Effective theories of the fractional quantum Hall effect at generic filling fractions

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We propose effective theories that describe the fractional quantum Hall effect (FQHE) states for the generic filling fractions $\nu=1/\sum_I (1/m_I)$ where m_I are integers. The theories describe the microscopic FQHE states proposed by Jain. We calculate charges and statistics of quasiparticles in these states. The structure of the edge states is derived directly from the underlying effective theory in the bulk. Our results are shown to be consistent with those obtained from the microscopic theory.

I. INTRODUCTION

The fractional quantum Hall effect (FQHE) at general filling factor ν ($\nu\neq 1/l$), is explained by the hierarchy schemes (see Refs. 1–6). There are many different hierarchy schemes. One may wonder whether the different hierarchy schemes lead to the same FQH states or not. If different hierarchy schemes lead to different FQH states, then we should determine which of the FQHE states constructed thus far are actually realized in nature.

As it is pointed out in Refs. 7 and 8, the filling fraction is not sufficient to characterize the FQH states. There can be many different FQH states for a given filling factor. The FQH states contain some universal internal structures which are independent of arbitrary perturbations. Such universal internal structures are called the topological orders in the FQH states. Characterizations of the topological orders are discussed in Refs. 7 and 8 using the ground-state degeneracy and properties of the edge states. However, complete classification of the topological structure of the generic FQH states remains an unsolved problem. Different hierarchy schemes may give rise to FQH states with different topological orders. Therefore, it is very important to find some physical properties of the FQH states which are related to their internal topological structures and can be tested experimentally. An important characteristic of the topological orders is the spectrum of the fractional statistics and fractional charges of the quasiparticles. The spectrum of the quantum numbers of the quasiparticles might completely classify the topological orders in the FQH states.

Quite recently an effective Ginsburg-Landau (GL) approach to the FQHE was proposed in Refs. 9–12 as a way to build the mean theories of the FQHE. This approach has been proved to be a useful tool in understanding the qualitative properties of the FQHE at filling fractions $\nu=1/l$.[11,12]

In this paper we construct the effective theories for the FQH states at generic filling fractions. We show that schemes proposed recently by Jain have a natural description in terms of the effective Ginsburg-Landau approach. The relation between our theories and microscopic construction of Jain's states are discussed. We also work out an effective theory that describes states that do not belong to the Jain scheme. Using the effective theories, we can easily calculate charges and statistics of the quasiparticles. These quantities have not been determined from microscopic considerations as yet. We also determine the structure of the edge states (Refs. 8 and 13–19) directly from the effective theory in the bulk of the sample. The dynamics of edge excitations is determined by the effective theory in the bulk of the sample up to the choice of their velocities.

The paper is arranged as follows. In Sec. II we describe the effective theory based on the "composite fermions" picture proposed by Jain.[3] We derive the charge and statistics of quasiparticles in these states. In Sec. III we consider a general effective theory based on another hierarchical scheme proposed by Jain. This scheme permits one to obtain naturally the general filling fractions $\nu=1/\sum_L (1/m_I$. In Sec. IV we consider the dual description of the effective theory proposed in Sec. III. The results obtained in Sec. III are rederived and confirmed. In Sec. V we give a new derivation of the dynamics of edge excitations. We show that the topological Chern-Simons theory, which describes FQHE states in the bulk of the sample, naturally gives rise to the edge excitations. The propagator of the edge excitations has scaling properties. In Sec. VI we discuss the results. We also give an example of an effective theory that cannot be included in the hierarchical schemes of Refs. 3 and 4.

II. COMPOSITE FERMIONS IN THE GINSBURG-LANDAU APPROACH

In this section we shall construct an effective theory for the FQH states with filling fraction $\nu=m/(mp+1)$, where p is an even integer. The construction is based on the hierarchy scheme proposed by Jain.[3]

Let us recall the basic ideas of this construction. We start from a 2D electron gas in the presence of a transverse magnetic field with a filling fraction

$\nu = m/(mp+1)$. We consider a "composite" object that consists of an electron and a flux tube with p units of flux (p is an even number). The resulting object displays a Fermi statistics because p is even. The 2D gas of such "composite" fermions detects an effective "magnetic" field $B - p\phi_0 m$ and an integer filling fraction $\nu = m$. In this way the $\nu = m/(mp+1)$ FQH state of electrons is related to the $\nu = m$ integral quantum Hall (IQH) state of the composite objects.

Let us now show that this construction can be described naturally in the mean-field approach. We start from the Lagrangian for the electrons in the magnetic field given by

$$\mathcal{L}_0 = \psi^\dagger i(\partial_0 - ieA_0)\psi + \frac{1}{2M}\psi^\dagger(\partial_i - ieA_i)^2\psi , \tag{1}$$

where ψ is the anticommuting variables that describe the electrons. The density of the electrons is given by

$$n_e = \nu\frac{eB}{hc} = \frac{m}{mp+1}\frac{eB}{hc} . \tag{2}$$

Let us bind the p units of the flux to the electrons. The binding of the flux to the electrons can be realized in the Lagrangian language by introducing a "fictitious" U(1) gauge field a_μ with a Chern-Simons term. The effective Lagrangian for the composite object is given by

$$\mathcal{L}_2' = \psi'^\dagger i(\partial_0 + ia_0 - ieA_0)\psi' + \frac{1}{2m}\psi'^\dagger(\partial_i + ia_i - ieA_i)\psi' + \frac{1}{p}\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} , \tag{3}$$

where ψ' is the *fermionic* field for the composite object. Using the similar arguments, similar to those given in Refs. 9–12, we find that Eq. (3) can be recovered from Eq. (5) by integrating out the a_μ field. However, here we will regard a_μ as a slow varying field. The vacuum expectation value of $b \equiv (\partial_1 a_2 - \partial_2 a_1)$ is determined by the electron density n_e. From the equation of motion $\delta\mathcal{L}_2'/\delta a_0 = 0$, we find that

$$n_e = \frac{1}{p}\frac{\bar{b}}{2\pi}, \quad \bar{b} = \langle b \rangle . \tag{4}$$

The fermions ψ' see (in average) an effective magnetic field:

$$\begin{aligned} -\bar{b} + eB &= \left[-pn_e + \frac{mp+1}{m}n_e\right]hc \\ &= \frac{1}{m}n_e hc = \frac{1}{\nu^*}n_e hc . \end{aligned} \tag{5}$$

Therefore, the effective filling factor ν^* for the ψ' fermions (i.e., the composite objects) is m. This agrees with the result in Ref. 3. In the mean-field theory, i.e., ignoring the fluctuations of b, we find that the ground state of the ψ' fermions in Eq. (5) is an integral quantum Hall (IQH) state with the first m Landau levels filled by the ψ' particles.

The GL effective theory of the IQH state with the filled first Landau level is given by[7,20]

$$\mathcal{L} = i\phi_1^\dagger(\partial_0 + ia_{10} - ieA_0)\phi_1 + \frac{1}{2M}\phi_1^\dagger(\partial_i + ia_{1i} - ieA_i)^2\phi_1 + \frac{1}{4\pi}a_{1\mu}\partial_\nu a_{1\lambda}\epsilon^{\mu\nu\lambda} - V(|\phi|) . \tag{6}$$

A hole in the first Landau level corresponds to a vortex in the ϕ_1 field. In the IQH state with m filled Landau levels, the fermions in the different Landau levels are independent. Therefore, the effective theory for the $\nu = m$ IQH state is given by

$$\mathcal{L} = \sum_{I=1}^{m}\left[i\phi_I^\dagger(\partial_0 + ia_{I0} - ieA_0)\phi_I + \frac{1}{2M}\phi_I^\dagger(\partial_i + ia_{Ii} - ieA_i)^2\phi_I + \frac{1}{4\pi}a_{I\mu}\partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda} - V(|\phi_I|)\right] . \tag{7}$$

Because in the mean-field theory the ground state of the ψ' particle is the $\nu = m$ IQH state, we may replace the first two terms in Eq. (5) with Eq. (9), with eA_μ in Eq. (9) replaced by $a_\mu + eA_\mu$. The final effective theory of the FQH is

$$\mathcal{L}_2 = \sum_{I=1}^{m}\left[i\phi_I^\dagger(\partial_0 + ia_{I0} + ia_0 - ieA_0)\phi_I + \frac{1}{2M}\phi_I^\dagger(\partial_i + ia_{Ii} + ia_i - ieA_i)^2\phi_I + \frac{1}{4\pi}a_{I\mu}\partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda} - V(|\phi_I|)\right] + \frac{1}{p}\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda E^{\mu\nu\lambda} . \tag{8}$$

Let us note that in this construction a number of different bosonic fields is equal to a number of filled Landau levels. The physical picture is that the electrons in each Landau level form an independent incompressible state. This state is described as a "condensate" of bosons. The quasiparticles are vortices in the boson field, and all have a finite gap. In the following we will study the ground state of the effective theory (8). From the equations of motion $\delta\mathcal{L}_2/\delta a_0 = \delta\mathcal{L}_2/\delta a_{I0} = 0$, we find that

$$\begin{aligned} n_I &= \frac{b_I}{hc}, \quad I = 1, \ldots, m , \\ \sum_{I=1}^{m} n_I &= \frac{1}{p}\frac{b}{hc} = n_e = \frac{m}{mp+1}\frac{eB}{hc} , \end{aligned} \tag{9}$$

where $b_1 = \partial_1 a_I - \partial_2 a_{I1}$, and $n_I = |\phi_I|^2$.

The potential energy of the theory (8) is minimized at

$$\langle \phi_I \rangle = \text{const} , \tag{10}$$

and

$$-b_I - b + eB = 0 . \tag{11}$$

The Eqs. (9) and (11) can be satisfied if

$$n_I=\bar{n}_I\equiv\frac{1}{m}n_e\ ,$$
$$b_i=\bar{b}_i\equiv\frac{1}{m}n_e hc\ , \tag{12}$$
$$b=\bar{b}\equiv pn_e hc\ .$$

The Eqs. (10) and (12) describe the mean-field ground state of the $\nu=m/(mp+1)$ FQH state.

The effective theory (10) can be used to determine the quantum numbers of the quasiparticles. The quasiparticle excitations correspond to the vortices in the ϕ_I fields. Consider the excitations that consist of vortices in the ϕ_I field with integer winding numbers l_I. The excitation carries an additional flux of $a_{I\mu}$ and a_μ gauge field, given by

$$\Phi_I=\int d^2x(b_I-\bar{b}_I)\ ,$$
$$\Phi=\int d^2x(b-\bar{b})\ . \tag{13}$$

The flux Φ_I and Φ should satisfy

$$\Phi_I+\Phi=l_I hc\ . \tag{14}$$

The excitation also carries the charges of the gauge fields $a_{I\mu}$ and a_μ. From Eq. (9) we find those charge to be

$$q_I=\frac{\Phi_I}{hc}\ ,$$
$$q=\sum_{I=1}^{m}q_I=\frac{1}{p}\frac{\Phi}{hc}\ , \tag{15}$$

where

$$q_I=\int d^2x(n_I-\bar{n}_I)\ ,$$
$$q=\int d^2x(n-n_e)\ . \tag{16}$$

Equations (14) and (15) imply that the charges satisfy a system of linear equations:

$$p\sum_J q_J+q_I=l_I\ . \tag{17}$$

Solving Eq. (17), we obtain

$$q_I=\left[l_I-\frac{p}{mp+1}\sum_{I=1}^{m}l_I\right]\ ,$$
$$q=\frac{1}{mp+1}\sum_{I=1}^{m}l_I\ . \tag{18}$$

Note that qe is also the electric charge carried by the excitation. We find that the quasiparticle excitations in the FQH state are labeled by m integers l_I, $I=1,\ldots,m$. The electric charge of the excitations is given by $[(\sum_{I=1}^{m}l_I)/(mp+1)]e$. The statistics of the excitation are given by

$$\theta=\frac{1}{2\hbar c}\left[q\Phi+\sum_{I=1}^{m}q_I\Phi_I\right]\pi$$
$$=\left[1-\frac{p}{pm+1}\right]\left[\sum_{I=1}^{m}l_I\right]^2\pi\ . \tag{19}$$

We see that for the FQH states described by Eq. (8) the statistics of the quasiparticles only depend on their charges.

The fundamental excitations are characterized by the integers $l_I=1$, and $l_J=0$ for $J\neq I$. The charges and statistics of those excitations are given by

$$q_e=\frac{1}{mp+1},\quad \theta=\left[1-\frac{p}{mp+1}\right]\pi\ . \tag{20}$$

Note that for the Laughlin states ($m=1$) we recover the known results:

$$q=\frac{1}{p+1},\quad \theta=\frac{1}{p+1}\pi\ , \tag{21}$$

thus proving the self-consistency of our approach.

III. DECOMPOSITION OF THE FERMION AND HIERARCHICAL FQHE

Let us now describe a more general construction for the FQHE states also proposed by Jain in Ref. 4. This scheme permits one to obtain general filling fractions:

$$\nu=\frac{1}{\sum_{I=1}^{P}(1/m_I)}\ . \tag{22}$$

Let us briefly review the basic construction. We begin by dividing an electron into particles of P distinct species labeled by $I=1,\ldots,P$. We solve the problem for each of these fictitious particles and in the end enforce the constraint that the coordinates of the fictitious particles belonging to the same electron are equal, i.e., $z_{Ij}=z_j$ for all I and j. Here index j labels the real electrons in the sample, and index I labels the fictitious particles into which we divided an electron.

In order to solve the problem, the following physically plausible assumption is made. An incompressible electron state is obtained if the particles of each of the fictitious species are in the incompressible state. To make use of this assumption we require the quasiparticles to satisfy the following conditions. (1) The particles of all species must be fermions since in this case we can use their fermionic statistics to produce an incompressible state. (2) Each of the fictitious particles sees the physical magnetic field. (3) the densities of all species must be equal to the electron density. (4) A sum of electric charges of the particles of different species must be equal to the charge of the electron e.

In the Jain construction, each ground state is described by a sequence of numbers $(m_1,\ldots,m_I)$. If we choose the charges of the fictitious fermions to be $e_I=\nu/m_I$, we find the filling fraction of the Ith fictitious fermions to be m_I. Therefore, m_I can be regarded as the number of Landau levels filled by the fictitious particles of the Ith type. From the equation $\sum_I e_I=1$, we see that the filling fraction is then given by the formula (22).

Let us now describe this theory in the mean-field approach. We have P species of the fictitious fermions. Each of them has a filling fraction m_I and a density equal to the electron density. This determines the charge of the fictitious fermions to be $e_I=(1/m_I)/\sum_J(1/m_J)$. We can introduce a set of independent bosonic fields ϕ_{Il}, $l=1,\ldots,m_I$, to describe the IQH state of the Ith fictitious particle. These fields are the "order parameters" of the incompressible QH states at each filled Landau level. Here index I labels the species, and index l labels different Landau levels.

The incompressible QH state is described by the condensates of these bosonic fields. Each of these fields interacts with a slowly varying gauge field a_{Il} and has a fractional electric charge e_I.

We also have to impose a constraint that the densities and the currents of all species are equal. This is done by introducing an auxiliary gauge field $a_{I\nu}$. The fields $a_{I\nu}$ have no Chern-Simons term and are simply the Lagrange multipliers. But the fields a_{Il} are not all independent, because we only require that the densities of the different species are equal to each other. This only gives rise to $P-1$ constraints instead of P constraints. Actually, $a_{I\nu}$ satisfies the condition

$$\sum_I a_{I\nu}=0 \ . \tag{23}$$

From these considerations, we can write the following effective theory:

$$\begin{aligned} L=&\sum_{Il} i\phi_{Il}^{\dagger}(\partial_0+ia_{I0}+ia_{Il0}-ie_I A_0)\phi_{Il} \\ &+\sum_{Il} \frac{1}{2M}\phi_{Il}^{\dagger}(\partial_i+ia_{Ii}+ia_{Ili}-ie_I A_i)^2\phi_{Il} \\ &+\sum_{Il}\frac{1}{4\pi}\epsilon_{\mu\nu\lambda}a_{Il\mu}\partial_\nu a_{Il\lambda} \ . \end{aligned} \tag{24}$$

Let us first consider the ground states of our theory. In the same way as it was done in Sec. II, we get the following densities of the bosonic condensates n_{Il}:

$$n_{Il}=\frac{b_{Il}}{hc} \ , \tag{25}$$

$$n_I=\sum n_{Il}=n_e=\nu B \ . \tag{26}$$

Minimizing the potential energy, we obtain

$$\langle \phi_{Il}\rangle=\text{const} \ , \tag{27}$$

$$-b_{Il}-b_I+e_I B=0 \ , \tag{28}$$

where $b_{Il}=\epsilon_{ij}\partial_i a_{Ilj}$, and $n_I=|\phi_{Il}|^2$. In the ground state, the values of the Lagrange multipliers are zero: $\bar{b}_I=0$. Hence we obtain

$$m_I e_I=\nu \tag{29}$$

for the filling fraction. Note that there is no summation over I in Eq. (29). Using the condition $\sum_I e_I=e$, we immediately get Eq. (22) for the filling fraction ν.

Now let us consider the low-lying excitations of this ground state. They are described by the holes in the filled Landau levels or by the excited electrons in empty Landau levels. A hole in the Landau level l of the fictitious particles of Ith type corresponds to a vortex in ϕ_{Il} field.

Let us consider the excitations consisting of vortices in the ϕ_{Il} fields with winding numbers M_{Il}. We have

$$\Phi_{Il}=\int d^2x(b_{Il}-\bar{b}_{Il}) \ , \tag{30}$$

$$\Phi_I=\int d^2x b_I \ . \tag{31}$$

The fluxes Φ_{Il} and Φ_I satisfy the equations

$$\Phi_{Il}+\Phi_I=M_{Il}hc \ . \tag{32}$$

From the equations of motion (25)–(28), we can determine the charges carried by these excitations relative to the gauge fields a_{Il} and a_I:

$$q_{Il}=\frac{\Phi_{Il}}{hc}=\int d^2x(n_{Il}-\bar{n}_{Il}) \ , \tag{33}$$

$$Q=q_I=\sum_l q_{Il}=\int d^2x(n_I-n_e) \ . \tag{34}$$

Equation (34) means that in the quasiparticle the densities of the fictitious fermions for different species are all the same. The quasiparticle in the theory receives the contributions from all the fictitious fermions.

The constraint (23) implies that

$$\sum_I \Phi_I=0 \ . \tag{35}$$

Using Eqs. (32)–(35), we easily get the expression for the densities of the fictitious fermions in the quasiparticle:

$$q_I=Q=\nu\sum_{Il}\frac{M_{Il}}{m_I}=\nu\sum_I\frac{N_I}{m_I} \ , \tag{36}$$

where $N_I=\sum_l M_{Il}$. Note that Q depends only on N_I, but not on each of M_{Il} separately. Using Eq. (36), we can easily find Φ_{Il} and q_{Il}:

$$\Phi_{Il}=\left[M_{Il}+\frac{\nu}{m_I}\sum_J\frac{N_J}{m_J}-\frac{N_I}{m_I}\right]hc \ , \tag{37}$$

$$q_{Il}=\frac{\Phi_{Il}}{hc}=M_{Il}+\frac{\nu}{M_I}\sum_J\frac{N_J}{m_J}-\frac{N_I}{m_I} \ . \tag{38}$$

It is easy to see that the statistics of excitations associated with each of the species are given by

$$\begin{aligned}\theta_I&=\frac{1}{2\hbar c}\left[q_I\Phi_I+\sum_l q_{Il}\Phi_{Il}\right] \\ &=-\left[\frac{\nu^2C}{m_I^2}-\frac{\nu N_I}{m_I^2}-\left[\sum_l M_{Il}^2-\frac{N_I^2}{m_I}+\frac{\nu^2C^2}{m_I}\right]\right]\pi \ ,\end{aligned} \tag{39}$$

where $C=\sum_I(N_I/m_I)$. The full statistics of the quasiparticle are determined by

$$\begin{aligned}\theta=\sum_I\theta_I&=\left[\sum_{Il}M_{Il}^2-\sum_I\frac{N_I^2}{m_I}+\nu C^2\right]\pi \\ &=\frac{1}{2\hbar c}\sum_{Il}q_{Il}\Phi_{Il} \ .\end{aligned} \tag{40}$$

Note that as far as statistics are concerned, we are interested only in θ mod2π. Hence we can rewrite Eq. (40) in the simpler form:

$$\theta=\left[\sum_I N_I^2\left(1-\frac{1}{m_I}\right)+\nu C^2\right]\pi \ . \tag{41}$$

Note that the statistics also depend only on the numbers N_I, not on M_{Il} separately. We would like to stress that it makes no sense to regard the quasiparticle as the one made of the single type of the fictitious fermions. It follows from Eq. (36) that even if N_I is nonzero only for one particular I, the densities of the fictitious fermions of all other species in the quasiparticle are also nonzero and equal to q_I. The physical quasiparticles are labeled by numbers $\{M_{Il}\}$. Note that sometimes two quasiparticles labeled by different M_{Il} may have the same statistics and charge, but are still distinct from each other. The vortex in the ϕ_{Il} field causes polarization in the Landau levels of other species due to the interaction with the auxiliary field $a_{I\nu}$. The electric charge carried by quasiparticles is equal to

$$q_e=\sum_I q_I e_I = Q=\nu\sum_I \frac{N_I}{m_I} \ . \tag{42}$$

Let us consider several important types of quasiparticle excitations. First, we consider the case $N_I=1$ and $N_J=0$ for $J\neq I$. We find that

$$\theta=\left[\frac{\nu}{m_I^2}+1-\frac{1}{m_I}\right]\pi, \quad q_e=\frac{\nu}{m_I}e \ . \tag{43}$$

This result looks different from the result of Jain in Ref. 4, where it was argued that $\theta=(\nu/m_I^2)\pi$ for the excitations with the charge given by Eq. (43). Note that for the case of the integer quantum Hall effect states ($\nu=m_1=1/(1/m_1)$), the number of species is equal to one. In this case we obtain

$$\theta=\pi, \quad q_e=e \ . \tag{44}$$

These excitations correspond to electrons and have Fermi statistics.

Another important case is when all $N_I=1$. In this case the quasiparticle contains one fictitious fermion from each of the species. The charge and the statistics of such an excitation are found to be

$$q_e=e, \quad \theta=\pi \ . \tag{45}$$

As expected, these are just the quantum numbers of an electron, and the quasiparticle corresponds to the original electrons.

Let us now check the self-consistency of our approach and its connection with the microscopic wave functions proposed by Jain. The general wave function that corresponds to the state with the filling fraction (22) is given by[4]

$$\Psi(z_1,z_2,\ldots,z_n)=\prod_{I,i}\chi_{m_I}(z_{Ii}) \ , \tag{46}$$

where χ_{m_I} are the wave functions of the fictitious particles I that fill m_I Landau levels (LL's), and we put $z_{I_j}=z_j$ for all I in the final expressions. Note that a product $\prod_{I=1}^{m}\chi_1$ corresponds to the Laughlin wave functions.

Let us construct quasiparticles in the Jain wave functions. There are many different ways to construct the same quasiparticle. A quasiparticle can be constructed by putting m_{I_1} holes in the LL of the I_1th fictitious fermions, one hole in each LL. The corresponding wave function of the I_1th fictitious fermion is given by

$$\chi^h_{m_{I_1}}(z_{Ii})=\prod_i (z_{I_1 i}-z_0)\chi_{m_{I_1}}(z_{I_1 i}) \ . \tag{47}$$

The total electron wave function for such a quasiparticle is given by

$$\Psi^h(z_i)=\left[\prod_{I\neq I_1}\chi_{m_I}(z_{Ii})\right]\chi^h_{m_{I_1}}(z_{I_1 i})|_{z_{I_i}=z_i} \ . \tag{48}$$

The same quasiparticle can be constructed by creating m_{I_2} holes in the LL's of the I_2th type of fictitious fermions (again one hole in each LL). From Eqs. (47) and (48) one easily sees that these two constructions lead to the identical electron wave functions and hence identical quasiparticle excitations. In the effective theory the first construction leads to the quasiparticle labeled by the integers

$$M_{I_1 l}=1 \ , \qquad M_{Il}=0, \quad I\neq I_1 \ . \tag{49}$$

The second construction leads to

$$M_{I_2 l}=1 \ , \qquad M_{Il}=0, \quad I\neq I_2 \ . \tag{50}$$

From the above considerations, we see that the quasiparticles in the effective theory labeled by Eqs. (49) and (50) must be *identical.*

Using Eqs. (36) and (40), we immediately see that the charge and statistics of these excitations are the same and are given by

$$q_{el}=\nu, \quad \theta=(m_{I_i}-m_{I_i}+\nu)=\nu\pi, \quad i=1,2 \ . \tag{51}$$

Therefore, our results are consistent with the microscopic theory. Moreover, the quasiparticle with the charge $q_{el}=\pm\nu$ can also be obtained from the microscopic wave function by adiabatically turning on a unit magnetic flux. The statistics of such a quasiparticle is $\theta=\nu\pi$.[4,21] This further confirms the self-consistency of our approach.

Let us now compare the scheme discussed in this section to the "composite" fermion approach that was considered in Sec. II. The two effective theories given by Eqs. (8) and (24) were constructed using different physical ideas and have different Lagrangians. Nevertheless, both theories have the ground states described by the same wave functions. Hence they must correspond to two different ways of describing the same FQHE states.

Let us consider the theory (24) in the case of the filling fraction:

$$\nu=\frac{m}{mp+1}=\frac{1}{p+1/m}=\frac{1}{\sum_{i=1}^{p}1+1/m}\ . \tag{52}$$

Now, m_I in (24) are given by $m_I=1$, $I=1,\ldots,p$, and $m_{p+1}=m$. The most general excitations of this theory are labeled by integers $M_{p+1,l}$ and $M_{I1}\equiv N_I$ $(I=1,\ldots,p)$. The charges and statistics of these excitations are equal to

$$\begin{aligned} q&=\frac{1}{mp+1}\left[\sum_l M_{p+1,l}+m\sum_I N_I\right],\\ \theta&=\left[1-\frac{p}{mp+1}\right]\left[\sum_l M_{p+1,l}+\sum_I N_I\right]^2 . \end{aligned} \tag{53}$$

The spectrum of the charges and statistics given by Eq. (53) coincide with those given by Eqs. (8) and (19). The quasiparticles labeled by L_l in the "composite fermion" approach correspond to the quasiparticles with $M_{p+1,l}=L_l$, and all $N_I=0$ in this framework. Note that the integer labels N_I are redundant in this particular case. The quasiparticles labeled by $M_{p+1,l}=L_l$, $N_I=0$, and $M_{p+1,l}=L_l-\sum_I n_I$, $N_I=n_I$, are actually equivalent (i.e., they have the same electron wave function).

We conclude that both theories (8) and (24) describe the same ground state despite their different appearances. The spectrum of low-lying excitations of the first theory is identical to that of the second theory. Both theories are actually equivalent as long as the redundancy of N_I is taken into account.

It would be very interesting to check both numerically and experimentally whether the excitation spectrum discussed in this section can or cannot be realized in nature.

IV. DUAL EFFECTIVE THEORY OF THE HIERARCHY FQH STATES

In Sec. III we have studied the fractional statistics of the quasiparticles in the hierarchical FQH states using the effective GL theory. As pointed out in Refs. 7 and 22, the GL theory has a dual form. In this section we are going to rederive the previous results using the dual theory. Some physical properties of the FQH states are more transparent in the latter approach.

As before, we decompose electrons into P kinds of fictitious fermions, each with electric charge e_I, $I=1,\ldots,P$. The fictitious fermions are described by the Lagrangian

$$\mathcal{L}=\sum_I\left[\psi_I^\dagger i(\partial_0+ie_IA_0)\psi_I+\frac{1}{2m}\psi_I^\dagger(\partial i+ie_IA_i)^2\psi_i\right] . \tag{54}$$

The fictitious fermions all have the same density, which is the density of the electrons. For the electron filling fraction $\nu=(\sum_I 1/m_I)^{-1}$, if we choose

$$e_I=\frac{\nu}{m_I}\ , \tag{55}$$

the Ith fictitious fermions will have a filling fraction $\nu_I=m_I$. The ground state of ψ_I is described by $|m_I|$ filled Landau levels. Here we allow m_I to be negative integers.

The effective theory (in the dual form) for the charge e^* fermions with m filled Landau levels is studied in Refs. 7 and 20. the Lagrangian of this theory has the form

$$\begin{aligned}\mathcal{L}=\sum_{l=1}^{m}&\left[-\frac{1}{4\pi}\frac{e^*}{|e^*|}a_{l\mu}\partial_\nu a_{l\lambda}\epsilon^{\mu\nu\lambda}+\frac{1}{g^2}(f_{l\mu\nu})^2+\frac{e^*}{2\pi}A_\mu\partial_\nu a_{l\lambda}\epsilon^{\mu\nu}\right]\\ &+\sum_{l=1}^{m}\left[\Phi_l^\dagger i(\partial_0+ia_{l0})\Phi_l+\frac{1}{2m}\Phi_l^\dagger(\partial_i+ia_{li})^2\Phi_l\right] .\end{aligned} \tag{56}$$

The fermion number current is given by

$$j_\mu=\sum_{l=1}^{m}\frac{1}{2\pi}\partial_\nu a_{l\lambda}\epsilon^{\mu\nu\lambda} \tag{57}$$

in the effective theory. Φ_l is a bosonic field that describes the hole excitations in the lth Landau level. Because of the Chern-Simons term, the hole described by Φ_l has fermionic statistics even though Φ_l is a bosonic field. The chemical potentials for Φ_l's are such that it costs finite energy to excite a hole.

According to the above discussion, we see that the effective theory for the fictitious fermions has a form as follows:

$$\mathcal{L}=\sum_{I=1,l=1}^{I=P,l=|m_I|}\left[-\frac{1}{4\pi}\frac{e_I}{|e_I|}a_{Il\mu}\partial_\nu a_{Il\lambda}\epsilon^{\mu\nu\lambda}+\frac{1}{g^2}(f_{Il\mu\nu})^2+\frac{e_I}{2\pi}A_\mu\partial_\nu a_{Il\lambda}\epsilon^{\mu\nu\lambda}\right] . \tag{58}$$

The hole excitations for the Ith fictitious fermion are described by

$$\mathcal{L}=\sum_{l=1}^{|m_I|}\left[\Phi_{Il}^\dagger i(\partial_0+ia_{Il0})\Phi_{Il}+\frac{1}{2m}\Phi_{Il}^\dagger(\partial i+ia_{Ili})^2\Phi_{Il}\right] . \tag{59}$$

Equations (58) and (59) are just the effective theory of the fictitious fermions. To obtain the effective theory of the electrons, we must recombine the fictitious fermions into electrons. This is achieved by imposing the constraint that the currents of the fictitious fermions are equal to each other, which in turn is equal to the electron current:

$$j_{I\mu}=\sum_{l=1}^{|m_I|}\frac{1}{2\pi}\partial_\nu a_{Il\lambda}\epsilon^{\mu\nu\lambda}=j_{J\mu}=j_{e\mu}\ . \tag{60}$$

Such a constraint can be easily realized in the effective theory by requiring that

$$\sum_{l=1}^{|m_I|}a_{Il\lambda}=\sum_{l=1}^{|m_J|}a_{Jl\lambda}\ . \tag{61}$$

Therefore not all $a_{Il\mu}$ are independent. There are only

$1+\sum_I(|m_I|-1)$ independent gauge fields. From Ref. 8 we see that the number of the independent gauge fields happens to be the number of the branches of edge excitation. Each independent gauge field corresponds to one branch of the edge excitations. In Sec. V we will derive those edge excitations directly from the effective theory (58).

The electrons are bound states of the fictitious fermions. The electron excitations are described by field $\Phi_e=\prod_I \Phi_{Il_I}$, where $1\le l_I \le |m_I|$. The Lagrangian for the electron excitations is given by

$$\mathcal{L}=\Phi_e^\dagger i\left[\partial_0+i\sum_{I=1}^{P} a_{Il_I0}\right]\Phi_e + \frac{1}{2m}\Phi_e^\dagger\left[\partial_i+i\sum_{I=1}^{P} a_{Il_Ii}\right]^2\Phi_e \ . \tag{62}$$

It is clear that the gauge fields generated by Φ_e automatically satisfy the constraint (61).

Now we are ready to ask what the allowed quasiparticle excitations are in the FQH state. In general, the quasiparticle excitations are described by the following effective Lagrangian:

$$\mathcal{L}=\Phi_q^\dagger i\left[\partial_0+i\sum_{I,l} q_{Il}a_{Il0}\right]\Phi_q + \frac{1}{2m}\Phi_q^\dagger\left[\partial i+i\sum_{I,l} q_{Il}a_{Ili}\right]^2\Phi_q \ . \tag{63}$$

The central issue is to determine the allowed charges q_{Il}. First, the charges should satisfy

$$\sum_{l=1}^{|m_1|} q_{Il}=q \ , \tag{64}$$

where q is a constant independent of the I. Equation (64) ensures that the gauge fields generated by Φ_q satisfy the constraint (61). Second, and more important, the phases induced by moving an electron around the quasiparticle must be multiples of 2π. Note that because of the Chern-Simons term, the charges q_{Il} induce flux of the gauge fields which causes a nonzero phase as the electron goes around the quasiparticle. We find that in addition to Eq. (64) the charges q_{Il} should also satisfy the condition

$$\sum_I \frac{e_I}{|e_I|} q_{Il_1}=\text{int} \ . \tag{65}$$

Notice that there are many different electron operators (similar to the electron operators in different Landau levels in the IQH states). This corresponds to the different choices of l_I. Therefore, (65) should be satisfied for arbitrary choices of l_I. The condition (65) has a simple physical interpretation in the microscopic theory. Equation (64) just implies that, after acting by the quasiparticle operator on the ground state, the resulting electron wave function is single valued. For example, we can act with the operator $\prod_i (z_i-z_0)^\alpha$ on a Laughlin wave function of a state with a filling fraction $1/m$. This operation creates a quasiparticle of charge α/m. The single-valuedness of the electron wave function requires α to be an integer. In this case the quasiparticle charge is quantized as multiplies of $1/m$. The quantization condition on α precisely corresponds to the quantization condition given by Eq. (65).

In the following we will solve Eqs. (64) and (65). Equation (65) implies that $q_{Il}-q_{Il'}=\text{int}$. Therefore, taking into account (64), we may write

$$b_{Il}=M_{Il}-\frac{1}{|m_I|}\sum_{l'} M_{Il'}+\frac{q}{|m_I|} \ , \tag{66}$$

where M_{Il} are integers. Substituting Eq. (66) into Eq. (65), we find that q must satisfy the equation

$$q=\nu\left[M_0+\sum_{Il}\frac{M_{Il}}{m_I}\right] \ , \tag{67}$$

where M_0 is an integer. Therefore, the quasiparticles are labeled by the integers M_0 and M_{Il}. The statistics of the quasiparticle can be obtained from the charge q_{Il}:

$$\theta=\pi\sum_{Il}\frac{e_I}{|e_I|}q_{Il}^2 = \left[\frac{q^2}{\nu}+\sum_I N_I^2\left[1-\frac{1}{m_I}\right]\right]\pi \ , \tag{68}$$

where $N_I=\sum_l M_{Il}$. The electric charge of the quasiparticle is given by q. Equations (67) and (68) completely agree with the results obtained in Sec. III. (Note that M_0 is redundant.)

Let us conisder a simple example. We choose $m_I=1$, $I=1,\ldots,p$ (p is even), and $m_{p+1}=-m$. The filling fractions for such states are $m/(mp-1)$. They included $\frac{2}{3}$, $\frac{2}{7}$, $\frac{2}{11}$, $\frac{3}{5}$, $\frac{3}{11}$, etc. The effective action takes a form as follows:

$$\begin{aligned}\mathcal{L}&=\sum_{l=1}^{l=m}\left[\frac{1}{4\pi}a_{l\mu}\partial_\nu a_{l\lambda}\epsilon^{\mu\nu\lambda}+\frac{1}{2\pi}A_\mu\partial_\nu a_{l\lambda}\epsilon^{\mu\nu\lambda}\right] - \sum_{l,l'=1}^{m}\frac{p}{4\pi}a_{l\mu}\partial_\nu a_{l'\lambda}\epsilon^{\mu\nu\lambda}\\ &=\sum_{l,l'=1}^{m}\frac{1}{4\pi}\Lambda_{ll'}a_{l\mu}\partial_\nu a_{l'\lambda}\epsilon^{\mu\nu\lambda}+\sum_{l=1}^{l=m}\frac{1}{2\pi}A_\mu\partial_\nu a_{l\lambda}\epsilon^{\mu\nu\lambda} \ ,\end{aligned} \tag{69}$$

where the matrix Λ has integral elements, which are given by

$$\Lambda_{lk}=\delta_{lk}-p \ . \tag{70}$$

Note that in Eq. (69) we have already solved the constraint (61). The fields $a_{l\nu}$ in Eq. (69) are all independent. The spectrum of the quantum numbers of the quasiparticles is given by

$$q=\frac{N}{mp-1}, \quad \theta=\left[1+\frac{p}{mp-1}\right]\pi N^2 \ . \tag{71}$$

In particular, for the $\nu=\frac{2}{3}$ FQH state ($p=2$, $m=2$), Λ is given by

$$\Lambda=\begin{pmatrix}-1 & -2\\ -2 & -1\end{pmatrix},$$

which has eigenvalues 1 and -3. After properly choosing the gauge fields, the effective theory of the $\nu=\frac{2}{3}$ FQH state takes the following simple form:

$$\frac{1}{4\pi}a_{1\mu}\partial_\nu a_{1\lambda}\epsilon^{\mu\nu\lambda}-\frac{3}{4\pi}a_{2\mu}\partial_\nu a_{2\lambda}\epsilon^{\mu\nu\lambda}+\frac{\sqrt{2}}{2\pi}a_\mu\partial_\nu a_{2\lambda}\epsilon^{\mu\nu\lambda}\ . \tag{72}$$

The charge $q=\frac{1}{3}$ quasiparticle has a statistics $\theta=\frac{5}{3}\pi$. The above effective action is consistent with the picture that the $\nu=\frac{2}{3}$ FQH state can be regarded as a $\nu=1$ IQH state plus a $\nu=-\frac{1}{3}$ FQH state (the minus sign in ν means the opposite signs of the charges).

V. EDGE STATES

In the previous sections we have considered the effective theory of the fractional quantum Hall states in the bulk of the sample. Let us now consider the excitations of this theory on the edge.

The QH states, as incompressible states, are very different from another kind of incompressible states—band insulators. The excitations in the band insulators always have finite energy gaps, even when they are on the boundaries of the insulators. But in the QH states, the excitations on the edges are always gapless because of the requirement of the gauge invariance.[13,15] These gapless edge excitations play a very important role in the understanding of many low-energy properties of the QH states.[13,14,16–18,23] The dynamical properties of the edge excitations are found to be described by the chiral Kac-Moody algebra (which is "one half" of the Tomonaga model).[15,19] The electrons on the edge are strongly correlated and form a new kind of states—chiral Luttinger liquid.[18] In this section we will derive the low-energy effective theory of the edge excitations directly form the bulk effective action of the FQH states. Our derivation demonstrates that the edge excitations are closely related to the bulk properties (or, more precisely, to the topological orders) in the FQH states. In addition, to recover the results in Refs. 8, 15, 18, and 19, we show that the FQH states in general contain both left-moving and right-moving edge excitations. In particular, the $\nu=\frac{2}{3}$ FQH state have one branch of left movers and one branch of right movers. We also discuss the correlation functions of the quasiparticles on the edges.

We will start from the dual form of the effective theory. This form is defined by Eqs. (58) and (61). In this case, the vortices (quasiparticles) of the theory are in one-to-one correspondence to the Wilson lines of the Chern-Simons theory. These Wilson lines represent world lines of the quasiparticles of the effective theory considered above. Thus our effective GL theory (24) is equivalent to the topological Chern-Simons theory, and quasiparticles are flux sources in this theory.[7]

Let us now consider the simplest case of the filling fraction $\nu=1/q$. Such a FQH state is described by U(1) Chern-Simons theory with the following action:[7]

$$S=\frac{q}{4\pi}\int a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}d^3x \tag{73}$$

[see Eq. (69) with $m=1$]. This Chern-Simons theory is called to have level q. The statistics of the quasiparticles are π/q. Suppose that our sample has a boundary. For simplicity we shall assume that the boundary is the x axis and the sample is the lower half-plane. The Chern-Simons action is not invariant under gauge transformations due to the boundary effects. We must choose boundary conditions for our gauge fields that cancel this noninvariance. For the Chern-Simons theory with the action given by Eq. (73), we obtain[24] an effective conformal theory on the edge with the action

$$S=S_{\rm conf}=\int\partial_t\phi\,\partial_x\phi\,dx\,dt\ . \tag{74}$$

Here $a_i=\partial_i\phi$ $(i=t,x)$. This result is derived by choosing the gauge condition $a_0=0$ on the boundary. This approach, however, has a setback. It is easy to see that a Hamiltonian associated with the action (74) is zero and that the boundary excitations described by Eq. (74) have no dynamics (i.e., their velocity is zero). Hence this action cannot be used to describe any physical edge excitations connected with the FQHE. The edge excitations in the FQH states always have finite velocities.

The appearance of finite velocities and of nontrivial dynamical of edge excitations is a boundary effect. The bulk effective theory defined by Eq. (58) does not contain the information about the velocities of the edge excitations. The reason is that the Chern-Simons action given by Eq. (73) contains no dimensional parameters and has a zero Hamiltonian. Therefore, the vortices in the bulk of the sample have no dynamics (or, equivalently, an infinite mass). Hence the notion of velocity has no meaning for these excitations. The inclusion of the Maxwell terms $f_{\mu\nu}^2$ makes the total Hamiltonian in the bulk of the sample nonzero. The vortices have finite gap and and acquire dynamics. This, however, does not help, since the Maxwell terms contain higher derivatives and cannot generate the linear dispersion relations on the boundary of the sample. The edge velocities in the QH states are actually determined by the edge potentials. To determine the dynamics of the edge excitations form the effective theory, we must find a way to input the information about the edge velocity. The edge velocities must be treated as the external parameters, and the problem is how to put these parameters in the theory.

Let us now note that the condition $a_0=0$ is not a unique choice of the boundary conditions that cancel gauge noninvariance on the edge. There are many other choices of boundary conditions different from $a_0=0$ that also cancel the gauge noninvariance on the boundary. The noninvariance of the Chern-Simons action on the boundary comes from the terms

$$\delta S=\delta a^i a^j\epsilon^{ij}\ , \tag{75}$$

where i,j labels the components of the gauge field in the x-t plane. It is easy to see that we can satisfy the condition $\delta S=0$ if we choose the boundary conditions in the form

$$a_\tau = a_0 + v a_x = 0 \ . \tag{76}$$

Here, a_x is the component of the vector potential parallel to the boundary of the sample, and v is a parameter which has a dimension of velocity.

It is convenient to choose new coordinates that satisfy

$$\tilde{x} = x - vt \ , \quad \tilde{t} = t, \ \ \tilde{y} = y \ . \tag{77}$$

In these coordinates the components of the gauge field are given by

$$\tilde{a}_{\tilde{t}} = a_t + v a_x , \ \ \tilde{a}_{\tilde{x}} = a_x , \ \ \tilde{a}_{\tilde{y}} = a_y \ . \tag{78}$$

It is easy to see that the form of the Chern-Simons action is preserved in the new coordinates:

$$S = \frac{q}{4\pi} \int d^3x a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} = \frac{q}{4\pi} \int d^3x \tilde{a}_{\tilde{\mu}} \partial_{\tilde{\nu}} \tilde{a}_{\tilde{\lambda}} \epsilon^{\tilde{\mu}\tilde{\nu}\tilde{\lambda}} \ . \tag{79}$$

This action can be rewritten as

$$S = \frac{-q}{4\pi} \int \tilde{a}_{\tilde{i}} \frac{\partial}{\partial \tilde{t}} \tilde{a}_{\tilde{j}} \epsilon^{\tilde{i}\tilde{j}} d^2\tilde{x}\, d\tilde{t} + \frac{q}{2\pi} \int \tilde{a}_{\tilde{t}} \partial_{\tilde{i}} \tilde{a}_{\tilde{j}} \epsilon^{\tilde{i}\tilde{j}} d^2\tilde{x}\, d\tilde{t} \ . \tag{80}$$

We recognize that $\tilde{a}_{\tilde{t}}$ is just a Lagrangian multiplier which enforces the constraint

$$\frac{\delta S}{\delta a_{\tilde{t}}} = \frac{q}{4\pi} \tilde{f}_{\tilde{i}\tilde{j}} \epsilon^{\tilde{i}\tilde{j}} = 0 \ . \tag{81}$$

This constraint is solved by introducing a new scalar field ϕ, $a_{\tilde{i}} = \partial_{\tilde{i}} \phi$ $(\tilde{i} = \tilde{x}, \tilde{y})$. Substituting this into Eq. (79), we get the edge action:

$$S = \frac{q}{4\pi} \int d\tilde{t}\, d\tilde{x}\, \partial_{\tilde{t}} \phi\, \partial_{\tilde{x}} \phi \ . \tag{82}$$

In terms of the old physical coordinates the above action acquires a form

$$S = \frac{q}{4\pi} \int dt\, dx (\partial_t + v \partial_x) \phi\, \partial_x \phi \ . \tag{83}$$

It is easy to see that the theory (83) contains only left-moving excitations. The equations of motion of the theory (83) have two solutions. The first solution has a form $\phi(x,t) = \phi(x - vt)$ and satisfies an equation $(\partial_t + v\, \partial_x)\phi = 0$. This solution corresponds to the left-moving chiral bosons. The second solution satisfies the equation $\partial_x \phi = 0$, i.e., ϕ is a function of time t only. Such solutions correspond to the gauge fields on the boundary $a_i = 0$ $(i = t, x)$. The physical Hilbert space on the edge is labeled by the fields a_x. Hence the second solution does not correspond to any physical excitations and shall be excluded. Another way to see that the second solution must be excluded is to demand the consistent quantization of the theory. Then, it is easy to see that if we take the solutions of the form $\phi(t)$ into account, the Poisson brackets will become degenerate.

After eliminating the unphysical degrees of freedom, it is straightforward to quantize the theory (83). [We need to use Dirac brackets in order to take into account a constraint $\pi = (q/4\pi)\partial_x \phi$.] The canonical momentum $\pi(x)$ is equal to $\pi = \delta L / \delta \phi_t = (q/4\pi)\partial_x \phi$. The coordinate ϕ and momentum π obey the commutation relations:

$$[\pi(x), \phi(y)] = \delta(x - y) \ , \quad [\pi(x), \pi(y)] = \frac{q}{4\pi} \delta'(x - y) \ , \quad [\phi(x), \phi(y)] = \frac{2\pi}{q} \mathrm{sgn}(x - y) \ . \tag{84}$$

The Hamiltonian of the theory (83) is given by

$$H = -\frac{qv}{4\pi} \int dx\, \partial_x \phi\, \partial_x \phi \ . \tag{85}$$

The Hilbert space contains only left-moving degrees of freedom (or right-moving degrees of freedom if $v < 0$). Equations (84) and (83) describe chiral bosons, i.e., free left- (or right-) moving phonons (edge density waves).

The velocity of the edge excitations are given by v, which enters into our theory through the gauge fixing condition. Note that the Chern-Simons action is gauge invariant only for those gauge transformations which are zero at the edge. Under those gauge transformations the gauge fixing conditions (76) with different v cannot be transformed into each other. They are physically inequivalent. This agrees with our result that v in the gauge fixing condition is physical and actually determines the velocity of the edge excitations.

The Hamiltonian is bounded from below only when $vq < 0$. The consistency of our theory requires v and q to have opposite signs. Therefore, the sign of the velocity (the chirality) of the edge excitations are determined by the sign of the coefficient in front of the Chern-Simons terms. This result implies that the $\nu = \frac{2}{3}$ FQH state described by (72) have two branches of the edge excitations with *opposite* velocities.

Note that this result is consistent with the following observation made in Ref. 25. Suppose that the Chern-Simons theory with the action (73) describes the left movers on the edge. Then, the Chern-Simons theory that can be obtained from Eq. (73) by the transformation $k \to -k$, $a(x) \to -a(-x)$ describes the right-moving degrees of freedom.

A relativistic form of the edge action can be obtained by choosing new coordinates as

$$v\tau = vt + x \ , \quad \bar{\tau} = -vt + x \ , \quad y = y \ . \tag{86}$$

We also choose the components of the gauge fields as

$$a_\tau = a_0 + v a_x \ , \quad a_{\bar{\tau}} = -v a_0 + a_x \ , \tag{87}$$

and a_y. Note that this transformation conserves the form $a_i dx^i$. The effective action in the new coordinates acquires the form given by Eq. (80) (after the change of x to $\bar{\tau}$). Integrating a_τ, we get the constraint $\tilde{f}_{\bar{\tau}y} = 0$. This constraint can be solved by introducing a bosonic field ϕ:

$a_{\bar\tau}=\partial_{\bar\tau}\phi$, $a_y=\partial_y\phi$. In terms of this field the action (80) becomes

$$S=\frac{q}{4\pi}\int(\partial_\tau\phi\partial_{\bar\tau}\phi)d\tau\, d\bar\tau \ . \tag{88}$$

Returning to the physical coordinates (t,x), we immediately see that all choices of the gauge condition $a_0+va_x=0$ $(v\neq 0)$ lead to the nonzero Hamiltonian on the edge and to the standard action of the relativistic boson:

$$S=\frac{q}{4\pi}\int dx\, dt\left[\frac{1}{v}\left[\frac{\partial\phi}{\partial t}\right]^2-v\left[\frac{\partial\phi}{\partial x}\right]^2\right] . \tag{89}$$

Here we explicitly wrote a speed of sound waves v. Note that when we quantize the theory (89), we must restrict ourselves to the same Hilbert space we used in order to quantize the action given by Eq. (83). Hence the theory describes only left movers. In the field theory language the action (89) is just the action of the $U(1)$ Wess-Zumino-Witten model.

The model has a conserved current $J^\alpha(x-vt)$ given by

$$J_\alpha=\sqrt{2\pi q}\,\epsilon^{\alpha\beta}\partial_\beta\phi,\quad \alpha,\beta=t,x \ . \tag{90}$$

The Fourie modes J_n of the current $J=J_x$ form a U(1) Kac-Moody algebra of level $k=q/2$:

$$[J_n,J_m]=kn\delta_{n+m} \ . \tag{91}$$

The Hamiltonian given by Eq. (85) can be easily rewritten in terms of the generators of the Kac-Moody algebra:

$$H=\frac{2\pi v}{L}\sum :J_{-n}J_n: \ , \tag{92}$$

where L is the length of the edge. Equations (92) and (91) describe one-dimensional free phonons. Thees phonons are created by the generators J_n of the Kac-Moody algebra and propagate only in one direction.[18,19] Let us note that the analogous algebra describes density fluctuations in the case of Tomonaga model of the one-dimensional interacting electronic gas.[26]

We see that the results obtained in Refs. 15, 17, and 19 can be derived from the effective Chern-Simons action of the FQH states.

The quantum field theory defined by the action (83) or (89) has a conformal invariance. It is easy to see that the Hamiltonian H given by Eq. (92) is equal to the generator L_0 of the corresponding Virasoro algebra. (See, e.g., Ref. 27 for the detailed description of the properties of the conformally invariant systems.) To find the structure of low-lying excitations, we can use the general results of Ref. 28 where the author established the connection between the conformal field theories and corresponding $(1+1)$-dimensional quantum systems.

The low-energy excitations of the theory are in one-to-one correspondence with the primary vertex operators of the conformal theory and their descendants. The low-energy excitations are gapless with the energies given by

$$E_n=\frac{\Delta_m+n}{L}2\pi v \ , \tag{93}$$

where L is the length of the boundary and $\Delta_m=m^2/2q$ are the conformal dimensions. Note that the quasiparticles with $m=0$ correspond to the density waves and can be created without adding external charge to the system (they are created by the Kac-Moody current J). On the other hand, in order to create the quasiparticles with $m>0$, we must add to the system a finite charge m/q or to create a particle-antiparticle pair. The propagator of the quasiparticles is given by a correlation functions of two corresponding vertex operators of the conformal theory and has a power-like behavior:

$$G(z)\sim\frac{1}{(z-vt)^{\theta/\pi}} \ . \tag{94}$$

Here, $\theta=m^2/2q+n$. Note that $\theta\,\mathrm{mod}2\pi$ is the statistics of the corresponding quasiparticle in the bulk.

We conclude that in the FQH state with filling fraction $1/q$ has one branch of edge excitations. The quasiparticles on the edge have energies given by Eq. (93) and propagator given by Eq. (94).

In the important case of an electron, i.e., $m=q$ (this means that the electron can be composed from m elementary quasiparticles), we get the scaling behavior obtained in[18]

$$G(x)\sim\frac{1}{x^m} \ . \tag{95}$$

This result has been confirmed by numerical calculations using Laughlin wave functions for the case $\nu=\frac{1}{2}$.[29]

We thus see that the dynamical properties of edge excitations directly follow from the topological theory that describes the quasiparticles inside the bulk.

All these results can be extended to the case of the general FQHE states proposed in Secs. II–IV. The relevant Chern-Simons theory was constructed in Sec. IV. From the general analysis given above, we immediately see that there are precisely $\sum_I(|m_I|-1)+1$ branches of edge excitations. Each branch corresponds to an independent gauge field that induces a chiral boson theory on the edge. The full-edge Hamiltonian is equal to

$$H=\sum_{n,s}\frac{2\pi v_s}{L}j_n^s j_n^s \ , \tag{96}$$

where index s labels different branches. There is again a one-to-one correspondence between the quasiparticles on the bulk and on the edge. The propagator of quasiparticle is given once again by

$$G(x)\sim\frac{1}{x^{\theta/\pi}} \ , \tag{97}$$

where θ is given by Eq. (40). A detailed proof of these statements and a description of edge excitations of the general FQHE states constructed in Secs. II–IV will be given elsewhere.

VI. CONCLUSIONS AND DISCUSSION

We have found that the FQHE states constructed by Jain have a natural description in terms of the GL effective theory. The relevant effective theories are given by Eqs. (8) and (24). These theories also have a dual description given in Sec. IV. The effective theories we have constructed permit the one to find a structure of low-lying excitations of the FQHE states constructed by Jain and to calculate the charge and statistics of these excitations. We have shown that the effective theories given by Eqs. (8), (24), and (58) [with a constraint (61)] describe the same FQHE state, although they were constructed using different physical approaches.

Our approach provides an easy way to calculate the charge and statistics of all low-lying quasiparticles of the theory, even without knowing the exact form of the wave functions of the excitations. The charges and statistics of the quasiparticles in the generic FQH states are given by Eqs. (40) and (42) (where m_I can be negative).

Let us note that in the microscopic theory developed in Refs. 3 and 4, it is relatively easy to calculate the quantum numbers of the quasiparticle induced by turning on a unit flux. The wave function of such a quasiparticle is given by (47) and (48). In this case our results for the charge and statistics [see Eq. (51)] coincide with the prediction from the microscopic theory. For the quasiparticle with the electric charge ν/m_I the statistics is given by Eq. (43). This result differs from that obtained in Ref. 4 by a term $1-1/m_I$. For the IQH states our theory also reproduces the standard results.

The effective-field theories studied in this article describe the FQHE states constructed by Jain. However Jain's states may not include all the possible FQH states. Let us given an example of the states that cannot be realized using Jain's scheme. Namely, we shall construct the FQH states with $\nu=\frac{2}{5}$ by assuming the electrons form charge $2e$ pairs. The effective filling factor for the charge $2e$ bosons is $\nu^*=\frac{1}{4}\nu=\frac{1}{10}$ (a factor $\frac{1}{2}$ comes from the increase of the charge, and the other factor $\frac{1}{2}$ comes from the decrease of the particle density). The FQH state of the electron pairs can be described by the Laughlin wave function:

$$\psi \sim \sum_{i<j} (Z_i - Z_j)^{10} \exp\left[-\frac{1}{4}\sum_i 2e\frac{B}{\hbar c}|Z_i|^2\right], \tag{98}$$

where Z_i are the center-of-mass coordinates of the electron pairs. The GL effective theory of the FQH state (98) is given by

$$\mathcal{L}_1 = i\phi^\dagger(\partial_t + ia_0 + i2eA_\mu)\phi + \frac{1}{2m}\phi^\dagger(\partial_i + ia_i + i2eA_i)^2\phi + \frac{1}{10}\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} - V(|\phi|), \tag{99}$$

where a_μ is the "fictitious" U(1) gauge field and ϕ is the field corresponding to the charge $2e$ electron pairs. The charge $e/5$ quasiparticles in the FQH state described by Eq. (98) [or (99)] have fractional statistics $\theta=\pi/10$. The edge states for such a FQH state contain one branch. It costs finite energy to create an electron even at the edges due to the pairing between electrons. However, it costs infinitesimal energy to create an electron pair on the edges. The electron pair creation operator Ψ can be shown to have the following propagator:

$$\langle \Psi^\dagger(x,t)\Psi(0,0)\rangle \propto \left[\frac{1}{x-vt}\right]^{10}, \tag{100}$$

along the edges. The $\nu=\frac{2}{5}$ FQH states described by Eq. (99) are definitely different from the ones described by Eq. (8) or (24).

We also showed that the effective theory (24) has a natural dual description. There are $\sum_I(|m_I|-1)+1$ independent gauge fields in the dual theory. The corresponding Chern-Simons theory naturally gives rise to the edge excitations. The number of the branches of edge excitations is equal to the number of the independent gauge fields. The dynamics of edge excitations in our approach arises from the need to impose a special gauge condition on the boundary of the sample to preserve gauge invariance. Other approaches to edge excitations can be found in Refs. 13–19.

The propagators of the quasiparticles have a powerlike behavior:

$$G(x) \sim \frac{1}{x^{\theta/\pi}}, \tag{101}$$

where θ is the statistics of the quasiparticles given by Eq. (43). In the case of the filling fraction $\nu=1/\sum_k(1/m_k)$, where all m_k are positive, and all edge excitations are left movers and have definite chirality. In the more general case when some of the integers m_k can be negative, the edge excitations contain both left and right movers.

Let us note that in our approach both theories in the bulk and on the edge are described by the same action. The action (83) [or (89)] of the edge theory is equal to the action (73) of the Chern-Simons theory in the bulk. As it was shown in Sec. V, the latter depends only on the values of the gauge fields on the boundary of the sample. This naturally suggests that both the bulk and edge states are described by the same microscopic wave functions. This in turn means that the scaling behavior of the quasiparticle propagators given by Eq. (101) can be obtained using microscopic wave functions. The numerical calculations carried out in Ref. 29 for the case of the filling fraction $\nu=\frac{1}{2}$ indeed support this suggestion.

Finally, we must note that it is yet not clear what is the relation between the FQHE states studied in this paper and those obtained by the standard hierarchy construction of Refs. 1 and 2. In order to answer this question, we must find the charges and statistics of the quasiparticles in the states constructed in Refs. 1 and 2.

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