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Quantum Hall effect in a self-similar system

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Hall conductivity is calculated for a self-similar system (Sierpińsky gasket) in magnetic fields to probe the transport properties specific to quasicrystalline systems. We have found anomalously large fluctuations in the Hall conductivity arising from a hierarchy of interference of wave functions.

The interest in quasicrystals[1] with unusual symmetry has been heightened by recent advances both in preparing single-phase quasicrystalline alloys and in fabricating submicrometer semiconductor structures. In the physics of quasicrystals, a most important problem is the identification of observable properties specific to this class of structures. One such property has been identified as large quantum fluctuations in electrical conductivity in the two-dimensional Fibonacci structure or Penrose tiling.[2] A similar study has also been done on one-dimensional systems.[3,4] Here we report an unusual structure in the quantum fluctuation in the Hall conductivity in a two-dimensional self-similar system.

Although a quasicrystal is characterized by several features, including strange local symmetries (fivefold in the Penrose tiling and icosahedral in the three-dimensional quasicrystal) and recursive construction (self-similarity), it is conceivable that some of the electronic properties are essentially the consequence of self-similarity alone. In fact, Ninomiya[5] has pointed out that the Fibonacci lattice can be constructed by introducing progressively larger scale modulations in real space, so that its electronic energy spectrum may be understood as arising from multiple band folding with hierarchical Bragg reflections. This accounts for the singular energy spectrum in quasicrystals and in almost periodic (or Harper) systems.[6,7] Thus it is an interesting problem to identify the effect of self-similarity on the electronic-transport properties.

Here we consider the Sierpińsky gasket[8] in magnetic fields as a simple system with an explicit self-similarity for studying the Hall conductivity. The electronic structure of the tight-binding system on (triangular) Sierpińsky gaskets has been studied by a number of authors, and results have been obtained for the energy spectrum in magnetic fields H,[9,10] and for the wave functions in the absence[11,12] and in the presence[13] of H.

Here we study the Hall conductivity for two reasons: (i) The quantum Hall effect is a sensitive probe of the electronic structure in two-dimensional systems; and (ii) the Hall conductivity is well defined even for finite systems unlike the longitudinal conductivity or transmission coefficients, for which the decay rate of the wave function (Lyapunov exponent) has to be defined with care for systems with singular spectra. The quantum Hall effect in self-similar systems is particularly interesting, since it has been shown that the Hall conductivity reflects the topology (dependence on the phase twist in the boundary condition) of the wave function,[14,15] so that the hierarchical interference of wave functions in self-similar systems is expected to have a drastic effect there. Semiclassically, the Hall conductivity reflects the curvature of the dispersion relation within the effective-mass approximation. Quantum mechanically, however, the problem is rather that of the nature of Landau-quantized electrons in a multiply folded band with many band extrema and saddle points.

We consider a lattice on the square Sierpińsky gasket up to the third generation with 512 sites (Fig. 1), and take the tight-binding Hamiltonian,

$$\mathcal{H}=\sum_{i,j} t_{ij}\exp\left[i\frac{e}{2\hbar c}\mathbf{H}\cdot(\mathbf{r}_i\times\mathbf{r}_j)-i\frac{e}{\hbar c}\mathbf{A}\cdot(\mathbf{r}_i-\mathbf{r}_j)\right]c_i^\dagger c_j\,,$$

where $c_i^\dagger$ creates a state at $\mathbf{r}_i$, the transfer energy $t_{ij}=-1$ for nearest-neighbor (ij) and zero otherwise, and $\mathbf{H}$ is the magnetic field. For usual lattices, the original tight-binding band coalesces into p Landau levels when $\tilde{H}=q/p$, where $\tilde{H}$ is the magnetic flux, in units of the magnetic-flux quantum $\phi_0=ch/e$, contained within the unit cell of the crystal.[16] For the Sierpińsky gasket we specify $\tilde{H}$ by the magnetic flux penetrating the smallest cell in Fig. 1.

We employ periodic boundary conditions to exclude edge states. Specifically, we take the generalized periodic boundary condition. This may be thought of as looking at either the Bloch states with a wave number $\mathbf{k}=(e/\hbar c)(A_x,A_y)$ for a periodic system in which the finite sys-

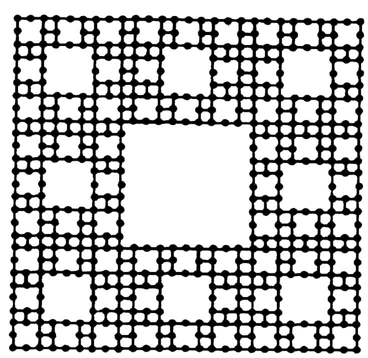


FIG. 1. Tight-binding lattice on a Sierpińsky gasket in the third generation with 512 sites. Solid circles represent atomic sites, which are connected by the nearest-neighbor transfer.

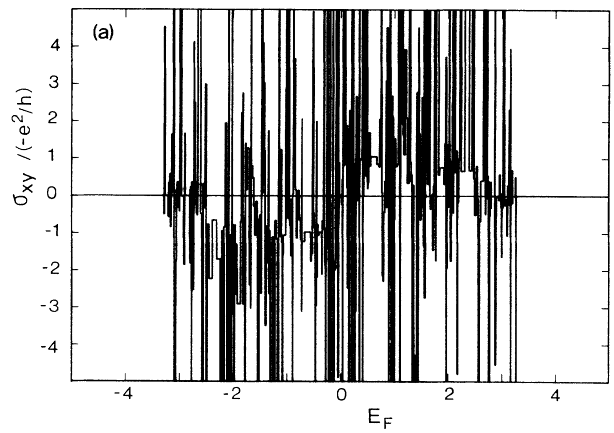
tem (of size L) is considered as a unit cell, or as the states on a torus with two external Aharonov-Bohm fluxes $(\phi_x,\phi_y)=(LA_x,LA_y)$, where $\mathbf{A}=(A_x,A_y)$ is the vector potential. The electronic states are a doubly periodic function of (A_x,A_y) with period ϕ_0/L.

We have numerically calculated all the eigenenergies and wave functions as well as the Hall conductivity σ_{xy} from the Kubo formula for each value of $\mathbf{A}$ for a given $\tilde{H}$. In Fig. 2(a) we show a result for σ_{xy} at $T=0$ against the Fermi energy E_F (the highest energy of the occupied states) for $\tilde{H}=\frac{1}{9}$ with $\mathbf{A}=\mathbf{0}$. The result shows that σ_{xy} has anomalously large fluctuations as E_F is varied.

When averaged over $\mathbf{A}$, the Hall conductivity is expressed as

$$\frac{\langle\sigma_{xy}\rangle}{e^2/h}=\frac{1}{2\pi i}\sum_{\alpha}^{\text{occ}}\int\int\left[\left\langle\frac{\partial u^\alpha}{\partial A_x}\middle|\frac{\partial u^\alpha}{\partial A_y}\right\rangle-\left\langle\frac{\partial u^\alpha}{\partial A_y}\middle|\frac{\partial u^\alpha}{\partial A_x}\right\rangle\right]dA_x dA_y\,,$$

where $u^\alpha(\mathbf{A})$ is the αth eigenstate, and $\langle\cdots|\cdots\rangle$ represents an inner product. This quantity gives an integer, which is a topological invariant (first Chern character) in terms of differential geometry.[14,15,17] The integral corresponds to the Hall conductivity for filled bands when E_F lies in a gap. The result[18] for $\langle\sigma_{xy}\rangle$ against the number of occupied electrons shows that the fluctuation in the Hall conductivity is still large even for the averaged $\langle\sigma_{xy}\rangle$. More specifically, if we plot the contribution of each state to $\langle\sigma_{xy}\rangle$ against the eigenstate number in Fig. 2(b), a remarkable behavior emerges in the Hall conductivity in which progressively larger values of $\langle\sigma_{xy}\rangle$ appear at larger intervals of eigenstate number. We can characterize this behavior in Fig. 3(a) by looking at the autocorrelation of the fluctuation, $F(\Delta\alpha)=\langle\sigma_{xy}(\alpha)\sigma_{xy}(\alpha+\Delta\alpha)\rangle$, where $\langle\cdots\rangle$ represent an average over the eigenstates and $\sigma_{xy}(\alpha)$ is the contribution of the αth eigenstate to $\langle\sigma_{xy}\rangle$.



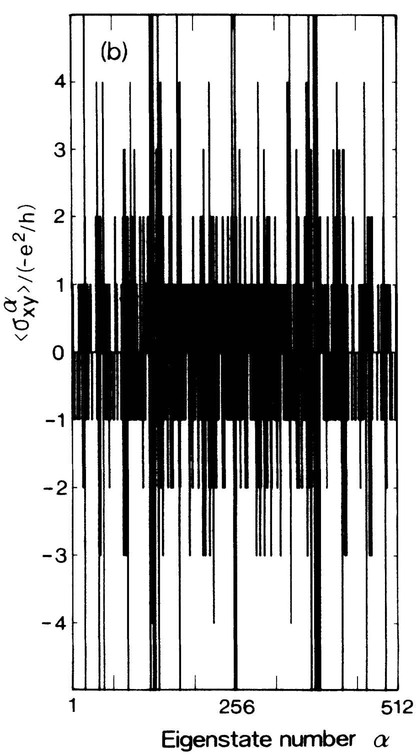


FIG. 2. (a) Hall conductivity σ_{xy} of the system described in Fig. 1 is plotted against E_F (the highest energy of the occupied state) for $\tilde{H}=\frac{1}{9}$ with $\mathbf{A}=\mathbf{0}$. (b) The contribution of each state to the Hall conductivity averaged over (A_x,A_y) and $\langle\sigma_{xy}^\alpha\rangle$, against the eigenstate number α.

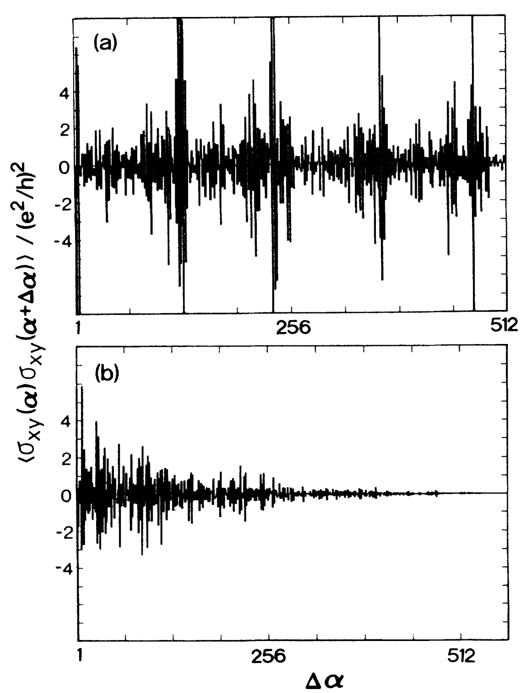


FIG. 3. The autocorrelation function, $\langle\sigma_{xy}(\alpha)\sigma_{xy}(\alpha+\Delta\alpha)\rangle$, for (a) the Sierpiński gasket with $\tilde{H}=\frac{1}{9}$ and (b) a typical disorder system on a 24×24 square lattice with a periodic boundary condition for $\tilde{H}=\frac{1}{12}$ and the distribution width of 0.5 in the site energies.

The correlation has strong structures which do not decay with $\Delta\alpha$ in contrast to the behavior in usual disordered quantum Hall systems on a square lattice with periodic boundary conditions [Fig. 3(b)]. A similar feature in the fluctuation autocorrelation is also obtained for $\sigma_{xy}(\mathbf{A}=0)$.

We can indeed visualize these large fluctuations as coming from the interference of the wave functions on various length scales, which makes the Landau quantization frustrated. Figure 4(a) depicts a typical dependence of σ_{xy} on $\mathbf{A}$ for a given eigenstate. It is seen that σ_{xy} as a function of boundary condition, or $\mathbf{A}$, fluctuates in a singular and somewhat self-similar manner. The wave function itself has a complex dependence on $\mathbf{A}$, reflecting the multiple interference of the wave function, and we can see this [Fig. 4(b)] from the phase[19] (argument of a_i/a_0) of a coefficient a_i relative to that of a reference site ($i=0$) when we expand an eigenstate as

$$u^{\alpha}(\mathbf{A})=\sum_j a_j^{\alpha}(\mathbf{A})c_j^{\dagger}|0\rangle .$$

For the second-generation Sierpiński gasket (with 64 sites), the result for σ_{xy} has a much less wild dependence on E_F or on $\mathbf{A}$, which indicates that the interference effect grows rapidly with the number of generations of the self-similarity.

Thus, unusually large quantum fluctuations in the Hall conductivity are shown to be an observable property of a self-similar system. In other words, the mesoscopic effect,[20] which is usually considered for disordered systems and comes from the interference of wave functions scattered by impurities, appears in self-similar systems with interference superposed on various length scales and can be probed by the Hall effect.[21] The fluctuation in the Hall conductivity is shown to change drastically when the value of $\tilde{H}$ is varied (down to $\frac{1}{27}$ in this study) in the self-similar system. This is reasonable, since the interference should be sensitive to the magnetic length ($\propto H^{-1/2}$). When H is varied continuously, we can expect a large magnetic fingerprint in the Hall effect in a finite self-similar system.

For an infinite self-similar system, the Hall conductivity is expected to be singular with the energy spectrum conforming a Cantor set. In real situations, however, there is always some cutoff lengths, such as the inelastic or phase-coherence lengths, or the domain size over which the fabricated sample realizes self-similarity. Thus the hierarchy would be truncated at a finite level.

In addition, randomness, if present in the system, will interfere with the effect arising from self-similarity. The effects of temperature and disorder on the conductivity fluctuation and its scaling properties should be relevant in real systems. For disordered quantum Hall systems with usual band structures, the fluctuations[22] in the Hall conductivity and its scaling properties[23] have been studied. The result shows that, although the root-mean-square fluctuation, $\delta\sigma_{xy}$, is of the order of e^2/h in accordance with the universal conductance fluctuation, $\delta\sigma_{xy}$ decreases sharply as the averaged Hall conductivity approaches a quantized Hall plateau near an integer Landau-level filling. This correlation is specific to quantum Hall systems, in which the fluctuation is dominated by the localization of states due to disorder, with the localization length being a continuous function of energy within a broadened Landau level. In the present case, by contrast, the crystal structure gives rise to a complicated quantum Hall effect in the disorder-free limit, which in turn makes the scaling properties peculiar, since the introduction of disorder exerts different effects on the different stages of the hierarchy. This problem will be discussed elsewhere.

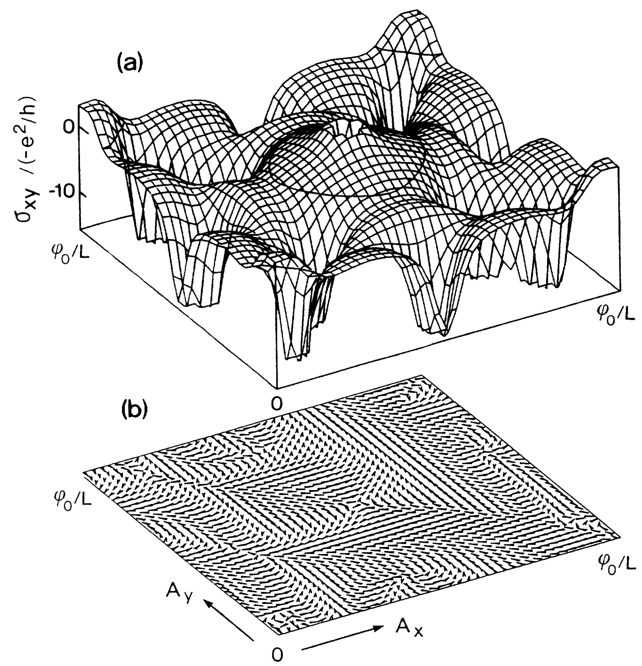


FIG. 4. An example of the (A_x,A_y) dependence of (a) σ_{xy} and (b) the phase of the wave function at a particular site [$r_i=(20,19)$ here] relative to that of a reference site [$\mathbf{r}_i=(0,0)$] is shown for an eigenstate (128th eigenstate here) for $\tilde{H}=\frac{1}{9}$.

Another problem for self-similar systems is the following. It has been suggested[24] that some of the eigenstates in quasicrystals are critical (borderline between localized and extended), or multifractal to be precise. It is an interesting problem to consider whether critical states can carry Hall current in the thermodynamic limit. For disordered quantum Hall systems there is an indication that they can, since the Kubo-formula approach[25] shows that a Landau level carries a nonzero Hall current in the quantum ($H\rightarrow\infty$) limit, while the current-carrying delocalized states, which exist at the center of the broadened Landau level in this limit, are shown to be fractal.[26]

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[18]Here $\langle \sigma_{xy} \rangle$ is numerically calculated on a (41×41) mesh in the $0\leqq A_x,A_y<\phi_0/L$ unit cell. The numerical integration gives an integer for each eigenstate within typically few percent error.

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