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Persistent currents in mesoscopic metallic rings: Ensemble average

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A magnetic flux induces a persistent current in a mesoscopic metallic ring with period ϕ_0, the flux quantum. We show that the ensemble average of the current (on disorder or on the number of particles) is finite with a $\phi_0/2$ periodicity. We find from numerical calculations that in the diffusive regime, this average is reduced compared with the typical current ev_D/L by a factor $(M_{\rm eff})^{-1/2}$, where v_D is the diffusion velocity along the ring and $M_{\rm eff}$ is the number of conducting channels in the ring. The recent experimental discovery of the persistent currents, which shows the $\phi_0/2$ periodicity, is in reasonable agreement with our results.

Some years ago, Büttiker, Imry, and Landauer suggested the existence of a persistent current in a ring of mesoscopic size pierced by a magnetic flux.[1] This current is an *equilibrium* property of the ring, given by the flux derivative of the total energy of the ring. There has recently been a growing experimental interest in the detection of this persistent current. This quest has stimulated theoretical descriptions of this current.[2-11] One of the most important goals has been to go beyond the simple picture of a one-dimensional (1D) ring[1] and to study this current in a ring of finite width.[5,7] Three regimes can then be distinguished, depending on the strength of disorder W. In the ballistic regime, the perimeter L of the ring is smaller than the elastic mean free path $l_e(W)$. In the localized regime, the length L is larger than the localization length ξ. In between, there is the metallic or diffusive regime when $l_e < L < \xi$. The localization length has been found to be of the order of Ml_e, M being the number of channels (at least in the 3D case).[12] Recently, Cheung, Riedel, and Gefen found the typical current in these three regimes.[5-7] Let us summarize their results. In the ballistic case, the typical current $I_{\rm typ}=(\langle I^2\rangle)^{1/2}$ is of the order of $\sqrt{M}I_0$. $\langle\cdots\rangle$ is the ensemble average on different realizations of disorder or number of particles. I_0 is the one-dimensional current. It is given by $I_0=ev_F/L$, v_F being the Fermi velocity. In a metal of mesoscopic size, I_0 is of the order of 10^{-7} A. In the localized case, the typical current decreases exponentially with the length L and inversely with the number of channels $I_{\rm typ}=(I_0/M)\exp(-L/2\xi)$. In the metallic regime it is proportional to the elastic mean free path and *independent of the number of channels*. $I_{\rm typ}=ev_D/L$, where $v_D=v_Fl_e/L$ is the diffusion velocity along the ring. The magnetic moment for a typical mesoscopic metallic ring is thus expected to be, at most, 10^2–10^3 Bohr magnetons.

Experimentally, two kinds of investigations are possible. One is the study of a single loop. It requires an extremely sensitive detector. The other is the detection of the magnetic moment of a large number $\mathcal{N}$ of independent loops. This requires a less sensitive detector but raises the problem of ensemble averaging. Stimulated by the second type of experiment, we have recently studied the ensemble average of the persistent current.[11] The current of a single ring has *a priori* a random sign which depends on the microscopic configuration of the disorder. As a result, one would naively expect the magnetic response of $\mathcal{N}$ rings to be zero (or of the order of $\sqrt{\mathcal{N}}$). It has been shown that it was not the case. *The ensemble average is finite (of order* $\mathcal{N}$*) and its periodicity is* $\phi_0/2$ *instead of* ϕ_0. Although the derivation of this result is simple in 1D,[5] a careful analysis was needed in the case of a multichannel ring. In the latter case, we derived this result from both analytical arguments and numerical simulations of the Anderson model in the two extreme regimes.[11] This important result is a central point concerning the interest of the $\mathcal{N}$-ring experiments. We showed the importance of fixing the number of electrons or the chemical potential μ in the calculation of the persistent current. Our results were obtained with a fixed number of electrons in each ring (this number being different from one ring to another). We found that, in the zero disorder limit, $\langle I\rangle$ is of the order of I_0, independent of the number of channels. In the strong disorder limit, $\langle I\rangle$ is of the order of the square of the typical current $I_{\rm typ}^2/I_0$. Calculations performed at fixed chemical potential give different results.[7,8] Averaging on the disorder at fixed μ yields in the diffusive and localized regimes a ϕ_0 periodic current with amplitude $I_0\exp(-L/l_e)$. This current vanishes when averaging on μ.

It is, of course, of interest to study the intermediate case, the metallic regime, and to estimate the average current in this case. This is essential to appreciate the interest of the $\mathcal{N}$-ring experiments and this is the main goal of this paper. The purpose of the present work is thus to study in detail the average current in the metallic regime.

We have used the Anderson model[13] to simulate the disorder in a ring of length L and finite width. The transfer term is taken as a constant t between first neighbors. The field effect is simply to change the boundary condition along the ring so that the transfer term gets a phase factor $\exp(2i\pi\phi/\phi_0)$ after one loop along the ring. Open boundary conditions are taken in the two other directions. The disorder is given by a random choice of the on-site energy between $-W/2$ and $W/2$. We are interested in the flux dependence of the total current $I(\phi)$ as a function of L, the number of channels M, and the number of electrons N in a ring. The current carried by each electron is $i_n = \partial E_n/\partial\phi$. At $T=0$ K, the total current in a ring with N electrons is thus $I_N(\phi)=\sum_{n=1}^{N} i_n(\phi)$.

In a previous paper, we studied small size systems. This allowed us to compute the persistent current in the ballistic and in the localized regimes. To investigate the diffusive regime ($l_e \ll L \ll \xi$) and study scaling in this regime, we must have the localization length ξ as large as possible compared to the elastic mean free path l_e. Since $\xi = M l_e$, this requires the study of systems with as many channels as possible. We have studied 3D systems of size $L\times\sqrt{M}\times\sqrt{M}$ with L up to 128 and M up to 100. For such large systems we have used a program operating on a Cray Research, Inc. Cray-2 computer based on a Lanczos algorithm well adapted to the diagonalization of sparse matrices.[14] We discuss the currents at zero temperature. From the harmonics expansion of the eigenvalues $E_n(\phi/\phi_0)=\sum_{p=0}^{\infty}\lambda_n^{(p)}\cos(2p\pi\phi/\phi_0)$, one obtains the harmonics $i_n^{(p)}$ of the band current $i_n^{(p)}=2p\pi\lambda_n^{(p)}/\phi_0$ and the harmonics $I_N^{(p)}$ of the total current from summation. The typical value of $\lambda_{\rm typ}^{(p)}=[\langle(\lambda_n^{(p)})^2\rangle_n]^{1/2}$, where $\langle\cdots\rangle_n$ is the average on the levels, is shown in Fig. 1. In the following, we focus our discussion on the two first harmonics of the currents. For this purpose we have computed the quantities $i_n^{(1)}=\pi[E_n(0)-E_n(\frac{1}{2})]/\phi_0$ and $i_n^{(2)}=\pi[E_n(0)+E_n\times(\frac{1}{2})-2E_n(\frac{1}{4})]/\phi_0$, where $E_n(\phi/\phi_0)$ is the energy of a single electron. Since the typical values of the $\lambda_n^{(p)}$ decay faster than $1/p$, these expressions yield good approximations of, respectively, the first and the second harmonics of the currents. The total currents $I_N^{(1)}$ and $I_N^{(2)}$ are then obtained from summation on the number of electrons.

Figure 2 shows the variation with N of the quantities $I_N^{(1)}$ and $I_N^{(2)}$ in the (a) diffusive and (b) localized regimes. It is seen that $\langle I_N^{(1)}\rangle_N$ is zero and $\langle I_N^{(2)}\rangle_N$ is finite and positive. $I_N^{(2)}$ takes less and less negative values when the disorder increases. Neglecting other harmonics, the average current $\langle I\rangle$ (which is periodic in $\phi_0/2$) is simply equal to $\langle I_N^{(2)}\rangle_N$. We also discuss the typical values of the first and second harmonics defined as $I_{\rm typ}^{(1)}=[\langle(I_N^{(1)})^2\rangle]^{1/2}$ and $I_{\rm typ}^{(2)}=[\langle(I_N^{(2)})^2\rangle]^{1/2}$. Note that this definition is only meaningful in the diffusive regime where we find a normal distribution for the currents. On the other hand, in the localized regime we find a lognormal distribution which should be characterized by the variance of the logarithm of the current. Here we will concentrate our discussion on the diffusive regime. Although the averages presented in this work are made on the number of electrons, we have checked that averaging at the same time on the disorder configurations (D) and on the number of particles (N) lead to the same result $\langle\cdots\rangle_N=\langle\cdots\rangle_{N,D}$. Now we discuss the dependence of the quantities $I_{\rm typ}^{(1)}$ and $\langle I\rangle$ with the different parameters of the ring.

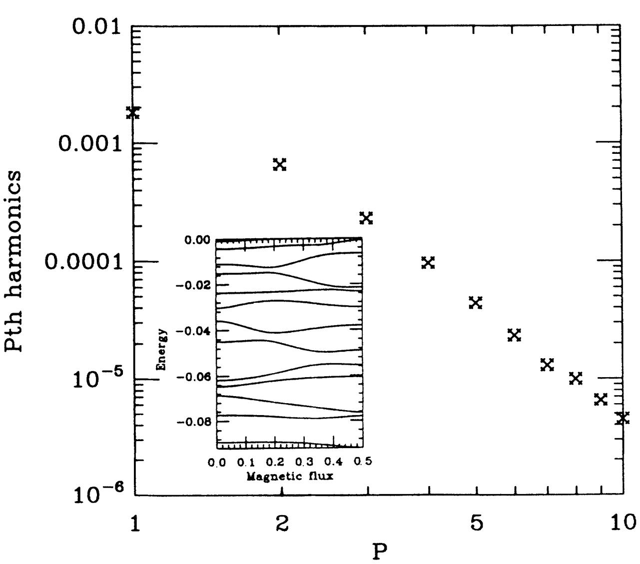

FIG. 1. Typical value of the pth harmonic $\lambda_{\rm typ}^{(p)}$ of the energy levels vs p, with size $64\times4\times4$ and $W/t=2$. Inset: flux dependence of the energy levels. ϕ is in ϕ_0 units.

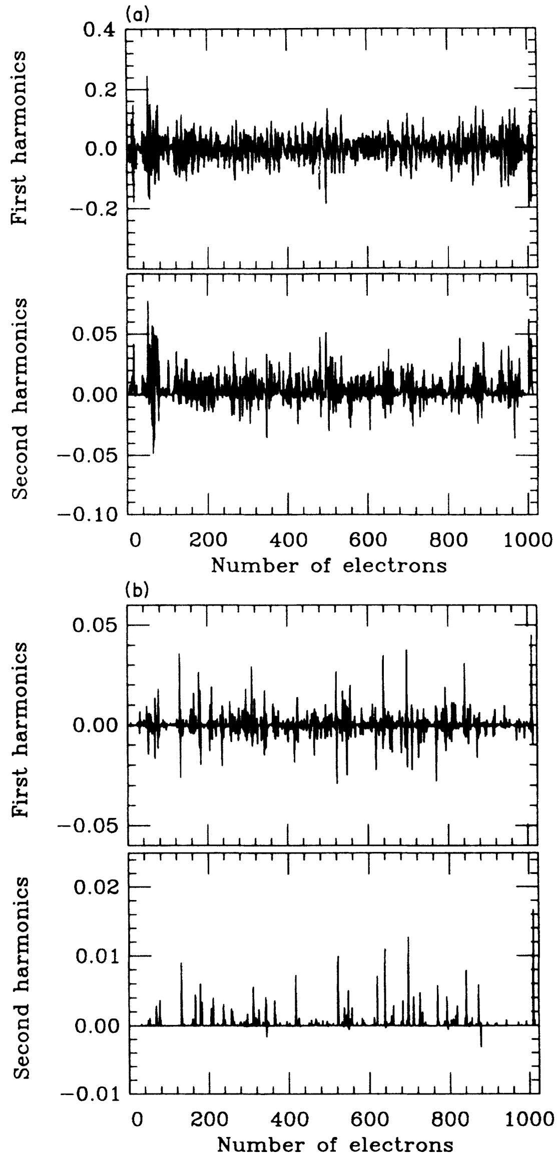

FIG. 2. Variation with N, number of electrons in the band, of the first $[I_N^{(1)}]$ and second $[I_N^{(2)}]$ harmonics of the current (a) in the diffusive regime ($W/t=3$), and (b) in the localized regime ($W/t=6$), for a size $64\times4\times4$.

Figure 3 shows the typical first and second harmonics of the current versus disorder W for various lengths and number of channels. The three regimes are easily visible in this picture. In the diffusive regime, both typical harmonics do not depend on the number of channels and vary as $1/W^2$ and thus as l_e (according to elementary scattering theory, $l_e \propto 1/W^2$) in agreement with the results of Ref. 7. More precisely, using the relation $l_e = At^2/W^2$, which results from the Born approximation that is valid when W is not too strong, we obtain the following results: $I_{\rm typ}^{(p)} = K^{(p)} I_0 l_e/L$, where $K^{(1)} = (25 \pm 2)/A$ and $K^{(1)}/K^{(2)} = 3.5 \pm 0.3$. The ratio of these typical values is a universal number, independent of the dimensions of the ring. The onset of the localized regime shows up as a departure from this power law to an exponential decay. This crossover occurs at a value of $1/W^2$ which increases linearly as L/M. We find that the crossover occurs just when $L \approx \xi$ and $L \approx 2\xi$, respectively, for the first and second harmonics.

The most striking and original result concerns the behavior of the average current $\langle I \rangle$. In the diffusive regime, we have found that it varies as $1/W^x$, where $x = 1 \pm 0.1$. It is thus proportional to $(l_e)^{1/2}$. Contrary to the typical current which is independent of M, it also decreases with the number of channels like $\sqrt{M}$. More generally, Fig. 4 shows that the average current varies like $I_0(l_e/ML)^{1/2}$. This result is different from those of Refs. 7 and 10, which performed averages on disorder at fixed chemical potential and found an exponential decay of $\langle I \rangle$ vs L/l_e. We

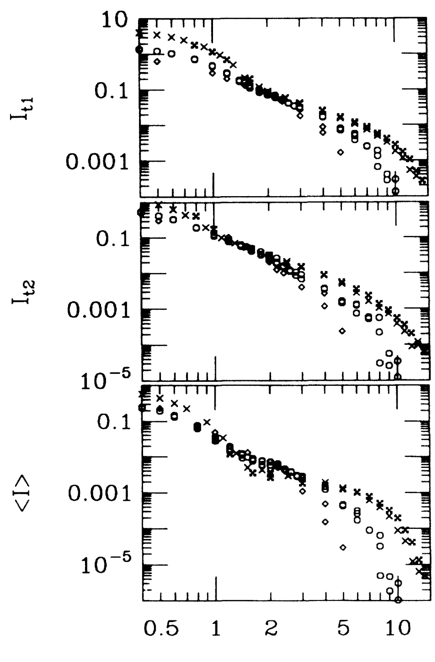

FIG. 3. Typical first $I_{\rm typ}^{(1)}$ and second $I_{\rm typ}^{(2)}$ harmonics of the current and average current $\langle I \rangle$ vs disorder for various sizes of the ring $64 \times \sqrt{M} \times \sqrt{M}$. Squares, $\sqrt{M} = 2$; circles, $\sqrt{M} = 4$; crosses, $\sqrt{M} = 8$; and stars, $\sqrt{M} = 10$. Note that in the diffusive regime, the typical currents decay as $1/W^2$ and that the average current decays as $1/W$.

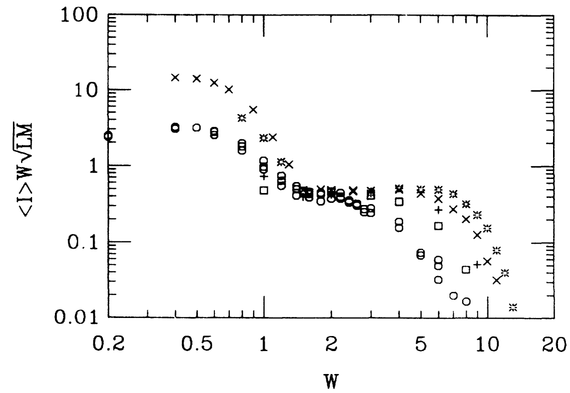

FIG. 4. Plot of the quantity $\langle I \rangle \times \sqrt{ML} \times W$ vs disorder W (in t units) for various sizes of the ring. Circles, $64 \times 4 \times 4$; crosses, $64 \times 8 \times 8$; squares, $128 \times 8 \times 8$; stars, $64 \times 10 \times 10$; and pluses, $128 \times 10 \times 10$. Note that $\langle I \rangle \times \sqrt{ML} \times W$ is independent of W only in the diffusive regime.

find that the typical and average current can be related in the diffusive regime through

$$\langle I \rangle = C I_{\rm typ}^{(1)}/(M_{\rm eff})^{1/2} .$$

The dimensionless coefficient C is $(0.02 \pm 0.005)\sqrt{A}$. $M_{\rm eff}$ is the effective number of conducting channels which is reduced compared to the number of channels: $M_{\rm eff} = Ml_e/L$.[7,15] In a typical mesoscopic metallic ring this number is of the order of 100. So the average current is only reduced by a factor of 10 compared with the typical current.

Our numerical simulations also show that the average current is also related to the typical first harmonic of the single level current by $\langle I \rangle \propto M \langle (i^{(1)})^2 \rangle$ (the currents are expressed in I_0 units). We have demonstrated a similar relation analytically in Ref. 11 using a perturbation expansion of the Anderson model in powers of t that is valid in the localized regime. (In this regime, the average has to be carried out in a different way because the current distribution is different.) We find here that the relation is valid whatever the disorder strength. We do not yet have any explanation why it should also hold in the diffusive regime. At this point it is worth noting that $i_{\rm typ}^{(1)} = [\langle (i^{(1)})^2 \rangle]^{1/2}$ is significantly different from the typical value of the band current $i_{\rm typ}$ at constant flux (e.g., $\phi = \phi_0/4$) computed in Ref. 7. We get from out numerical simulations $i_{\rm typ}^{(1)} = [\langle (i^{(1)})^2 \rangle]^{1/2} \sim I_{\rm typ}^{(1)}/M_{\rm eff}^{3/4}$, while $i_{\rm typ} \sim I_{\rm typ}^{(1)}/\sqrt{M_{\rm eff}}$. This difference results from the content in harmonics of the band current.[16]

To summarize, the main purpose of this work was to estimate the amplitude of the average current in the metallic regime. We find that it varies as a *power law* of the mean free path,

$$\langle I \rangle \sim D \frac{I_0}{\sqrt{M}} \left(\frac{l_e}{L} \right)^{1/2} ,$$

and not exponentially as obtained in the grand-canonical ensemble,[5,7,10] $D = 2CK^{(1)} \sim 1/\sqrt{A}$.[17] Contrary to the typical current, this average current varies with the number of channels. It can be written as $\langle I\rangle \sim I_{typ}/(M_{eff})^{1/2}$, where M_{eff} is the number of conducting channels. This result should motivate the search for persistent current in "many rings" experiments where we expect the magnetic moment to be proportional to $\mathcal{N}\langle I\rangle$ with period $\phi_0/2$, rather than $\sqrt{\mathcal{N}} I_{typ}$ with period ϕ_0. Indeed, during the completion of this paper, the experimental discovery of the persistent currents in an assembly of $\mathcal{N}\sim 10^7$ disconnected mesoscopic copper rings has been achieved and the period $\phi_0/2$ has been found.[18] We think that this is a remarkable confirmation of the results given in this Rapid Communication. Let us now compare our result with the experimental data. For each of these rings, we estimate the number of channels as $M = S\pi/\lambda_F^2 \sim 17\,000$. S is the transverse section of the ring and λ_F is the Fermi wavelength. Using $l_e \sim 200$ Å and $L\sim 2$ μm, we get $\langle I\rangle/I_0 \sim 8\times 10^{-4}/\sqrt{A}$. A has been estimated in 1D ($A\sim 105$) and in 2D ($A\sim 10$–15).[19] In 3D, using Born approximation $\hbar/\tau = 2\pi n(\varepsilon_F)W^2$, we get $A\sim 4$. With this latter value our estimate is less than 1 order of magnitude smaller than the experimental estimate.[18] Given the fact that the values of the numerical coefficients are probably model dependent, the amplitude of the measured average is in reasonably good agreement with our predictions.

After completion of this work, we learned that electron-electron interactions yield also a finite ensemble average with periodicity $\phi_0/2$, *even in the grand-canonical ensemble.*[20] Experiments performed on a connected array of rings would enable us to distinguish between the two different physical mechanisms.

Note added. From a recent work by Imry[21], the numerical ratio between $\langle I\rangle$ and $\langle i_1^2\rangle$ can be determined: $\langle I\rangle = (M/2\pi)\langle i_1^2\rangle/I_0$. This ratio agrees very well with our numerical calculations.

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