we would like to argue that the same results can be obtained from the microscopic wave functions. The electron ground-state wave function can be derived from the pseudo-wave-function (1) of the quasiholes using the "fractional-statistics transformation" (see, e.g., Refs. 2, 3, and 21)

$$\psi(\xi_1,\ldots,\xi_n)\sim\prod_n^{N_e}(\xi_n-\xi_0)\prod_{i<j}^{N_e}(\xi_i-\xi_j)^p\int\prod_{i<j}^{N_h}|\bar z_i-\bar z_j|^{2/p_1}\prod_{i<j}^{N_h}(\bar z_i-\bar z_j)^{p_2}\prod_{p,q}^{N_e,N_h}(z_p-\xi_q)$$
$$\times\prod_i^{N_h}\exp\left[-\frac{1}{2p_1}|z_i|^2\right]\prod_j^{N_e}\exp(-\tfrac{1}{4}|\xi_j|^2)d^2z_i\ . \tag{26}$$

Here ξ_i and z_i are the coordinates of the electrons and the quasiholes, respectively. The number of the quasiholes is given by

$$N_h=\frac{1}{p_2}N_e\ . \tag{27}$$

The filling fraction of the state (26) can be easily found from the angular-momentum argument.[4] The angular momentum L and the electron number N_e are connected by the equation $L=N_e(N_e-1)/2\nu$. The angular momentum can be determined as $L=[N_e(N_e-1)/2]p_1-[N_h(N_h-1)/2]p_2+N_eN_h$. Taking the limit $N_e,N_h\to\infty$ and using the condition (27) we find that the filling fraction is given by Eq. (12) (where we make a substitution $p_2\to-p_2$ since we consider the quasihole condensation).

In the microscopic theory there are two different types of the quasiparticles in the state described by the wave function (26). The quasiholes of the first type are described by the following electron wave function:

$$\psi(\xi_1,\ldots,\xi_n;\xi_0)\sim\prod_n^{N_e}(\xi_n-\xi_0)\prod_{i<j}^{N_e}(\xi_i-\xi_j)^{p_1}\int\prod_{i<j}^{N_h}|z_i-z_j|^{2/p_1}\prod_{i<j}^{N_h}(\bar z_i-\bar z_j)^{p_2}\prod_{p,q}^{N_e,N_h}(z_p-\xi_q)$$
$$\times\prod_i^{N_e}\exp\left[-\frac{1}{2p_1}|z_i|^2\right]\prod_j\exp(-\tfrac{1}{4}|\xi_j|^2)d^2z_i\ . \tag{28}$$

Here ξ_0 is the position of the quasihole. The quasiholes of the second type are described by the wave function

$$\psi_2(\xi_1,\ldots,\xi_n;\xi_0)\sim\prod_{i<j}^{N_e}(\xi_i-\xi_j)^{p_1}\int\prod_{i<j}^{N_h}|z_i-z_j|^{2/p_2}(\bar z_i-\bar z_j)^{p_2}(z_i-\xi_j)(\bar z_i-\bar\xi_0)\prod_i^{N_h}\exp\left[-\frac{1}{2p_1}|z_i|^2\right]\prod_j^{N_e}\exp(-\tfrac{1}{4})\xi_j|^2)d^2z_i\ . \tag{29}$$

The analogous wave functions can be written in the case of the quasiparticle, although they look more cumbersome.

The charge and statistics of the quasiparticle described by the wave function (28) can be found in the same way as it was done for the quasiparticles in the Laughlin state in Ref. 22. Adiabatically moving a quasiparticle rough the circle induces the phase which is proportional to the enclosed flux. The coefficient determines the electric charge of the quasiparticle. By adiabatically interchanging two quasiparticles we can obtain their statistics. By repeating the calculation of Ref. 22 we obtain that the quantum numbers of the quasiparticle (28) are given by

$$Q_{\rm el}=\nu e,\ \theta=\nu\pi\ . \tag{30}$$

These are the quantum numbers of the quasiparticle in the hierarchical state, labeled by $l_1=1$, $l_2=0$. Hence the quasiparticle with $l_1=1$, $l_2=0$ in the effective theory corresponds to the quasiparticle (28) in the microscopic theory.

However, the adiabatic approach of Ref. 22 cannot be directly applied to the wave function (29). The reason is the occurrence of the cross terms $\prod_{i<j}(z_i-\xi_j)$. According to Refs. 2 and 3 the second type of the quasiparticles in the hierarchical state can be described by the pseudo-wave-function

$$\psi(z_1,..,z_n)=\prod_{k<l}^{N_h}(\bar z_k-\bar z_l)^{-1/p_1}\prod_{k<l}^{N_h}(\bar z_k-\bar z_l)^{p_2}\prod_{k=1}^{k=N_h}(\bar z_k-\bar\xi_0)\exp(-q_h|z_k|^2/4l_0^2)\ . \tag{31}$$

The wave functions (29) and (31) are connected by the "fractional-statistic transformation." We assume (although we lack direct proof) that a quasiparticle in the incompressible anyon state described by the wave function (31) is the same quasiparticle as the one described by the electron wave function (29). Thus we can use the wave function (31) to find the quantum numbers of the quasiparticle described by Eq. (29). It is straightforward to obtain the quantum numbers of such a quasiparticle:

$$Q_{\rm el}=-\frac{1}{p_1p_2+1}e,\ \theta=-\frac{p_1}{p_1p_2+1}\pi\ . \tag{32}$$

(The alternative proofs can be obtained from the arguments given in Refs. 2 and 3 or using the methods of the conformal field theory Ref. 23.) Equation (32) describes the quantum numbers of the quasiparticles of the effective theory labeled by the integers $l_1=0$, $l_2=1$. We thus convinced ourselves that our effective theory has a direct correspondence with the microscopic theory. Let us note that in the framework of the microscopic theory it is straightforward to calculate the quantum numbers only of the quasiparticles given by the wave functions (28) and (31). More general quasiparticles are labeled by the arbitrary integers l_1 and l_2. It is not clear to us whether the above arguments can be used to calculate the quantum numbers of such general quasiparticles. It is difficult to obtain the cross term in θ [see Eq. (21)] using the microscopic wave functions. For such general quasiparticles the separation between the short-range properties of the wave functions and their long-range properties that define the quantum numbers of the quasiparticles becomes more complicated. On the other hand, the effective theory directly describes the long-range properties of the FQH state without taking into account its short-scale structure. This is the reason why it is so simple to find the quantum numbers of the quasiparticles using effective theory. The short-range properties in the electronic wave functions depend on the details of the model and complicate the calculation. The different construction schemes may lead to the same FQH states, although they have different wave functions. The only difference between these states are the unimportant short-range structures.

III. THE TOPOLOGICAL ORDERS IN THE GENERAL HIERARCHY STATE

In the previous chapter we discussed in details the second level of the FQH hierarchy. In this chapter we shall discuss the general construction. The effective Lagrangian for the most general state of the Haldane-Halperin hierarchy can be obtained in the same way as for the second level, by assuming that quasiparticles (or quasiholes) condense. We write explicitly the effective Lagrangian in the Ginsburg-Landau representation

$$\begin{aligned}\mathcal{L}=&\,i\Phi_e^\dagger(\partial_0+ia_0^0-ieA^0)\Phi_e+\frac{1}{2M_e}\Phi_e^\dagger(\partial_i+ia_0^i-ieA^i)^2\Phi_e+\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_0^\mu\partial^\nu a_1^\lambda\\&+\sum_{k=1}^{n-2}i\Phi_q^{+k}(\partial_0+ia_{2k-1}^0+ia_{2k}^0)\Phi_q^k+\frac{1}{2M_q}\Phi_q^{+k}(\partial_i+ia_{2k-1}^i+ia_{2k}^i)^2\Phi_q^k\\&+\frac{1}{4\pi}p_k\epsilon^{\mu\nu\lambda}a_{2k-1}^\mu\partial^\nu a_{2k-1}^\lambda+\epsilon^{\mu\nu\lambda}\frac{1}{2\pi}a_{2k}^\mu\partial^\nu a_{2k+1}^\lambda+i\Phi_q^{+(n-1)}(\partial_0+ia_{2n-3}^0+ia_{2n-2}^0)\Phi_q^{n-1}\\&+\frac{1}{2M_q}\Phi_q^{+(n-1)}(\partial_i+ia_{2n-3}^i+ia_{2n-2}^i)^2\Phi_q^{n-1}+\frac{1}{4\pi}\frac{1}{p_n}\epsilon^{\mu\nu\lambda}a_{2n-2}^\mu\partial^\nu a_{2n-2}^\lambda-\frac{1}{4\pi}p_{n-1}\epsilon^{\mu\nu\lambda}a_{2n-3}^\mu\partial^\nu a_{2n-3}^\lambda\ .\end{aligned} \tag{33}$$

Here n is the level of the hierarchy (Laughlin state corresponds to the first level of the hierarchy). The order parameters Φ_q^k describe the condensates of the quasiparticles that form the hierarchy state. We denote $\Phi_e \equiv \Phi_q^0$. The fields a_i are the fictitious U(1) gauge fields that are bound to the quasiparticles of the preceding states in the way described in Sec. II. We assume that each level of hierarchy can be represented by means of the condensation of the "composite" objects that contain a quasiparticle of the previous level and the flux tube with the even number of flux units. The ground-state properties of this Lagrangian are easily found using the equations of motion:

$$\begin{aligned}
&a_{2k-1}+a_{2k}=0\ ,\\
&a_0-eB=0\ ,\\
&n_e\equiv n_0=\nu B\ ,\\
&n^{(k)}=-a_{2k-2}-p_k a_{2k-1}=-a_{2k+1}\quad k=1,\ldots,n-2\ ,\\
&n^{(n-1)}=-a_{2n-4}-p_{n-1}a_{2n-3}=+\frac{1}{p_n}a_{2n-2}\ .
\end{aligned} \tag{34}$$

Here $n_i=\langle \Phi_q^{i\dagger}\Phi_q^i\rangle$ is the "density" of the ith type of the quasiparticles that take part in the formation of the hierarchy state. The field $a_i=\epsilon_{pq}\partial_p a_i^q$ is the strength of the fictitious gauge field, $i=0,\ldots,2n-2$.

These equations have solutions only if the following consistency condition is obeyed:

$$\nu=\cfrac{1}{p_1-\cfrac{1}{p_2-\cfrac{1}{\cdots-\cfrac{1}{p_n}}}}\ . \tag{35}$$

The electron density and the values of the gauge-field strength in the ground state are given by

$$n^{(k)}=a_{2k+2}=-a_{2k-3}=\frac{D_{n-k-1}(p_{k+2},\ldots,p_n)}{D}B\ . \tag{36}$$

Here D is the denominator of the filling fraction ν given by Eq. (35). D_{n-k} is the denominator of the filling fraction

$$\nu_{n-k}=\cfrac{1}{p_{k+1}-\cfrac{1}{p_{k+2}-\cfrac{1}{\cdots-\cfrac{1}{p_n}}}}\ . \tag{37}$$

Let us now consider the quantum numbers of the quasiparticles associated with the Lagrangian (33). The equations of motion for charges and fluxes of the auxiliary gauge field have the form

$$Q_0=-\frac{\Phi_1}{2\pi}\ , \tag{38}$$

$$Q_{2k}=Q_{2k-1}\ , \tag{39}$$

$$Q_{2k}=-\frac{\Phi_{2k+1}}{2\pi}\quad k=1,\ldots,n-2\ , \tag{40}$$

$$Q_{2k-1}=-\Phi_{2k-2}+p_k\Phi_{2k-1}\frac{1}{2\pi}\ , \tag{41}$$

$$Q_{2n-2}=\frac{\Phi_{2n-2}}{p_n 2\pi}\ , \tag{42}$$

$$\Phi_{2k-1}+\Phi_{2k}=l_k 2\pi\ , \tag{43}$$

$$\Phi_0=l_0\ . \tag{44}$$

We can eliminate the fluxes Φ_{2k} from these equations using Eqs. (43) and (44). We obtain the system

$$\Lambda_{ij}\Phi_{2j-1}=l_{i-1}(i=1,\ldots,n)\ . \tag{45}$$

Here the matrix Λ is given by the equation

$$\Lambda=\begin{bmatrix}
p_1 & -1 & \cdots & 0 & 0 & 0\\
-1 & p_2 & -1 & 0 & \cdots & 0\\
0 & -1 & p_3 & -1 & 0 & \cdots\\
\vdots & \vdots & \vdots & \ddots & & \\
0 & 0 & 0 & \cdots & -1 & p_n
\end{bmatrix}\ . \tag{46}$$

The solution of the system (45) amounts to inverting the matrix Λ. The inverse of the matrix Λ, the matrix B has the matrix elements

$$\begin{aligned}
B_s^k&=\frac{D_{n-k}(p_{k+1},\ldots,p_n)D_{s-1}(p_1,\ldots,p_{s-1})}{D},\quad s<k\\
B_s^k&=\frac{D_{n-s}(p_{s+1},\ldots,p_n)D_{k-1}(p_1,\ldots,p_{k-1})}{D},\quad s\geq k\ .
\end{aligned} \tag{47}$$

Here $D_k(p_{s_1},\ldots,p_{s_k})$ is the denominator of the filling fraction

$$\nu_i=\cfrac{1}{p_{s_1}-\cfrac{1}{p_{s_2}-\cfrac{1}{\cdots-\cfrac{1}{p_{s_k}}}}}\ . \tag{48}$$

The numbers D_i satisfy the recursion relation

$$\begin{aligned}
D_k(p_{i_1},\ldots,p_{i_k})&=p_{i_1}D_{k-1}(p_{i_2},\ldots,p_{i_k})\\
&\quad -D_{k-2}(p_{i_3},\ldots,p_{i_k})\ ,\\
D_0=1,\ D_{-1}&=0\ .
\end{aligned} \tag{49}$$

Note that $D\equiv D_n(p_1,\ldots,p_n)$.

The charges of the quasiparticles are given by

$$Q_{\rm el}=\sum_{j=1}^{j=n}B_{1j}l_{j-1}\ . \tag{50}$$

Note that B_{1j} coincide with $n^{(j)}$ given by Eq. (36). After some algebra we can also show that the statistics of the quasiparticles are given by the formula

$$\theta=\sum_{s=0}^{s=2n-2} Q_s\Phi_s=\sum_{i,j=1}^{i,j=n-2} B_{ij}l_{i-1}l_{j-1}\pi \ . \tag{51}$$

In particular the statistics of the fundamental quasiparticles that are labeled by integers $l_{i-1}=1$, $l_k=0|_{k\neq i-1}$ are given by

$$\theta_i=\frac{D_{n-i}(p_{i+1},\ldots,p_n)D_{i-1}(p_1,\ldots,p_{i-1})}{D}\pi \ . \tag{52}$$

In the case $n=2$ (the second level of the hierarchy) we recover the results of Sec. II.

It is convenient to regard the incompressible electron state as being made of several different incompressible components. The density of each of these components is given by Eq. (36). The fundamental quasiparticles can be viewed as the quasiparticles in these parent fluids.

The quasiparticle quantum numbers found for the hierarchy states can be compared with the quantum numbers of the quasiparticles in the states constructed by Jain in Ref. 7 for the same filling fractions. We shall compare the hierarchical states and the Jain states with the filling fraction $\nu=k/(kp+1)$, where p is an even integer and k is an arbitrary integer. The charge and the statistics of the quasiparticles of the corresponding Jain state are given by[12]

$$Q_{\rm el}=\frac{\sum_{i=1}^{k} n_i}{pk+1}e,\ \theta=\left[\sum_{i=1}^{k} n_i^2-\frac{p}{pk+1}\left[\sum_{i=1}^{k} n_i\right]^2\right]\pi \ . \tag{53}$$

As it was pointed out to the authors by Read,[24] the filling fraction $\nu=k/(kp+1)$ can be represented in the form

$$\nu_k=\cfrac{1}{p+1-\cfrac{1}{2-\cfrac{1}{\cdots-\frac{1}{2}}}} , \tag{54}$$

where these are $k-1$ levels in the continued fraction. This is the way in which the Jain state is represented in the hierarchical scheme. For the hierarchical state (54) the symmetric matrix B_n^s is given by

$$B_n^s=\frac{(k-n+1)[p(s-1)+1]}{pk+1} \quad s\le n \ . \tag{55}$$

The charge of the quasiparticle is given by

$$Q_{\rm el}=\frac{\sum_{n=1}^{k}(k-n+1)l_n}{pk+1} \ . \tag{56}$$

It can be shown that, after a redefinition of integers,

$$\begin{aligned} &n_1=l_1 \ ,\\ &n_2=l_1+l_2 \ ,\\ &\ldots,\\ &n_k=l_1+\cdots+l_k \ , \end{aligned} \tag{57}$$

that the charge and the statistics of the general quasiparticle in the Jain states [see Eq. (53)] and in the hierarchical state [see Eqs. (56), (55), and (51)] coincide. The analogous results can be proven for the second main sequence of the Jain states. These are the states with the filling fraction $\nu=k/(kp-1)$. They have the representation

$$\nu_k=\cfrac{1}{p-1+\cfrac{1}{2-\cfrac{1}{-\cdots\frac{1}{2}}}} , \tag{58}$$

where there are $k-1$ levels in the continued fraction. Thus the quantum numbers of the quasiparticles in the Jain states and in the hierarchical states are identical. This shows that these states are in fact equivalent FQH states.

Note however that, in general, some of the Jain states cannot be obtained from the hierarchy construction. The simplest examples are the states with the filling fractions $\frac{1}{2}=1/(1+\frac{1}{2}+\frac{1}{2})$ and $\frac{2}{3}=1/(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})$. It would be interesting to find which of the general Jain state with the filling fraction $\nu=1/\sum_k 1/m_k$ can be obtained from the hierarchy construction. The Jain states with the filling fraction $\nu=1/\sum_k 1/m_k$ contain $\sum_k(|m_k|-1)+1$ branches of edge excitations.[11,12] In the next section we will see that the number of the edge branches is equal to the level of the state in the hierarchy. Hence if the Jain state with the filling fraction $\nu=1/\sum_k 1/m_k$ is equivalent to some hierarchy state, this state must lie at the level $\sum_k(|m_k|-1)+1$ of the hierarchy.

The hierarchy states have one more interesting property. The charge $Q_{\rm el}$ and the statistics of an arbitrary quasiparticle satisfy the equations

$$\theta=D_{n-1}D\left[\frac{Q_{\rm el}}{e}\right]^2 \bmod 2\pi \ . \tag{59}$$

As far as only the charge and the statistics are concerned the quantum numbers of all the quasiparticles can be generated using only one quasiparticle labeled by the integers $l_n=1$, $l_i=0$, $i\neq n$. This quasiparticle has the following quantum numbers:

$$Q_{\rm el}=\frac{1}{D}e,\ \theta=\frac{D_{n-1}(p_1,\ldots,p_{n-1})}{D}\pi \ . \tag{60}$$

This formula can be easily proved using the Eq. (56) for the electric charges

$$Q_{\rm el}=\frac{\sum l_i D_{n-i}(p_{i+1},\ldots,p_n)}{D}e \tag{61}$$

and the identity

$$D_{s-1}(p_1,\ldots,p_{s-1})-D_{n-s}(p_{s+1},\ldots,p_n)D_{n-1}(p_1,\ldots,p_{n-1})=-D_{n-s-1}(p_{s+1},\ldots,p_{n-1})D\ . \tag{62}$$

This identity can be obtained by induction over n. Note, however, that this does not mean that all quasiparticles in the theory can be considered as the bound states of the quasiparticle (60). The reason is that the charge and statistics are not all quantum numbers that characterize the quasiparticle. For example in the integer quantum Hall states the electrons in the different Landau levels have the same charge and statistics but they are definitely different.

Finally, let us note that the construction of the hierarchical FQHE states described above has two natural generalizations. First, we can construct the hierarchy starting from an arbitrary Jain state. It would be interesting to consider the hierarchy that begins from the Jain states that cannot be obtained using the hierarchy construction, e.g., the $\frac{1}{2}$ and $\frac{2}{3}$ states mentioned above. We can built this new hierarchy and calculate the quantum numbers of the quasiparticles in the same way as it was done above for the case of the Haldane-Halperin hierarchy states.

Second, let us note that in the Haldane-Halperin hierarchy construction we assumed that a particular type of quasiparticles condenses each time when we move into the next level of the hierarchy. This quasiparticle is labeled by the integers $l_i=1$, $l_j=0|_{j\neq i}$ and has quantum numbers given by Eq. (60). More general hierarchical states can be obtained through the condensation of other quasiparticles. This possibility will be discussed in the next section.

IV. DUAL THEORY AND THE EDGE STATES

In this section we are going to study the edge states[25–29,11,12] in the hierarchy FQH states. For this purpose it is most convenient to use the dual effective theory in which the incompressible fluids are described by the gauge fields. Furthermore, the dual effective theory is much simpler than the Ginzburg-Landau effective theory. In the following we are going to derive the dual effective theories for the hierarchy FQH states.

First, let us review the dual effective theory for the Laughlin state.[30] Consider the anyon system in the magnetic field

$$\mathcal{L}_0=\psi^\dagger_{\text{anyon}}i(\partial_0-ieA_0)\psi_{\text{anyon}}+\frac{1}{2M}\psi^\dagger_{\text{anyon}}(\partial_i-ieA_i)^2\psi_{\text{anyon}}\ , \tag{63}$$

where ψ_{anyon} is the field that describes anyons with the fractional statistics θ. At the filling fraction $\nu=(\theta/\pi+m)^{-1}$, where m is an even integer, the anyon ground state is given by the Laughlin wave function

$$\left[\prod_{i,j}(z_i-z_j)^{\theta/\pi+m}\right]\exp-1/4\sum_i|z_i|^2\ . \tag{64}$$

The effective theory for such a state is given by (in its dual form)

$$\mathcal{L}=\left[-\left[\frac{\theta}{\pi}+m\right]\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}+\frac{1}{g^2}(f_{\mu\nu})^2+\frac{e}{2\pi}A_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}\right]+\left[\Phi^\dagger i(\partial_0-ia_0)\Phi+\frac{1}{2M}\Phi^\dagger(\partial_i-ia_i)^2\Phi\right]\ , \tag{65}$$

where the bosonic field Φ describes the quasiparticles in the Laughlin state. The effective theory (65) for the fermion case (i.e., $\theta/\pi=1$) is first given in Ref. 10. The generalization to the anyon cases is straightforward using the idea of the adiabatic continuation.[20] The anyon number current is given by

$$j_\mu=\frac{1}{2\pi}\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}\ . \tag{66}$$

From the equation of motion $\delta\mathcal{L}/\delta a_0=0$ we find the filling fraction to be

$$\nu\equiv\frac{j_0}{eB}=\frac{1}{\theta/\pi+m}\ . \tag{67}$$

The quasiparticle can be shown to carry an electrical charge $q_e=e/(\theta/\pi+m)$ and a statistics $\theta_q=\pi/(\theta/\pi+m)$.

Now consider an electron system. In this case $\theta/\pi=1$ and $\nu=1/(m+1)$. Let us increase the filling fraction by creating the quasiparticles. In the mean-field theory the quasiparticle gas behaves like bosons in the "magnetic" field $b=\partial_i a_j\epsilon_{ij}$, as one can see from the second term in (65). When the boson density satisfies

$$\Phi^\dagger\Phi=\frac{1}{p_2}\frac{b}{2\pi}\ , \tag{68}$$

where p_2 is even, the bosons have a filling fraction $1/p_2$. The ground state of the bosons can be again described by a Laughlin state. The final electronic state that we obtained is nothing but a second-level hierarchy FQH state constructed by Haldane and Halperin.

The boson Laughlin state can be described by a condensate of the composite objects that are bound states of the bosons and even number of units of flux. The binding of the flux to the boson can be realized by introducing a gauge field b_μ. We write the second term of Eq. (65) as

$$\Phi'^\dagger(\partial_0+ia_0+ib_0)\Phi'+\frac{1}{2M}\Phi'^\dagger(\partial_i+ia_i+ib_i)^2\Phi'-V(\Phi')+\frac{1}{p_2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}b_\nu\partial_\mu b_\lambda\ , \tag{69}$$

where Φ' describe the composite objects. The second term in Eq. (65) is recovered after integrating out the b_μ field. The Laughlin state of the bosons is described by the mean-field vacuum $\Phi'=\text{const}$. The quasiparticles in the new FQH state are the vortices in the Φ' field and the original bosons that are not bounded with the flux. These "bare" bosons are described by

$$\Phi^\dagger i(\partial_0 - ia_0)\Phi + \frac{1}{2M}\Phi^\dagger(\partial_i - ia_i)^2\Phi \ . \quad (70)$$

The total Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &-(1+m)\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} + \frac{1}{g^2}(f_{\mu\nu})^2 \\ &+ \frac{e}{2\pi}A_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} \\ &+ \Phi^\dagger i(\partial_0 - ia_0)\Phi + \frac{1}{2M}\Phi^\dagger(\partial_i - ia_i)^2\Phi \\ &+ \left[\Phi'^\dagger(\partial_0 + ia_0 + ib_0)\Phi' + \frac{1}{2M}\Phi'^\dagger(\partial_i + ia_i + ib_i)^2\Phi' \right. \\ &\left. - V(\Phi') + \frac{1}{p_2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}b_\nu\partial_\mu b_\lambda\right] \ . \quad (71)\end{aligned}$$

Equation (71) is a mixture of the Ginzburg-Landau theory and the dual theory.

The boson Laughlin state can also be described by a dual effective theory of a form (65):

$$\begin{aligned}\mathcal{L} = &-\frac{p_2}{4\pi}\tilde{a}_\mu\partial_\nu\tilde{a}_\lambda\epsilon^{\mu\nu\lambda} + \frac{1}{2\pi}a_\mu\partial_\nu\tilde{a}_\lambda\epsilon^{\mu\nu\lambda} + \tilde{\Phi}^\dagger i(\partial_0 - i\tilde{a}_0)\tilde{\Phi} \\ &+ \frac{1}{2M}\tilde{\Phi}^\dagger(\partial_i - i\tilde{a}_i)^2\tilde{\Phi} \ . \quad (72)\end{aligned}$$

This is just the dual form of the terms in square bracket in Eq. (71). Note that in Eq. (72) a_μ is treated as the background field just as A_μ in Eqs. (63) and (65). In Eq. (72) $\tilde{a}_\mu$ describes the boson condensate and $\tilde{\Phi}$ describes the new quasiparticles corresponding to the vortices in the boson wave functions [i.e., the vortices in the Φ' field in Eq. (71)]. The current $\tilde{j}_\mu$ of the quasiparticles in the original Laughlin state can be expressed in terms of $\tilde{a}_\mu$:

$$\tilde{j}_\mu = \frac{1}{2\pi}\partial_\nu\tilde{a}_\lambda\epsilon^{\mu\nu\lambda} \ . \quad (73)$$

The total effective theory has a form

$$\begin{aligned}\mathcal{L} = &\left[-\frac{p_1}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} + \frac{e}{2\pi}A_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}\right] \\ &+ \left[-\frac{p_2}{4\pi}\tilde{a}_\mu\partial_\nu\tilde{a}_\lambda\epsilon^{\mu\nu\lambda} + \frac{1}{2\pi}a_\mu\partial_\nu\tilde{a}_\lambda\epsilon^{\mu\nu\lambda}\right] , \quad (74)\end{aligned}$$

where $p_1 = m+1$ is an odd integer. The quasiparticle excitations in the new state are described by the Lagrangian

$$\begin{aligned}\mathcal{L} = &\Phi^\dagger i(\partial_0 - ia_0)\Phi + \frac{1}{2M}\Phi^\dagger(\partial_i - ia_i)^2\Phi \\ &+ \tilde{\Phi}^\dagger i(\partial_0 - i\tilde{a}_0)\tilde{\Phi} + \frac{1}{2M}\tilde{\Phi}^\dagger(\partial_i - i\tilde{a}_i)^2\tilde{\Phi} \ , \quad (75)\end{aligned}$$

where the field Φ describes the original quasiparticles (the vortices in the electron wave function). In the following, we use the dual theory (74) and (75) to calculate the quantum numbers of the quasiparticles. A similar calculation can also be performed starting from the effective theory (71) (see Sec. II).

The total filling fraction can be determined from the equation of motion $\delta\mathcal{L}/\delta a_0 = \delta\mathcal{L}/\delta\tilde{a}_0 = 0$:

$$eB = p_1 b - \tilde{b}, \ b = p_2\tilde{b} \ . \quad (76)$$

We find

$$\nu = \frac{b}{eB} = \left[p_1 - \frac{1}{p_2}\right]^{-1} \ . \quad (77)$$

Equation (74) can be written in a more compact form by introducing $(a_{1\mu}, a_{2\mu}) = (a_\mu, \tilde{a}_\mu)$:

$$\mathcal{L} = -\sum_{I,I'}\frac{1}{4\pi}\Lambda_{II'}a_{I\mu}\partial_\nu a_{I'\lambda}\epsilon^{\mu\nu\lambda} + \frac{e}{2\pi}A_\mu\partial_\nu a_{1\lambda}\epsilon^{\mu\nu\lambda} \ , \quad (78)$$

where the matrix Λ has integral elements:

$$\Lambda = \begin{bmatrix} p_1 & -1 \\ -1 & p_2 \end{bmatrix} \ . \quad (79)$$

Let us consider a generic quasiparticle that consists of $l_1\Phi$ quasiparticles and $l_2\tilde{\Phi}$ quasiparticles. Such a quasiparticle carries l_1 units of the charge of the a_μ field and l_2 units of the charge of the $\tilde{a}_\mu$ field. The statistics of such a quasiparticle is given by

$$\theta = \pi l^T\Lambda^{-1}l = \frac{1}{p_2p_1 - 1}(p_2l_1^2 + p_1l_2^2 + 2l_1l_2) \ . \quad (80)$$

The electric charge of the quasiparticle is

$$Q_{\text{el}} = e\Lambda^{-1}_{1I}l_I = e\frac{p_2l_1 + l_2}{p_2p_1 - 1} \ . \quad (81)$$

These results are in full agreement with the results obtained in the Sec. II.

The above construction can be easily generalized to the level n hierarchy FQH states with the filling fraction

$$\nu = \cfrac{1}{p_1 - \cfrac{1}{p_2 - \cfrac{1}{\cdots - \cfrac{1}{p_n}}}} \ , \quad (82)$$

where p_1 is odd and $p_i|_{i>1}$ are even. The effective theory still has a form (74) but now I runs from 1 to n. To obtain the form of the matrix Λ, let us assume that at level $n-1$ the effective theory is given by Eq. (74) with $a_{I\mu}$, $I = 1, \ldots, n-1$, and $\Lambda = \Lambda^{(n-1)}$. The quasiparticles carry integer charges of the $a_{I\mu}$ gauge fields. If we assume that the nth-level hierarchy state is obtained by the "condensation" of the quasiparticles with the $a_{I\mu}$ charge $l_I|_{I=1,\ldots,n-1}$, the level n effective theory will be given by Eq. (74) with n gauge fields. The nth gauge field $a_{n\mu}$ comes from the new condensate. The matrix Λ is given by

$$\Lambda^{(n)} = \begin{bmatrix} \Lambda^{(n-1)} & -l \\ -l^T & p_n \end{bmatrix} \ . \quad (83)$$

The new condensate gives rise to a new kind of quasiparticles that carry the integer charge of the new gauge field $a_{n\mu}$. Hence a generic quasiparticle always carries integral charges l_I of the $a_{I\mu}$ field. The electric charge and the statistics of the quasiparticle are given by the following general formulas:

$$\theta=\pi l_I \Lambda_{IJ}^{-1} l_J,\; q_e=e\Lambda_{1I}^{-1} l_I \;. \tag{84}$$

In the hierarchy construction one always assumes that the quasiparticles with the quantum numbers given by Eq. (60) "condense." Therefore for the hierarchy FQH states Λ is given by

$$\Lambda_{II'}=p_I\delta_{I,I'}-\delta_{I,I'-1}-\delta_{I,I'+1} \;. \tag{85}$$

Now Λ^{-1} is the B matrix we calculated in Sec. III. Once again our results are in full agreement with those obtained using Ginsburg-Landau theory in the previous chapters.

From the above discussion we see that more general hierarchy states can be obtained by assuming other quasiparticles condense. The choice of the condensing quasiparticle in the Haldane-Halperin hierarchy scheme is valid if the quasiparticle has the smallest energy gap. However, one cannot exclude the possibility (at least there are no reliable arguments to exclude such possibility) that a different type of quasiparticles has the smallest energy gap. In this case, such quasiparticles shall condense and we will obtain a new hierarchy state. It is straightforward to find filling fractions and the topological orders in this new hierarchy state using the above general formulas.

In Ref. 12 we have shown that each independent gauge field in the dual effective theory gives rise to one branch of the edge excitations. Hence the nth-level hierarchy state will have n branches of the edge excitations. The signs of the edge velocities are given by the signs of the eigenvalues of the matrix Λ. Let us consider the $\nu=(p_1-1/p_2)^{-1}$ FQH state in more detail. The two eigenvalues of Eq. (79) have the same sign if $p_2>0$. In this case the edge states have two branches moving in the same direction. This is consistent with the picture that the FQH states with $p_2>0$ are formed through the quasielectron condensation. The quasielectrons and the electrons have the same charges. The wave functions for quasielectrons are also holomorphic, like the electron wave function. Thus the quasielectron condensate and the electron condensate give rise to the edge velocities with the same sign. When $p_2<0$, two eigenvalues have the opposite sign and the two edge branches will move in opposite directions. This is also consistent with the fact that the FQH states with $p_2<0$ are formed by the quasihole condensate. The wave function for the quasiholes is antiholomorphic, which gives rise to the opposite sign of the edge velocity. In particular $\nu=\frac{2}{5}$ state ($p_1=3$, $p_2=2$) has two edge branches with the same sign of the edge velocities. While the $\nu=1-1/n$ state ($p_1=1$, $p_2=1-n$) has two edge branches with the opposite edge velocities. These results agree with those obtained in Ref. 11, 28, and 12. Actually one can show that the effective theories for these FQH states are identical to those obtained in Ref. 12 after a proper field redefinition. This further indicates that the states with $\nu=\frac{2}{5},\frac{2}{3}$ obtained by the Jain construction are the same as those obtained by the hierarchy construction. The same thing is true even for more general FQH state with filling fraction $\nu=k/(pk\pm 1)$. Using the transformation (57) we can show that the dual effective theory for those Jain states are identical to the dual effective theories for the corresponding hierarchy states.

The quasiparticle propagator along the edge has the following general form:

$$G(x)\propto\prod_i\frac{1}{(x-v_{Ri}t)^{\alpha_i}}\prod_i\frac{1}{(x+v_{Li}t)^{\bar{\alpha}_i}} \;, \tag{86}$$

where v_{Ri} (v_{Li}) are the edge velocities of the right- (left-) moving excitations. The exponent α_i and $\bar{\alpha}_i$ are the quadratic functions of the quasiparticle charges. Let us call $h=\frac{1}{2}\sum_i\alpha_i$ the right dimension of the quasiparticle operator and $\bar{h}=\frac{1}{2}\sum_i\bar{\alpha}_i$ the left dimension of the quasiparticle operator. We find that $2h-2\bar{h}$ is related to the quasiparticle statistics

$$2h-2\bar{h}=\frac{\theta}{\pi} \;, \tag{87}$$

where θ is given by the Eq. (51). Therefore $2h-2\bar{h}$ is universal and independent of the electron interactions and edge potentials. From Eq. (87) and the fact that h and $\bar{h}$ are quadratic functions of the quasiparticle charges, we find that

$$2h-2\bar{h}=l_I\Lambda_{IJ}^{-1}l_J \tag{88}$$

for the quasiparticle with the $a_{I\mu}$ charge l_I. If the edge excitations contain only right (or left) movers the Eq. (88) completely determines the total dimension of the quasiparticle operator on the edge (which is given by $h+\bar{h}$).

Let us consider the $p_2/(p_2p_1-1)$ states as the example. In this case the charges of the quasiparticles are given by integers (l_1, l_2) satisfying $p_2l_1+l_2=p_2p_1-1$. These quasiparticles correspond to electrons. Thus the electron propagator on the edge has a form

$$G_{\mathrm{ele}}(x,t)\propto\frac{1}{(x-v_Rt)^{2h}}\frac{1}{(x+v_Lt)^{2\bar{h}}} \;, \tag{89}$$

where h and $\bar{h}$ satisfy

$$\begin{aligned}2h-2\bar{h}&=\frac{1}{p_2p_1-1}(p_2l_1^2+p_1l_2^2+2l_1l_2)\\&=p_2l_1^2-2(p_2p_1-1)l_1+p_1(p_2p_1-1) \;.\end{aligned} \tag{90}$$

When $p_2>0$ we further have $\bar{h}=0$. For $p_2>0$, $|2h-2\bar{h}|$ has a minimum near $l_1=p_1$:

$$\begin{aligned}&l_1=p_1-1:\;\; 2h-2\bar{h}=p_2-2+p_1 \;,\\&l_1=p_1:\;\; 2h-2\bar{h}=p_1 \;,\\&l_1=p_1+1:\;\; 2h-2\bar{h}=p_2+2+p_1 \;.\end{aligned} \tag{91}$$

This agrees with the results obtained in Ref. 11.

For the case of the general hierarchical state obtained through the condensation of the quasiparticles (all $p_i > 0$) we can also prove that $2h$ has a minimum value p_1. This is achieved for the following quantum numbers of the electron state: $l_1 = p_1$, $l_2 = -1$, all other $l_i = 0$.

In Ref. 11 we have discussed how to experimentally measure the exponent $(2h + 2\bar{h})$ in the electron propagator on the edge. In the following we will discuss how to measure the exponent in the quasiparticle propagator on the edge. We will concentrate on the following example. On a background of the $\nu = \frac{1}{3}$ FQH state there are two islands in which the electrons form a $\nu = \frac{2}{5}$ FQH state. The $\frac{2}{5}$ FQH state in each island can be regarded as formed by the anyon Laughlin state on the background of the $\frac{1}{3}$ FQH state. The anyons are just the quasiparticles in the $\frac{1}{3}$ FQH state. There is one branch of the edge excitations on the edge of each island. The quasiparticle wave function is given by

$$\Psi \propto \prod_{i,j} (z_i - z_j)^{5/3} . \tag{92}$$

Thus the quasiparticle propagator on the edge has a form

$$G(x,t) \propto \left[\frac{1}{x - vt} \right]^{5/3} . \tag{93}$$

When the chemical potentials on the two islands are different, the quasiparticle can tunnel from one island to the other. The differential conductance is expected to have a form

$$\frac{dI}{dV} \propto V^{2h + 2\bar{h} - 1} = V^{2/3} . \tag{94}$$

Another quantum number that reflects the topological orders in the FQH states is the ground-state degeneracy on the torus.[10,13] From the topological theory, one can show that the ground-state degeneracy n_G of the theory (78) is equal to the determinant of the matrix Λ. It is easy to prove that

$$n_G = D , \tag{95}$$

where D is the denominator of the filling fraction.

V. DISCUSSIONS

In this paper we construct the effective theories for the hierarchy FQH states. Using the effective theories we evaluate the charge and the statistics of the quasiparticles in the hierarchy FQH states. The quasiparticles for the nth-level hierarchical FQH state with a filling fraction (12) are labeled by n integers $l_1, \ldots, l_n$. The charges and statistics of the quasiparticles in such states are given by Eqs. (50) and (52).

We showed that at the filling fractions $\nu = k/(kp \pm 1)$ the Jain states and the hierarchy states have the same topological order. Moreover, the dual theories obtained for those states in the hierarchy and the Jain schemes are identical after a field redefinition. This result is quite remarkable since the effective theory and the microscopic wave functions obtained from these two constructions have very different form. Hence these Jain states are identical to the hierarchy states. Note, however, that there exist Jain states that cannot be obtained using the hierarchy scheme. The simplest examples are the states with the filling fractions $\frac{1}{2}$: $\nu = 1/(1 + 1/2 + 1/2)$ and $\frac{2}{3}$: $\nu = 1/\frac{1}{2} + \frac{1}{2} + \frac{1}{2})$. Also there are hierarchical states that cannot be obtained from the Jain construction. The simplest example is the level-two state with a filling fraction $\frac{4}{5}$. We conclude that the set of Jain states has a nonzero overlap with the set of hierarchical states. This overlap includes all Jain states obtained in Ref. 7 by binding flux to fermions. The latter states have filling fractions $k/(kp \pm 1)$. However, there exist states that can be obtained only from one of the two schemes, either the Jain or the Halperin-Haldane hierarchical scheme.

We discuss the relation between the effective theories and the microscopic wave functions. Our results are consistent with the known results derived from the microscopic theory. We also study the edge excitations in the hierarchy FQH states. The number of the edge branches is found to be equal to the level of the hierarchy state. We also consider the quasiparticle and the electron tunneling in the hierarchical edge states as the possible way to explore the properties of these states. The I-V curve for the quasiparticle tunneling is given by Eq. (94).

Combining the results in Ref. 12 and in this paper, we gain some basic understanding about the structures in the high-level FQH states. The universal properties in the bulk and on the edge are closely related. Our results give rise to physical and sometimes practical ways to characterize the different FQH states. These results bring us closer to the complete classification of all possible QH states.

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