

Galvanomagnetic phenomena and surface roughness in thin metallic films

D. Calecki

*Groupe de Physique des Solides, Universités Paris 7 et Paris 6, 2 place Jussieu,
75251 Paris CEDEX, France*

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We study the galvanomagnetic properties of an electron gas in quasibidimensional structures such as thin films, submitted to a magnetic induction perpendicular to their surfaces. The confinement generates electronic quantum states that are gathered in a subband scheme. We determine the electron distribution functions in the different subbands by solving the appropriate set of Boltzmann equations. Then we derive the general expressions of transport coefficients and discuss the variations of the Hall effect and magnetoresistance with film thickness in the special case in which electrons are scattered by impurities and surface roughness. In thin metallic films where the correlation length describing surface roughness is less than the electron Fermi wavelength, we find a Hall constant proportional to thickness d and a magnetoresistance that follows a d^6 law.

I. INTRODUCTION

The effect of surfaces on the electrical properties of thin metallic films is a very old problem; Fuchs and Sondheimer¹ were the first to incorporate surfaces via boundary conditions in a classical Boltzmann equation to determine the electron distribution function and the variations of conductivity with film thickness. Their theory has been improved many times² but only a quantum treatment can account for size effects appearing when the electron mean free path becomes higher than the film thickness. Quantum-size effects were studied first by Sandomirskii³ but he assumed perfect plane surfaces. The surface roughness was introduced in a quantum theory by Prange and Nee⁴ for interpreting surface impedance and skin effect in the presence of high magnetic fields. More recently Tesanovic *et al.*⁵ and, with a similar theory, Trivedi and Aschroft,⁶ examined the behavior of the electrical conductivity by incorporating deviations from ideal plane surfaces in boundary conditions on the electron Hamiltonian. We⁷ have also studied this problem by introducing the effect of surface roughness as a perturbation of a perfect Hamiltonian describing an electron moving between the two plane surfaces; we found that it was equivalent to a perturbation localized on the ideal surfaces.

The influence of surface roughness on galvanomagnetic phenomena such as Hall effect or magnetoresistance has never been investigated either theoretically or experimentally. The purpose of the present work is to extend the ideas developed in Ref. 7, introducing a magnetic induction \mathbf{B} perpendicular both to the film surfaces and the applied electric field. We assume B sufficiently small for neglecting the Landau level quantization. We want to obtain variations of the Hall and magnetoresistance coefficients with the film thickness. These variations are governed not only by the electron wave functions shape but also by the number of filled subbands. Moreover we want to determine how galvanomagnetic effects depend on the roughness parameters such as the root-mean-

square deviations from the average thickness and the roughness correlation length.

In Sec. II we introduce the model describing an electron in a metallic film submitted to an electric field \mathbf{E} and a magnetic induction, respectively, parallel and perpendicular to its surfaces. Electrons, scattered by defects such as impurities and surface roughness, make transitions between quantum states of a subbands set. Equations satisfied by the different electron distribution functions associated with each subband are derived. In Sec. III we solve the preceding set of equations and we discuss the validity of the notion of relaxation time. Then, in Sec. IV, we calculate the general expressions of the transport coefficient of an electron gas filling the subband set; particular attention is paid to Hall and magnetoresistance coefficients. The results are applied to a film with perfect surfaces but with embedded impurities in Sec. V. In Sec. VI we explain the model describing scattering by surface roughness; we exhibit the variations of the Hall constant and of the magnetoresistance with thickness. Detailed results are given in the limit of small roughness correlation length as compared with the electron Fermi wavelength. We conclude in Sec. VII with some comments.

II. QUANTUM AND STATISTICAL BACKGROUND

In an ideal quasibidimensional structure, the motion of a charge carrier is separated into two independent parts. The first one, which is parallel to both limiting structure plane surfaces, is a free-particle motion. The associated wave function is a normalized plane wave, $S^{-1/2}e^{ik \cdot \rho}$, and corresponds to a kinetic energy $\hbar^2 k^2 / 2m$; ρ is the projection of the carrier position vector \mathbf{r} onto a surface plane and \mathbf{k} is a two-dimensional wave vector quantized by the usual periodic conditions; S is the sample surface area and m is the carrier effective mass. The second part of the motion is along the z axis, perpendicular to both surfaces, and is determined by a unidimensional potential $V(z)$ which confines carriers between the two parallel surfaces. The associated localized wave functions and

energies are, respectively, labeled $\zeta_\nu(z)$ and ε_ν ($\nu=1,2,\dots,\nu_{\max}$). Thus the carrier Hamiltonian in this ideal quasibidimensional structure is simply

$$H_0 = \frac{P_x^2}{2m} + \frac{P_z^2}{2m} + V(z) \quad (1)$$

and we have, for the stationary states and energies

$$H_0|\nu\mathbf{k}\rangle = \varepsilon_{\nu\mathbf{k}}|\nu\mathbf{k}\rangle$$

with

$$\langle \mathbf{r}|\nu\mathbf{k}\rangle = \psi_{\nu\mathbf{k}}(\mathbf{r}) = S^{-1/2} e^{i\mathbf{k}\cdot\rho} \zeta_\nu(z) \quad (2)$$

and

$$\varepsilon_{\nu\mathbf{k}} = \varepsilon_\nu + \frac{\hbar^2 k^2}{2m}. \quad (2')$$

The dispersion law $\varepsilon_{\nu\mathbf{k}}$ defines the parabolic subband ν . We observe that ε_ν depends on d .

In a real quasibidimensional structure, carriers interact with different defects: impurities, phonons, surface roughness, etc. This interaction produces transitions between states $|\nu\mathbf{k}\rangle$; in the Born approximation framework, we can introduce a transition probability per unit time $W_{\nu,\nu'}(\mathbf{k},\mathbf{k}')$ for a carrier to pass from the state $|\nu\mathbf{k}\rangle$ of subband ν to the state $|\nu'\mathbf{k}'\rangle$ of subband ν' .

When carriers are in thermal equilibrium at temperature T , the mean number of carriers in a state $|\nu\mathbf{k}\rangle$ is given by the Fermi-Dirac distribution function $f_0(\varepsilon_{\nu\mathbf{k}}) = [1 + \exp(\varepsilon_{\nu\mathbf{k}} - \varepsilon_F)/k_B T]^{-1}$. Here ε_F is the Fermi energy which depends on the volume carrier density n and also on the film thickness d . It is deduced from the relation

$$n = \sum_{\nu=1}^{\nu_{\max}} n_\nu = \sum_{\nu=1}^{\nu_{\max}} \int_0^\infty \mathcal{N}_\nu(\varepsilon) f_0(\varepsilon) d\varepsilon, \quad (3)$$

where n_ν is the density of electrons filling subband ν , while

$$\mathcal{N}_\nu(\varepsilon) = \frac{Sm}{\pi\hbar^2} Y(\varepsilon - \varepsilon_\nu) \quad (4)$$

$$-\frac{e}{\hbar} \left[\mathbf{E} + \frac{\hbar\mathbf{k}}{m} \times \mathbf{B} \right] \cdot \nabla_{\mathbf{k}} f_\nu(\mathbf{k}) = \sum_{\nu'=1}^{\nu_{\max}} \sum_{\mathbf{k}'} \{ W_{\nu',\nu}(\mathbf{k}',\mathbf{k}) f_{\nu'}(\mathbf{k}') [1 - f_\nu(\mathbf{k})] - W_{\nu,\nu'}(\mathbf{k},\mathbf{k}') f_\nu(\mathbf{k}) [1 - f_{\nu'}(\mathbf{k}')] \}. \quad (9)$$

We consider only elastic collisions between electrons and defects; this approximation is convenient for scattering by impurities, surface roughness, and acoustical phonons. In this case

$$W_{\nu,\nu'}(\mathbf{k},\mathbf{k}') = W_{\nu',\nu}(\mathbf{k}',\mathbf{k}) \\ = P_{\nu,\nu'}(|\mathbf{k}-\mathbf{k}'|) \delta(\varepsilon_{\nu\mathbf{k}} - \varepsilon_{\nu'\mathbf{k}'}), \quad (10)$$

where $P_{\nu,\nu'}(|\mathbf{k}-\mathbf{k}'|) = (2\pi/\hbar) |\langle \nu\mathbf{k}|H_i|\nu'\mathbf{k}'\rangle|^2$ depends only on $|\mathbf{k}-\mathbf{k}'|$. The last important restriction will be to limit the calculations to first order in the electric field E . Thus we look for the solution of the Boltzmann equations

is the density of states in subband ν . In Eq. (4), $Y(\varepsilon)$ is the unit step function.

In the following we will often make use of the number, ν_F , of subbands whose minimum ε_ν is below ε_F . At fixed density n , ν_F is a function of d . For instance, in the ideal case where the film surfaces act on carriers as an infinite repulsive potential, $V(z)$ is an infinite well and $\varepsilon_\nu = (\hbar^2 \pi^2 / 2md^2) \nu^2$. In that case $\nu_F = \text{Int}[\varepsilon_F / (\hbar^2 \pi^2 / 2md^2)]^{1/2}$. Another useful definition of ν_F is given by the double inequality

$$\nu_F(4\nu_F^2 - 3\nu_F - 1) \leq \frac{12n}{\pi} d^3 \leq \nu_F(4\nu_F^2 + 9\nu_F + 5). \quad (5)$$

It is clear that for a fixed n , ν_F is a discontinuous function of d . If $\nu_F \gg 1$, the discontinuities of ν_F are very small and the exact variation law can be replaced by the approximate continuous function:

$$\nu_F \simeq \left(\frac{3n}{\pi} \right)^{1/3} d. \quad (6)$$

In CoSi₂ metallic films, the electron density n is about $3 \times 10^{22} \text{ cm}^{-3}$ and $\nu_F \simeq 0.3d$ (\AA). This gives $3 < \nu_F < 60$ when $10 < d < 200 \text{ \AA}$.

Once ν_F is known, we can immediately evaluate n_ν by

$$n_\nu = \frac{n}{\nu_F} + \frac{\pi}{2d^3} \frac{(\nu_F + 1)(2\nu_F + 1)}{6} - \frac{\pi}{2d^3} \nu^2. \quad (7)$$

Again, if $\nu_F \gg 1$,

$$n_\nu \simeq \frac{\pi}{2d^3} \left[\nu_F^2 + \frac{\nu_F}{2} - \nu^2 \right]. \quad (8)$$

When a static electric field \mathbf{E} parallel to the sample surfaces and a perpendicular magnetic induction \mathbf{B} are externally applied to carriers, their stationary distribution function $f_\nu(\mathbf{k})$ in state $|\nu\mathbf{k}\rangle$ differs from $f_0(\varepsilon_{\nu\mathbf{k}})$. If we assume a sufficiently weak magnetic induction for discarding the Landau level quantization, the functions $f_\nu(\mathbf{k})$ obey the following system of semiclassical Boltzmann equations:

(9) in the form

$$f_\nu(\mathbf{k}) = f_0(\varepsilon_{\nu\mathbf{k}}) + \phi_\nu(\mathbf{k}), \quad (11)$$

where $\phi_\nu(\mathbf{k})$ is linear in the electric field E .

Calculations become simpler by introducing the new set of functions $g_\nu(\mathbf{k})$ related to $\phi_\nu(\mathbf{k})$ by

$$\phi_\nu(\mathbf{k}) = -\frac{e\hbar}{m} \frac{\partial f_0(\varepsilon_{\nu\mathbf{k}})}{\partial \varepsilon_{\nu\mathbf{k}}} k E g_\nu(\mathbf{k}). \quad (12)$$

Finally, to first order in E , the system of coupled equations that describe the behavior of the set $g_\nu(\mathbf{k})$ is

$$k \cos \theta - \frac{e}{m} k(\mathbf{k} \times \mathbf{B}) \cdot \nabla_{\mathbf{k}} g_{\nu}(\mathbf{k}) \\ = \sum_{\nu', \mathbf{k}'} W_{\nu, \nu'}(\mathbf{k}, \mathbf{k}') [k' g_{\nu'}(\mathbf{k}') - k g_{\nu}(\mathbf{k})], \quad (13)$$

where θ is the angle between the two vectors \mathbf{k} and \mathbf{E} .

It has been shown elsewhere⁸ that in absence of magnetic induction, the equivalent set of Eqs. (13) can be decoupled and solved by introducing relaxation times $\tau_{\nu}(\varepsilon)$ attached to each subband ν . We will show that in the presence of a magnetic induction we can again decouple the set of Eqs. (13) and introduce the same $\tau_{\nu}(\varepsilon)$ set, but the interpretation is less simple than in the $B = 0$ situation.

III. DISTRIBUTION FUNCTIONS AND RELAXATION TIMES

The functions $g_{\nu}(\mathbf{k}) = g_{\nu}(k, \theta)$ are periodic in the angle between \mathbf{E} and \mathbf{k} , with period 2π . Thus

$$g_{\nu}(\mathbf{k}) = \sum_{n=-\infty}^{+\infty} \bar{a}_{\nu, n}(k) e^{in\theta}. \quad (14)$$

As $k = [(2m/\hbar^2)(\varepsilon_{\nu\mathbf{k}} - \varepsilon_{\nu})]^{1/2}$, we can consider the Fourier coefficient $\bar{a}_{\nu, n}(k)$ as a function of $\varepsilon_{\nu\mathbf{k}}$ and define new coefficients

$$a_{\nu, n}(\varepsilon_{\nu\mathbf{k}}) = \bar{a}_{\nu, n}(k).$$

These coefficients obey the set of equations derived from Eq. (13):

$$k \left[\cos \theta + i\omega_c \sum_{n=-\infty}^{+\infty} n a_{\nu, n}(\varepsilon_{\nu\mathbf{k}}) e^{in\theta} \right] = \sum_{\nu', \mathbf{k}'} \sum_{n=-\infty}^{+\infty} W_{\nu, \nu'}(\mathbf{k}, \mathbf{k}') [k' a_{\nu', n}(\varepsilon_{\nu'\mathbf{k}'}) e^{in\theta'} - k a_{\nu, n}(\varepsilon_{\nu\mathbf{k}}) e^{in\theta}]. \quad (15)$$

$\omega_c = eB/m$ is the cyclotron pulsation. The set of functions $e^{in\theta}$ is a basis for periodic functions; thus we deduce from Eq. (15) and from the property upon which $W_{\nu, \nu'}(\mathbf{k}, \mathbf{k}')$ depends only on k, k' and $\cos(\theta' - \theta) = \cos \alpha$ that all the coefficients $a_{\nu, n}(\varepsilon_{\nu\mathbf{k}})$ are zero except $a_{\nu, \pm 1}(\varepsilon_{\nu\mathbf{k}})$ which satisfy the linear relations

$$k \left[\frac{1}{2} \pm i\omega_c a_{\nu, \pm 1}(\varepsilon_{\nu\mathbf{k}}) \right] = \sum_{\nu', \mathbf{k}'} W_{\nu, \nu'}(\mathbf{k}, \mathbf{k}') [k' \cos \alpha a_{\nu', \pm 1}(\varepsilon_{\nu'\mathbf{k}'}) - k a_{\nu, \pm 1}(\varepsilon_{\nu\mathbf{k}})]. \quad (16)$$

Then, let us multiply the two sides of Eq. (16) by $k \delta(\varepsilon - \varepsilon_{\nu\mathbf{k}})$ and perform a summation upon \mathbf{k} . As electron collisions are assumed elastic [$\varepsilon_{\nu\mathbf{k}} = \varepsilon_{\nu'\mathbf{k}'}$ in Eq. (16)], we obtain

$$F_{\nu}(\varepsilon) \left[\frac{1}{2} \pm i\omega_c a_{\nu, \pm 1}(\varepsilon) \right] = - \sum_{\nu'=1}^{\nu_{\max}} C_{\nu\nu'}(\varepsilon) a_{\nu', \pm 1}(\varepsilon), \quad (17)$$

where

$$F_{\nu}(\varepsilon) = \sum_{\mathbf{k}} k^2 \delta(\varepsilon - \varepsilon_{\nu\mathbf{k}}) = \frac{Sm^2}{\pi\hbar^4} (\varepsilon - \varepsilon_{\nu}) Y(\varepsilon - \varepsilon_{\nu}) \quad (18)$$

and

$$C_{\nu\nu'}(\varepsilon) = \delta_{\nu, \nu'} \sum_{\mu=1}^{\nu_{\max}} \sum_{\mathbf{k}, \mathbf{k}'} k^2 W_{\nu, \mu}(\mathbf{k}, \mathbf{k}') \delta(\varepsilon - \varepsilon_{\mu\mathbf{k}'}) \\ - \sum_{\mathbf{k}, \mathbf{k}'} \mathbf{k} \cdot \mathbf{k}' W_{\nu, \nu'}(\mathbf{k}, \mathbf{k}') \delta(\varepsilon - \varepsilon_{\nu\mathbf{k}}). \quad (19)$$

Now it is interesting to define two matrices, $\underline{C}(\varepsilon)$ and $\underline{F}(\varepsilon)$, whose matrix elements are, respectively, $C_{\nu\nu'}(\varepsilon)$ and

$$F_{\nu\nu'}(\varepsilon) = F_{\nu}(\varepsilon) \delta_{\nu, \nu'}, \quad (20)$$

where $F_{\nu}(\varepsilon)$ is given by Eq. (18). This allows to obtain immediately

$$a_{\nu, 1}(\varepsilon) = a_{\nu, -1}^*(\varepsilon) \\ = -\frac{1}{2} \sum_{\nu'} \{ [\underline{C}(\varepsilon) + i\omega_c \underline{F}(\varepsilon)]^{-1} \underline{F}(\varepsilon) \}_{\nu\nu'}. \quad (21)$$

For the following it is important to define a new matrix $\underline{T}(\varepsilon)$ homogeneous to a time:

$$\underline{T}(\varepsilon) = \underline{C}^{-1}(\varepsilon) \underline{F}(\varepsilon). \quad (22)$$

This allows to transform the last expression for $a_{\nu, \pm 1}(\varepsilon)$ into

$$a_{\nu, 1}(\varepsilon) = a_{\nu, -1}^*(\varepsilon) \\ = -\frac{1}{2} \sum_{\nu'} \{ [1 + i\omega_c \underline{T}(\varepsilon)]^{-1} \underline{T}(\varepsilon) \}_{\nu\nu'}. \quad (23)$$

Finally when we substitute this last result in Eqs. (14), (12), and (11), the electron distribution functions $f_{\nu}(\mathbf{k})$ in the different subbands ν are given by

$$f_{\nu}(\mathbf{k}) = f_0(\varepsilon_{\nu\mathbf{k}}) + eE \frac{\hbar k}{m} \frac{\partial f_0(\varepsilon_{\nu\mathbf{k}})}{\partial \varepsilon_{\nu\mathbf{k}}} \frac{1}{2} \\ \times \sum_{\nu'} (e^{i\theta} \{ [1 + i\omega_c \underline{T}(\varepsilon_{\nu\mathbf{k}})]^{-1} \\ \times \underline{T}(\varepsilon_{\nu\mathbf{k}}) \}_{\nu\nu'} + \text{c.c.}). \quad (24)$$

We recall that this final expression for $f_{\nu}(\mathbf{k})$ is only valid if electric and magnetic fields are sufficiently weak (linear phenomena in E and no Landau quantization) and if electron collisions are elastic.

Now let us discuss the possible existence of a relaxation time $\tau_{\nu}(\varepsilon_{\nu\mathbf{k}})$ associated with each subband ν . The question is does the left-hand side of the linearized Boltzmann equation (9) equal

$$-\frac{f_v(\mathbf{k}) - f_0(\varepsilon_{v\mathbf{k}})}{\tau_v(\varepsilon_{v\mathbf{k}})} ?$$

In the absence of magnetic induction, $\omega_c = 0$ and Eq. (24) reduces to

$$f_v(\mathbf{k}) = f_0(\varepsilon_{v\mathbf{k}}) + eE \frac{\hbar k}{m} \frac{\partial f_0(\varepsilon_{v\mathbf{k}})}{\partial \varepsilon_{v\mathbf{k}}} \cos\theta \sum_{v'} [\mathcal{I}(\varepsilon_{v\mathbf{k}})]_{vv'}.$$

As, in this case, the left-hand side of the linearized Boltzmann equation is exactly

$$-\frac{eE}{\hbar} \cdot \nabla_{\mathbf{k}} f_0(\varepsilon_{v\mathbf{k}}) = -eE \frac{\hbar k}{m} \frac{\partial f_0(\varepsilon_{v\mathbf{k}})}{\partial \varepsilon_{v\mathbf{k}}} \cos\theta$$

it is trivial to show that a relaxation time exists having

$$\frac{f_v(\mathbf{k}) - f_0(\varepsilon_{v\mathbf{k}})}{\tau_v(\varepsilon_{v\mathbf{k}})} = -eE \frac{\hbar k}{m} \frac{\partial f_0(\varepsilon_{v\mathbf{k}})}{\partial \varepsilon_{v\mathbf{k}}} \frac{1}{2} \frac{\sum_{v'=1}^{v_{\max}} (e^{i\theta} \{ [1 + i\omega_c \mathcal{I}(\varepsilon_{v\mathbf{k}})]^{-1} \mathcal{I}(\varepsilon_{v\mathbf{k}}) \}_{vv'} + \text{c.c.})}{\sum_{v'} [\mathcal{I}(\varepsilon_{v\mathbf{k}})]_{vv'}}.$$

Nevertheless, in the presence of a magnetic field, it remains correct to use a relaxation time if and only if $v_{\max} = 1$, because in that case

$$L_1(\varepsilon_{1\mathbf{k}}) = -\frac{f_1(\mathbf{k}) - f_0(\varepsilon_{1\mathbf{k}})}{\tau_1(\varepsilon_{1\mathbf{k}})}.$$

In conclusion, when an electron gas is filling a system of subbands, as encountered in a quasibidimensional structure, the coupled linearized Boltzmann equations can always be decoupled and solved. Nevertheless we must pay attention to the fact that in the presence of a magnetic field B , the standard relaxation time method for solving the Boltzmann equation cannot be applied. This method is valid only if $B \neq 0$ and $v_{\max} = 1$ or if $B = 0$ and $v_{\max} \geq 1$.

IV. TRANSPORT COEFFICIENTS

The expression for the current density parallel to the surfaces of the quasibidimensional structure is

$$\mathbf{j} = -\frac{2e}{Sd} \sum_{v=1}^{v_{\max}} \sum_{\mathbf{k}} \frac{\hbar \mathbf{k}}{m} \phi_v(\mathbf{k}), \quad (26)$$

where $\phi_v(\mathbf{k}) = f_v(\mathbf{k}) - f_0(\varepsilon_{v\mathbf{k}})$ is given by Eq. (24). Factor 2 in Eq. (26) accounts for the electron spin.

We separate \mathbf{j} into its two components \mathbf{j}_{\perp} and \mathbf{j}_{\parallel} , respectively, parallel and perpendicular to the electric field \mathbf{E} .

$$j_{\parallel} = -\frac{2e\hbar}{md} \sum_{v=1}^{v_{\max}} \frac{1}{(2\pi)^2} \int d^2k k \cos\theta \phi_v(\mathbf{k}). \quad (27)$$

Integration on θ is straightforward; we get

$$j_{\parallel} = E \sum_{v=1}^{v_{\max}} \frac{n_v e^2}{m} \sum_{v'} \langle \{ [1 + \omega_c^2 \mathcal{I}^2(\varepsilon)]^{-1} \mathcal{I}(\varepsilon) \}_{vv'} \rangle_v. \quad (28)$$

the expression

$$\tau_v(\varepsilon) = \sum_{v'} [\mathcal{I}(\varepsilon)]_{vv'} = \sum_{v'} [\mathcal{C}^{-1}(\varepsilon)]_{vv'} F_{v'}(\varepsilon). \quad (25)$$

In the general case where $\omega_c \neq 0$, the left-hand side $L_v(\mathbf{k})$ of the linearized Boltzmann equation, derived from the result (24) for $f_v(\mathbf{k})$, is

$$L_v(\mathbf{k}) = -eE \frac{\hbar k}{m} \frac{\partial f_0(\varepsilon_{v\mathbf{k}})}{\partial \varepsilon_{v\mathbf{k}}} \frac{1}{2} \times \sum_{v'=1}^{v_{\max}} (e^{i\theta} \{ [1 + i\omega_c \mathcal{I}(\varepsilon_{v\mathbf{k}})]^{-1} \}_{vv'} + \text{c.c.})$$

which is clearly different from

Here $\langle \psi(\varepsilon) \rangle_v$ denotes the average of the function $\psi(\varepsilon)$ over all the energies of subband v and is defined by

$$\langle \psi(\varepsilon) \rangle_v = \frac{\int_0^{\infty} \psi(\varepsilon) \mathcal{N}_v(\varepsilon) (\varepsilon - \varepsilon_v) \frac{\partial f_0(\varepsilon)}{\partial \varepsilon} d\varepsilon}{\int_0^{\infty} \mathcal{N}_v(\varepsilon) (\varepsilon - \varepsilon_v) \frac{\partial f_0(\varepsilon)}{\partial \varepsilon} d\varepsilon}, \quad (29)$$

where $\mathcal{N}_v(\varepsilon)$ is the density of states in subband v previously defined [see Eq. (4)]. A similar derivation yields the following expression for the perpendicular component j_{\perp} of the current density:

$$j_{\perp} = E \omega_c \sum_{v=1}^{v_{\max}} \frac{n_v e^2}{m} \sum_{v'} \langle \{ [1 + \omega_c^2 \mathcal{I}^2(\varepsilon)]^{-1} \mathcal{I}^2(\varepsilon) \}_{vv'} \rangle_v. \quad (30)$$

Next, we introduce the three conductivity coefficients σ_0 , σ_1 , and σ_2 where σ_0 is the usual conductivity in absence of magnetic field, while σ_1 and σ_2 are the parts of conductivity which are, respectively, odd and even in \mathbf{B} . If \mathbf{u} is a unit vector, perpendicular to the surfaces, the expression for the current density is

$$\mathbf{j} = \sigma_0 \mathbf{E} + \sigma_1 (\mathbf{B}) \mathbf{u} \times \mathbf{E} + \sigma_2 (\mathbf{B}) \mathbf{u} \times (\mathbf{u} \times \mathbf{E}). \quad (31)$$

We deduce immediately from Eqs. (28) and (30) that

$$\sigma_0 = \sum_{v=1}^{v_{\max}} \frac{n_v e^2}{m} \sum_{v'} \langle \langle \mathcal{I}(\varepsilon) \rangle_{vv'} \rangle_v, \quad (32)$$

$$\sigma_1 = \omega_c \sum_{v=1}^{v_{\max}} \frac{n_v e^2}{m} \sum_{v'} \langle \{ [1 + \omega_c^2 \mathcal{I}^2(\varepsilon)]^{-1} \mathcal{I}^2(\varepsilon) \}_{vv'} \rangle_v, \quad (33)$$

$$\sigma_2 = \omega_c^2 \sum_{v=1}^{v_{\max}} \frac{n_v e^2}{m} \sum_{v'} \langle \{ [1 + \omega_c^2 \mathcal{I}^2(\varepsilon)]^{-1} \mathcal{I}^3(\varepsilon) \}_{vv'} \rangle_v. \quad (34)$$

We notice that, as expected, the conductivity σ_0 , at $\mathbf{B} = 0$, is simply

$$\sigma_0 = \sum_{\nu} \frac{n_{\nu} e^2}{m} \langle \tau_{\nu}(\epsilon) \rangle_{\nu}, \quad (35)$$

where $\tau_{\nu}(\epsilon)$ is the electron relaxation time related to subband ν and previously defined by Eq. (25).

In order to describe Hall and magnetoresistance effects in samples with a rectangular shape, it is necessary to express the electric field \mathbf{E} as a function of the current density \mathbf{j} . This is achieved by inverting Eq. (31):

$$\mathbf{E} = \rho_0 \mathbf{j} + \rho_1(B) \mathbf{u} \times \mathbf{j} + \rho_2(B) \mathbf{u} \times (\mathbf{u} \times \mathbf{j}), \quad (36)$$

where ρ_0 is the resistivity at zero magnetic field while, again, $\rho_1(B)$ and $\rho_2(B)$ are, respectively, odd and even in B . These three coefficients are related to σ_0 , σ_1 , and σ_2 by

$$\begin{aligned} \rho_0 &= \sigma_0^{-1}, \\ \rho_1 &= -\frac{\sigma_1}{(\sigma_1)^2 + (\sigma_0 - \sigma_2)^2}, \\ \rho_2 &= \sigma_0^{-1} \frac{\sigma_1^2 - (\sigma_0 - \sigma_2)\sigma_2}{(\sigma_1)^2 + (\sigma_0 - \sigma_2)^2}. \end{aligned} \quad (37)$$

From the exact expression for the resistivity ρ_0

$$\begin{aligned} \rho_0 &= \left[\sum_{\nu} \frac{n_{\nu} e^2}{m} \langle \tau_{\nu}(\epsilon) \rangle_{\nu} \right]^{-1} \\ &= \left[\sum_{\nu, \nu'} \frac{n_{\nu} e^2}{m} \langle [\underline{C}^{-1}(\epsilon)]_{\nu\nu'} F_{\nu'}(\epsilon) \rangle_{\nu} \right]^{-1} \end{aligned}$$

it is easy to discuss the validity of the Mathiessen's rule applied to metallic films. If electron scattering is produced by two types of defects, the matrix element $[\underline{C}(\epsilon)]_{\nu\nu'}$ defined by Eq. (19) is simply the sum $[\underline{C}_1(\epsilon)]_{\nu\nu'} + [\underline{C}_2(\epsilon)]_{\nu\nu'}$ of the two terms corresponding to the contribution of each type of scatterers. But it is clear that $\underline{C}^{-1}(\epsilon) \neq [\underline{C}_1(\epsilon)]^{-1} + [\underline{C}_2(\epsilon)]^{-1}$; thus the total resistivity ρ_0 cannot be the sum of the partial resistivity $\rho_{1,0}$ and $\rho_{2,0}$ due to each type of defects. Only in the special case where electrons fill only one subband, is the Mathiessen's rule valid.

Furthermore the Hall constant R_H and the magnetoresistance coefficient $\Delta R/R$ of a rectangular thin film are related to ρ_1 and ρ_2 by equations

$$R_H = \frac{\rho_1}{B} \quad \text{and} \quad \frac{\Delta R}{R} = -\frac{\rho_2}{\rho_0}. \quad (38)$$

For a degenerate electron gas and to lowest order in magnetic field, the expressions for R_H and $\Delta R/R$ can be simplified into

$$R_H = -\frac{1}{e} \frac{\sum_{\nu, \nu'}^{v_F} n_{\nu} T_{\nu\nu'}^2(\epsilon_F)}{\left[\sum_{\nu, \nu'}^{v_F} n_{\nu} T_{\nu\nu'}(\epsilon_F) \right]^2} \quad (39)$$

$$\frac{\Delta R}{R} = -\omega_c^2 \frac{\left[\sum_{\nu, \nu'}^{v_F} n_{\nu} T_{\nu\nu'}^2(\epsilon_F) \right]^2 - \left[\sum_{\nu, \nu'}^{v_F} n_{\nu} T_{\nu\nu'}^3(\epsilon_F) \right] \left[\sum_{\nu, \nu'}^{v_F} n_{\nu} T_{\nu\nu'}(\epsilon_F) \right]}{\left[\sum_{\nu, \nu'}^{v_F} n_{\nu} T_{\nu\nu'}(\epsilon_F) \right]^2}. \quad (39')$$

In Eqs. (38) and (39), v_F is the number of subbands whose minimum ϵ_{ν} is lower than the Fermi energy ϵ_F . We must pay attention to the condition for degeneracy of the electron gas: $(\epsilon_F - \epsilon_{\nu}) \gg k_B T$, which must be fulfilled for all subbands $\nu \leq v_F$.

At fixed electron density all the transport coefficients $\sigma_i, \rho_i, R_H, \Delta R/R$ depend on the film thickness d . This dependence is not only explicit in the calculation of the matrix elements $T_{\nu\nu'}(\epsilon)$ but also implicit in the number of occupied subbands v_F . Depending on the nature of electron scattering, the variations of transport coefficients with d can be either smooth or very strong. In order to illustrate this point, we perform, in the following sections, calculations on the conductivity, Hall, and magnetoresistance coefficients of a degenerate electron gas scattered by static impurities and by surface roughness.

V. HALL EFFECT AND MAGNETORESISTANCE IN AN IMPURITY SCATTERING MODEL

All transport coefficients are expressed through the matrix $\underline{T}(\epsilon)$ defined by Eq. (22). Thus, we have to evaluate the transition probability per unit time $W_{\nu, \nu'}(\mathbf{k}, \mathbf{k}')$ and, more precisely, the matrix $|\langle \nu \mathbf{k} | H_i | \nu' \mathbf{k}' \rangle|^2$ in which H_i is the Hamiltonian describing the interaction between an electron and N_i impurities embedded within the film. For simplicity the impurity scattering potential is schematized by a contact potential:

$$H_i = U \sum_{i=1}^{N_i} \delta(\mathbf{r} - \mathbf{r}_i), \quad (40)$$

where U describes the strength of an electron-impurity

interaction.

A straightforward derivation yields

$$\overline{|\langle \mathbf{v}\mathbf{k} | H_i | \mathbf{v}'\mathbf{k}' \rangle|^2} = \frac{U^2 n_i}{S} \int dz |\xi_{\mathbf{v},(z)} \xi_{\mathbf{v}',(z)}|^2. \quad (41)$$

In Eq. (41), the overbar denotes an average over the disorder of the impurity sites \mathbf{r}_i and $n_i = N_i/Sd$ is the impurity density. Result (41) is particularly simple because it does not depend on \mathbf{k} and \mathbf{k}' and, as a consequence, the matrix $\underline{T}(\epsilon)$ will be diagonal.

Again, for simplicity, we assume that electrons move along the direction perpendicular to surfaces as in an infinite potential well. Thus $\xi_{\mathbf{v},(z)} = (\sqrt{2}/d) \sin(\mathbf{v}\pi z/d)$ and we immediately get

$$\overline{|\langle \mathbf{v}\mathbf{k} | H_i | \mathbf{v}'\mathbf{k}' \rangle|^2} = \frac{U^2 n_i}{Sd} (1 + \frac{1}{2} \delta_{\mathbf{v},\mathbf{v}'}). \quad (42)$$

The derivation of $\underline{T}(\epsilon)$ is particularly simple; we obtain

$$T_{\mathbf{v}\mathbf{v}'}(\epsilon) = \delta_{\mathbf{v},\mathbf{v}'} \frac{d \hbar^3}{n_i U^2 m} \frac{1}{\frac{1}{2} + \nu_\epsilon}, \quad (43)$$

where ν_ϵ is the number of subbands whose minimum $\epsilon_{\mathbf{v}}$ is lower than ϵ . $T_{\mathbf{v}\mathbf{v}'}(\epsilon)$ becomes a step function of energy.

For the particular case of a degenerate electron gas and for $\nu \leq \nu_F$

$$T_{\mathbf{v}\mathbf{v}'}(\epsilon_F) = \frac{d \hbar^3}{n_i U^2 m} \frac{1}{\frac{1}{2} + \nu_F}. \quad (44)$$

Equation (44) shows that $T_{\mathbf{v}\mathbf{v}'}(\epsilon_F)$ is independent of ν ($\leq \nu_F$) while Eq. (43) proves that $T_{\mathbf{v}\mathbf{v}'}(\epsilon)$ does not depend on ϵ provided ϵ remains near ϵ_F . This last remark is important for the calculation of mean values such as $\langle T_{\mathbf{v}\mathbf{v}'}(\epsilon) \rangle_{\mathbf{v}}$ or $\langle T_{\mathbf{v}\mathbf{v}'}^p(\epsilon) / [1 + \omega_c^2 T_{\mathbf{v}\mathbf{v}'}^2(\epsilon)] \rangle_{\mathbf{v}}$ (with $p=2,3$) by a Sommerfeld's development around ϵ_F . All these mean values which occur in the σ_i coefficients are exactly equal to

$$T_{\mathbf{v}\mathbf{v}'}(\epsilon_F) \text{ or } \langle T_{\mathbf{v}\mathbf{v}'}^p(\epsilon_F) / [1 + \omega_c^2 T_{\mathbf{v}\mathbf{v}'}^2(\epsilon_F)] \rangle_{\mathbf{v}}$$

provided $(\epsilon_F - \epsilon_{\mathbf{v}_F}) \gg k_B T$.

Thus, in this impurity scattering model, the relaxation times $\tau_{\mathbf{v}}(\epsilon_F)$, calculated for the Fermi energy ϵ_F , are identical for all subbands:

$$\tau_{\mathbf{v}}(\epsilon_F) = T_{\mathbf{v}\mathbf{v}'}(\epsilon_F) = \tau_i = \frac{d \hbar^3}{n_i U^2 m} \frac{1}{\frac{1}{2} + \nu_F}.$$

We immediately deduce from Eq. (35) that at zero magnetic field, the conductivity is

$$\sigma_0(d) = \frac{ne^2}{m} \frac{\hbar^2 d}{n_i U^2 m} \frac{1}{\frac{1}{2} + \nu_F(d)}. \quad (45)$$

The ratio of $\sigma_0(d)$ to the bulk conductivity $\sigma_0(d = \infty)$ is quite simple:

$$\frac{\sigma_0(d)}{\sigma_0(d = \infty)} = \left[\frac{3n}{\pi} \right]^{1/3} \frac{d}{\frac{1}{2} + \nu_F(d)}. \quad (46)$$

This result is identical to the one obtained in Ref. 6. It

shows that, at a fixed low electron density, when ν_F is equal to 1 in the complete d variation ranges, the ratio $\sigma_0(d)/\sigma_0(d = \infty)$ is a linear function of d . This may be applied to a semiconducting quantum well. On the contrary, in a metallic film, $\nu_F \gg 1$, then $\nu_F \approx (3n/\pi)^{1/3} d$ and

$$\frac{\sigma_0(d) - \sigma_0(d = \infty)}{\sigma_0(d = \infty)} \approx -\frac{1}{2\nu_F} \approx -\frac{1}{2(3n/\pi)^{1/3} d} \quad (47)$$

the relative deviation of the conductivity from its bulk value is proportional to the inverse thickness d^{-1} .

In this impurity scattering model, the Hall effect and the magnetoresistance are trivial; from Eqs. (37) and (38), we get $R_H = -1/ne$ and $\Delta R/R = 0$. The thickness d is not an acting parameter on R_H or $\Delta R/R$.

VI. HALL EFFECT AND MAGNETORESISTANCE IN A SURFACE ROUGHNESS MODEL

The effect of surface roughness on the electrical conductivity of thin films has been experimentally studied in the past few years, in semiconducting GaAs-Ga_xAl_{1-x}As quantum wells⁹ and in CoSi₂ ultrathin metallic films.¹⁰ In the first case, the electron gas has a small density ($n \approx 10^{18} \text{ cm}^{-3}$) and, therefore, fills often one subband, and occasionally two. For metallic CoSi₂ films, the electron density is higher than 10^{22} cm^{-3} and the number of filled subbands ν_F increases with d and reaches several tenths for thickness d around 100 Å.

It has been already shown⁵⁻⁹ that the zero magnetic field conductivity, when limited by electron scattering on rough surfaces, may vary strongly with d . For instance, if $\nu_F = 1$ we obtain $\sigma \sim d^6$, while if $\nu_F \gg 1$, σ shows oscillations, generally too small to be experimentally observed, whose average obeys approximately a d^2 law. The question is now to predict the behavior of the Hall and magnetoresistance coefficients.

A. A model of the electron scattering by surface roughness

As in the case of impurity scattering, we need to know the Hamiltonian H_i describing the electron interaction with the two nonideal thin film surfaces. If these surfaces were ideal planes distant of d and perpendicular to the z axis, their equations would be $z = \pm d/2$. Moreover they would create two potential barriers whose heights would be V_+ and V_- , respectively. The electron Hamilton H_0 defined at the beginning of Sec. II becomes

$$H_0 = \frac{p^2}{2m} + V_+ Y \left[z - \frac{d}{2} \right] + V_- Y \left[z + \frac{d}{2} \right], \quad (48)$$

where $Y(z)$ is the usual step function.

In fact, the two real surfaces are not ideal planes; their equations depend on the projection ρ of the position vector \mathbf{r} unto a plane perpendicular to the z axis:

$$z = \pm \frac{d}{2} + f_{\pm}(\rho).$$

Here, the functions $f_{\pm}(\rho)$ describe entirely the roughness of each surface. Thus the electron Hamiltonian is

changed into $H = H_0 + H_i$, where

$$H_i = V_+ \left[Y \left[z - \frac{d}{2} - f_+(\boldsymbol{\rho}) \right] - Y \left[z - \frac{d}{2} \right] \right] + V_- \left[Y \left[z + \frac{d}{2} - f_-(\boldsymbol{\rho}) \right] - Y \left[z + \frac{d}{2} \right] \right]. \quad (49)$$

Indeed the surface roughness is far lower than the thickness $d: f_{\pm}(\boldsymbol{\rho}) \ll d$. Now we can develop H_i to first order

in $f_{\pm}(\boldsymbol{\rho})/d$

$$H_i \simeq -V_+ f_+(\boldsymbol{\rho}) \delta \left[z - \frac{d}{2} \right] - V_- f_-(\boldsymbol{\rho}) \delta \left[z + \frac{d}{2} \right]. \quad (50)$$

Finally H_i appears as a perturbation fully localized on the ideal plane surfaces $z = \pm d/2$.

Next we have to calculate the matrix element:

$$|\langle v\mathbf{k} | H_i | v'\mathbf{k}' \rangle|^2 = \frac{1}{S} \sum_{\eta=\pm} \sum_{\eta'=\pm} V_{\eta} V_{\eta'} \xi_{\nu}^* \left[\eta \frac{d}{2} \right] \xi_{\nu} \left[\eta \frac{d}{2} \right] \xi_{\nu'}^* \left[\eta' \frac{d}{2} \right] \xi_{\nu'} \left[\eta' \frac{d}{2} \right] \frac{1}{S} \int_s d^2 \rho \int_s d^2 \rho' e^{i(\mathbf{k}' - \mathbf{k}) \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')} f_{\eta}(\boldsymbol{\rho}) f_{\eta'}(\boldsymbol{\rho}'). \quad (51)$$

We remark that

$$\lim_{S \rightarrow \infty} \frac{1}{S} \int_s d^2 \rho \int_s d^2 \rho' e^{i(\mathbf{k}' - \mathbf{k}) \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')} f_{\eta}(\boldsymbol{\rho}) f_{\eta'}(\boldsymbol{\rho}') = \int_s d^2 \rho e^{i(\mathbf{k}' - \mathbf{k}) \cdot \boldsymbol{\rho}} \langle f_{\eta} f_{\eta'} \rangle, \quad (52)$$

where $\langle f_{\eta} f_{\eta'} \rangle$ is the correlation function of the two functions which describe the roughness

$$\langle f_{\eta} f_{\eta'} \rangle = \lim_{S \rightarrow \infty} \frac{1}{S} \int_s f_{\eta}(\boldsymbol{\rho}') f_{\eta'}(\boldsymbol{\rho} + \boldsymbol{\rho}') d^2 \rho'. \quad (53)$$

It is reasonable to assume that the roughnesses of the two different film surfaces are not correlated; therefore

$$\langle f_{\eta} f_{\eta'} \rangle = \delta_{\eta, \eta'} \langle f_{\eta} f_{\eta} \rangle. \quad (54)$$

Moreover, in the absence of a precise knowledge on roughness, we imagine that the autocorrelation $\langle f_{\eta} f_{\eta} \rangle$ is an isotropic ρ function and is essentially characterized by two physical parameters. The first one Δ_{η} describes the root-mean square of the height of the bumps on the surface η and the second one ξ_{η} is the roughness correlation length of the same surface. Finally, in this model, which has been introduced for the first time by Prange and Nee⁴ in their treatment of surface impedance, we assume that

$$\langle f_{\eta} f_{\eta} \rangle = \Delta_{\eta}^2 G \left[\frac{\rho}{\xi_{\eta}} \right], \quad (55)$$

where G is a function which has significant values only if $\rho \lesssim \xi_{\eta}$. We will give more details on G later.

As the right-hand side of Eq. (52) is the Fourier transform of $\langle f_{\eta} f_{\eta'} \rangle$, we can introduce the Fourier transform $F(k)$ of $G(\rho)$ and we get for the matrix element describing the interaction of an electron with two rough surfaces:

$$|\langle v\mathbf{k} | H_i | v'\mathbf{k}' \rangle|^2 = \frac{1}{S} \sum_{\eta=\pm} A_{v\nu', \eta}(d) \Delta_{\eta}^2 \xi_{\eta}^2 F(\xi_{\eta} |\mathbf{k} - \mathbf{k}'|), \quad (56)$$

where

$$A_{v\nu', \eta}(d) = V_{\eta}^2 \left| \xi_{\nu} \left[\eta \frac{d}{2} \right] \xi_{\nu'} \left[\eta \frac{d}{2} \right] \right|^2. \quad (57)$$

B. Transport coefficients

All transport coefficient calculations requires knowledge of the matrix elements $C_{v\nu'}(\varepsilon)$ defined previously by Eq. (19). With the last result (56) and for the Fermi energy $\varepsilon = \varepsilon_F$, we find

$$C_{v\nu'}(\varepsilon_F) = \frac{Sm^2}{4\pi^2 \hbar^5} \sum_{\eta=\pm} \Delta_{\eta}^2 \xi_{\eta}^2 \left[\delta_{v\nu'} k_{vF}^2 \sum_{\mu=1}^{v_F} A_{v\mu, \eta}(d) \int_0^{2\pi} d\alpha F(\xi_{\eta} k_{v\mu}(\alpha)) - k_{vF} k_{v'F} A_{v\nu', \eta}(d) \int_0^{2\pi} d\alpha \cos\alpha F(\xi_{\eta} k_{v\nu'}(\alpha)) \right]. \quad (58)$$

In Eq. (58), k_{vF} is the length of the Fermi wave vector associated with subband ν , such that

$$\frac{\hbar^2}{2m} k_{vF}^2 = \varepsilon_F - \varepsilon_{\nu}. \quad (59)$$

Here k_{vF} is also related to the density of electrons filling subband ν by

$$n_{\nu} = \frac{k_{vF}^2}{2\pi d}, \quad (60)$$

the other quantity $k_{v\nu'}(\alpha)$ is simply $|\mathbf{k}_{vF} - \mathbf{k}_{v'F}|$

$$k_{v\nu'}(\alpha) = (k_{vF}^2 + k_{v'F}^2 - 2k_{vF} k_{v'F} \cos\alpha)^{1/2}. \quad (61)$$

Unlike the impurity case treated just before, the matrix

$\underline{C}(\varepsilon_F)$ is generally nondiagonal. Nevertheless there exists a particular case where $\underline{C}(\varepsilon_F)$ is diagonal. To see this, let us assume that $\xi_\eta k_{v\nu'} \ll 1$ (this condition is fulfilled as soon as $\xi_\eta k_{1F} \ll 1$, since k_{vF} and $k_{v'F} \ll k_{1F}$ for all $v, v' \leq v_F$); then we can approximate $F(\xi_\eta k_{v\nu'}(\alpha))$ by $F(0)$ in Eq. (58) and the integral $\int_0^{2\pi} d\alpha \cos\alpha F(0)$ vanishes obviously. In conclusion, for the sufficiently low-correlation length we expect the matrix $\underline{C}(\varepsilon_F)$ to be diagonal.

Moreover although it is possible to evaluate exactly the term $A_{v\nu',\eta}(d)$ defined by Eq. (57), we prefer to replace it by its limit when $V_\eta \rightarrow \infty$. In this limit, the electron is confined in an infinite one-dimensional quantum well and

$$A_{v\nu',\eta}(d) = \lim_{V_\eta \rightarrow \infty} A_{v\nu',\eta}(d) = \left[\frac{\hbar^2 \pi^2}{md^3} \right]^2 v^2 v'^2. \quad (62)$$

The deviations from the limit $V_\eta \rightarrow \infty$ have been already studied¹¹ for a single subband structure, in a GaAs-Ga_xAl_{1-x}As quantum well.

In summary, in the low-correlation-length limit and for infinite potential barriers at the film surfaces, we can write the diagonal matrix $\underline{C}(\varepsilon)$ in the form

$$C_{v\nu'}(\varepsilon_F) = \delta_{v,\nu'} \frac{S\pi^4}{2\hbar} \frac{F(0)}{\pi} \times \frac{v_F(v_F+1)(2v_F+1)}{6d^6} \Delta^2 \xi^2 v^2 k_{vF}^2, \quad (63)$$

where $\Delta^2 \xi^2$ stands for the sum $\Delta^2 \xi^2 = \Delta_+^2 \xi_+^2 + \Delta_-^2 \xi_-^2$. All transport coefficients will depend on $\Delta^2 \xi^2$ and it will not be possible to distinguish between the roughness of the two different film surfaces by analyzing experimental transport results. In this limit of low-correlation length, $C_{v\nu'}(\varepsilon_F)$ is proportional to the value $F(0)$ of the Fourier transform of the autocorrelation function of $f(\rho)$. For a large variety of functions $G(\rho)$, $F(0)/\pi$ is approximately unity. For instance, $F(0) = \pi$ for the traditional autocorrelation function $G(\rho) = e^{-\rho^2}$, while $F(0) = 2\pi$ for $G(\rho) = e^{-\rho}$.

From expression (63) for $\underline{C}(\varepsilon)$, we immediately deduce that

$$T_{v\nu'}(\varepsilon_F) = \delta_{v,\nu'} \frac{m}{\pi^5 \hbar} \frac{\pi}{F(0)} \frac{6d^6}{v_F(v_F+1)(2v_F+1)} \frac{1}{\Delta^2 \xi^2} \frac{1}{v^2} \quad (64)$$

and that the relaxation time corresponding to subband v is simply $\tau_v(\varepsilon_F) = T_{v\nu'}(\varepsilon_F)$. This relaxation time decreases with v as v^{-2} and its variations with the thickness d are included in the factor $d^6/v_F(v_F+1)(2v_F+1)$ (we must not forget that, at fixed electron density n, v_F is a function of d).

Finally, inside the limits where the result (64) is valid, the basic conductivity coefficients σ_0 , σ_1 , and σ_2 , defined by Eqs. (32)–(34), take the simple expressions:

$$\sigma_0(d) = \frac{e^2}{m} \tau(d) \sum_{v=1}^{v_F(d)} \frac{n_v(d)}{v^2}, \quad (65)$$

$$\sigma_1(d) = \frac{e^2}{m} \tau(d) \omega_c \tau(d) \sum_{v=1}^{v_F(d)} \frac{n_v(d)}{v^4 + \omega_c^2 \tau^2(d)}, \quad (66)$$

$$\sigma_2(d) = \frac{e^2}{m} \tau(d) \omega_c^2 \tau^2(d) \sum_{v=1}^{v_F(d)} \frac{n_v(d)}{v^2 [v^4 + \omega_c^2 \tau^2(d)]}, \quad (67)$$

where the common factor $\tau(d)$ is homogeneous to a time and defined by

$$\tau(d) = \frac{m}{\pi^5 \hbar} \frac{\pi}{F(0)} \frac{1}{\Delta^2 \xi^2} \frac{6d^6}{v_F(v_F+1)(2v_F+1)}. \quad (68)$$

It is remarkable that all the coefficients σ_0 , σ_1 , and σ_2 are independent of the effective mass m of the charge carriers.

If the electron gas fills only one subband, we find exactly for the Hall and magnetoresistance coefficients: $R_H = -1/ne$ and $\Delta R/R = 0$. There is no variation of these two coefficients with the thickness d . Such results are valid to all orders in the magnetic induction B and they can be applied to an electron gas in a semiconducting quantum well, with a small electron density.

If the electron density increases, the second subband can be filled; when the Fermi energy crosses the bottom of the second subband, we predict a discontinuity of the conductivity which is rather important: $\Delta\sigma/\sigma = 4/5$; but there is no discontinuity of the Hall and magnetoresistance coefficients.

For metallic films, the electron density is such that $v_F > 2$ and it is very hard to satisfy the condition $\omega_c \tau(d) > 1$ for ordinary magnetic induction. For instance, in CoSi₂ films, it has been shown⁷ that the coefficients Δ and ξ are about a few Å, while the electron density $n \simeq 3 \times 10^{22} \text{ cm}^{-3}$; thus $v_F \simeq 0,3d$ (Å) and $10^{-4}B < \omega_c \tau(d) < 0,2B$ (B in T) as d increases from 10 to 100 Å. We conclude that only the low magnetic field range $\omega_c \tau(d) \ll 1$ is easily investigated in the metallic films case. In this low-field range we get

$$R_H = -\frac{1}{e} \frac{\sum_{v=1}^{v_F} n_v/v^4}{\left[\sum_{v=1}^{v_F} n_v/v^2 \right]^2} \quad (69)$$

and

$$\frac{\Delta R}{R} = \omega_c^2 \tau^2(d) \left[\frac{\sum_{v=1}^{v_F} n_v/v^6}{\sum_{v=1}^{v_F} n_v/v^2} - \frac{\left[\sum_{v=1}^{v_F} n_v/v^4 \right]^2}{\left[\sum_{v=1}^{v_F} n_v/v^2 \right]^2} \right]. \quad (70)$$

Equations (69) and (70) associated with Eqs. (5) and (7) allow one to compute exactly the variations of R_H and $\Delta R/R$ with thickness.

In the limiting case where $v_F \gg 1$, we know that v_F is a linear function of d : $v_F \simeq (3n/\pi)^{1/3}d$. Expressions (65), (69), and (70) reduce in this case to

$$\sigma_0 \simeq \frac{e^2}{4\pi^2\hbar} \frac{\pi}{F(0)} \frac{1}{\Delta^2\xi^2} \frac{d^2}{(3n/\pi)^{1/3}} \times \left[1 - \frac{1}{(3n/\pi)^{1/3}d} \left[1 + \frac{6}{\pi^2} \right] \right], \quad (71)$$

$$R_H \simeq -\frac{4}{5\pi} \frac{1}{e} \frac{d}{(3n/\pi)^{1/3}} \left[1 + \frac{1}{(3n/\pi)^{2/3}d} \left[\frac{12}{\pi^2} - \frac{1}{2} \right] \right], \quad (72)$$

$$\frac{\Delta R}{R} \simeq \frac{1}{525} \left[\frac{e}{\pi^2\hbar} \frac{\pi}{F(0)} \frac{1}{\Delta^2\xi^2} \right]^2 \times \frac{B^2 d^6}{n^2} \left[1 - \frac{1}{(3n/\pi)^{1/3}d} \left[\frac{8}{\pi^2} + 3 \right] \right]. \quad (73)$$

It was already known⁷ that, in the low-correlation-length range, the conductivity of metallic films obeyed approximately a d^2 law. Such a law is very different from the result $\sigma_0 \sim d^6$ valid for a semiconducting film with only one filled subband.

We remark on Eqs. (72) and (73) that R_H is approximately a linear function of d while $\Delta R/R$ presents a very strong variation in d^6 . We recall that in the opposite case where electrons fill one subband, $R_H = -1/ne$ and $\Delta R/R = 0$.

Let us discuss in more detail the Hall constant R_H . It is not influenced at all by the roughness parameters Δ and ξ (in the limit of the low-correlation length). Moreover the leading term in Eq. (72) is not proportional to the inverse electron density but to $n^{-2/3}$. Another expression for this leading term is $R_H \simeq -(1/ne) \frac{4}{15} v_F$. It differs from the bulk value by a factor $\frac{4}{15} v_F$ which varies between 1 and 10 in CoSi_2 when d increases from 10 to 100 Å. Such a variation must be detectable experimentally.

As far as we are concerned regarding the magnetoresistance $\Delta R/R$, its exact value depends on the roughness parameters Δ , ξ , and $F(0)$. The root-mean square of the height of the bumps on surfaces is about one or two interatomic distances; $\Delta \simeq 4$ Å in CoSi_2 . In such a metallic film the electron density is $n \simeq 3 \times 10^{22} \text{ cm}^{-3}$ and ξ can be very small for rather imperfect surfaces: $\xi \simeq 2$ Å. Then, with a 1 T magnetic induction, we find $10^{-7} < \Delta R/R < 10^{-1}$ when $10 < d < 100$ Å.

Let us conclude this section by two remarks. (i) In the presence of both impurity and roughness scattering, the total matrix time $\underline{T}(\epsilon)$ is easily deduced, in the limit of small correlation length, from the partial results (44) and (64):

$$\tau_{v,v'}(\epsilon_F) = \delta_{v,v'} \left[\frac{n_i m U^2}{\hbar^3 d} \left(v_F + \frac{1}{2} \right) + \frac{\pi^5 \hbar}{m} \frac{F(0)}{\pi} \frac{v_F(v_F+1)(2v_F+1)}{6d^6} \times \Delta^2 \xi^2 v^2 \right]^{-1}. \quad (74)$$

The results (72) and (73) concerning the Hall and magne-

toresistance are modified by the presence of the two scattering mechanisms; the variations of R_H with d remains rather smooth, while those of $\Delta R/R$ are still rapid. (ii) In the surface roughness scattering case, all of the preceding results were obtained analytically because we limited the developments to the low-correlation-length range. In fact, it is easy to find an exact expression for the matrix $\underline{C}(\epsilon)$ even if $\xi k_{vF} \gtrsim 1$. This has been done in Ref. 7 where a Gaussian autocorrelation function $G(\rho)$ was chosen. But the inverse matrix $\underline{C}^{-1}(\epsilon)$ which is necessary to obtain the matrix $\underline{T}(\epsilon)$ needs to be computed by numerical techniques. In the absence of experimental results on R_H and $\Delta R/R$ clearly related to roughness scattering, it does not seem necessary to explain this in detail.

VII. CONCLUSION

We have investigated galvanomagnetic properties of thin metallic films in which the electron quantum states are gathered in an energy set of subbands. More precisely we have studied quantum-size effects in the case where electrons are scattered elastically by surface roughness and impurities and are submitted to a magnetic induction perpendicular to film surfaces. We have solved the set of Boltzmann equations which govern the electron distribution functions in the different subbands; for that purpose, we have introduced a matricial relaxation time $\underline{T}(\epsilon)$ which replaces the usual relaxation time of standard transport theories. Nevertheless, contrary to the zero magnetic induction case, the collision part (right-hand side) of the Boltzmann equations cannot be replaced by the traditional relaxation term (unless the number of subbands in the set reduces to one). Using the matrix elements of $\underline{T}(\epsilon)$, we have given the expressions of the conductivity tensor and of the Hall and magnetoresistance coefficients.

When electrons interact with impurities through a contact potential, the Hall constant takes the trivial value ($-1/ne$) while there is no magnetoresistance. The origin of these results is explained by the scalar nature of the matricial relaxation time $\underline{T}(\epsilon_F)$ and by the fact that the relaxation times associated to the different filled subbands are identical.

When electrons are scattered by surface roughness, the transport coefficients depend on the shape and the characteristic parameters of the autocorrelation function describing roughness. At a sufficiently low-correlation length as compared with electron Fermi wavelength, the matrix $\underline{T}(\epsilon)$ is diagonal and calculations of transport coefficients become easy. We have shown that the variations of these coefficients with the film thickness depend strongly on the number v_F of subbands filled by the electron gas. In the case where $v_F = 1$, which is encountered in semiconducting quasibidimensional structures at low electron density, we recover the known results on the conductivity: $\sigma \sim d^6$; and in the case $v_F \gg 1$, which is the case of metallic films, $\sigma \sim d^s$ with $s \simeq 2$. We have studied the behavior of the Hall and magnetoresistance at low magnetic induction. When $v_F = 1$, the results are not surprising: $R_H = -1/ne$ and $\Delta R/R = 0$. But when

$v_F \gg 1$, we have found $R_H = -(1/ne)^{4/15} v_F \sim d$ while $\Delta R/R \sim d^{s'}$, with $s' \simeq 6$. The Hall constant does not depend on the parameters describing the surface roughness and does not give directly the value of the electron density; it is also proportional to the number of occupied subbands which varies with n and d . The magnetoresistance in metallic films depends very strongly on thickness, through a power law which is nearly the same as that obeyed by the conductivity in a semiconducting quasibidimensional structure with $v_F = 1$. Moreover, from the variations of the magnetoresistance with either the applied magnetic induction or the film thickness, it is possible to determine the product of the root-mean square of

the bumps on the film surface with the roughness correlation length. All the preceding results are valid only if the correlation length is neglected as compared with the electron Fermi wavelength. If not, the matrix relaxation time $\underline{T}(\epsilon)$ is no longer diagonal and we expect corrections to the different power laws discussed above; numerical computation will be easily made as soon as experimental results are published.

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