Exotic quantum effects in two space dimensions: The role of translation invariance

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A translation-invariant system of particles in two space dimensions has a state space with a continuous infinity of superselection sectors parametrized by a real number α. In sectors with $\alpha \neq 0$, plane-wave quantization of free particles is not possible; the "freest" Hamiltonian is that of charged particles moving in a constant transverse magnetic field.

Unusual features of particle quantum mechanics in two space dimensions, especially the possible occurrence of "fractional" statistics and angular momentum,[1] are now being put forward as the underlying cause of physical effects like the quantum Hall effect(s) and high-temperature superconductivity.[2] Laughlin, in particular, has shown in a recent paper[3] that a free gas of fractional-statistics particles behaves, in the Hartree-Fock approximation, as a gas of charged fermions in a constant magnetic field perpendicular to the plane and has proposed that this behavior would be sufficient to account for the superconductivity.

The present paper is an attempt at understanding the origins of possible unusual behavior of a two-dimensional quantum-mechanical system from the general point of view of its symmetries. As is known from the pioneering days,[4,5] the essential principle in the symmetry approach is that the state space of a system with symmetry group G must accommodate all projective unitary representations (PUR) of G and not only those PUR's which are actually UR's of G. Thus, half-odd-integer spin states of a rotation-invariant three-dimensional system carry PUR's of the rotation group SO(3) which are UR's, not of SO(3) but of SU(2). To determine the most general state space, one therefore needs to find all PUR's of the symmetry group of the system.

For particle quantum mechanics in n space dimensions, Bargmann[5] found a long time ago that the Euclidean group $E(n)$ has a continuous infinity of inequivalent PUR's for $n=2$ (and only for $n=2$). The question posed in the present paper is the following: What modification of the dynamics in two dimensions is required in each of the distinct PUR's (sectors) to ensure Euclidean invariance? It is most directly answered by invoking the general connection that exists between PUR's of a group G and central extensions of G by the group U(1) of phases. For this reason, I now state the essential facts pertaining to this connection.[6]

A central extension of G by U(1) is a group $\tilde{G}$ of which U(1) is a central subgroup [U(1) commutes with all elements of $\tilde{G}$] such that the factor group $\tilde{G}/\mathrm{U}(1)$ is G. G is itself not a subgroup of $\tilde{G}$ [except when $\tilde{G}$ is the trivial extension $G\times \mathrm{U}(1)$] but only a subset. Hence, a UR of $\tilde{G}$ is not, in general, a UR of G. However, it is always a PUR of G. Moreover, given a PUR of G, there exists a central extension $\tilde{G}$ and a UR of $\tilde{G}$ such that, when restricted to the subset G, it is the given PUR and this correspondence between PUR's and central extensions of G is one-to-one. It is this fact that makes the study of central extensions of symmetry groups unavoidable in quantum mechanics.

When G is a Lie group, it is simpler to deal, following Bargmann,[5] with its Lie algebra Lie G (and to indicate, where appropriate, that caution has to be used in passing to the group). Corresponding to $\tilde{G}$ there is a central extension of Lie $G=\{X,Y,Z,\dots\}$ obtained by modifying the Lie bracket $[X,Y]$ to

$$[X,Y]_\gamma=[X,Y]+i\gamma(X,Y)\ , \tag{1}$$

where γ is a real-valued antisymmetric bilinear function [in particular, $\gamma(X,0)=\gamma(0,X)=0$] satisfying the condition (to ensure that $[\ ,\ ]_\gamma$ fulfills the Jacobi identity)

$$\gamma([X,Y],Z)+\gamma([Y,Z],X)+\gamma([Z,X],Y)=0\ . \tag{2}$$

If $\gamma(X,Y)$ is of the form $i\beta([X,Y])$ for β some real linear function, Eq. (2) is automatically satisfied and $[\ ,\ ]_\gamma$ and $[\ ,\ ]$ define the same (isomorphic) Lie algebra: $X+\beta(X)$ and X have the same Lie brackets. Hence, solutions of Eq. (2), which are inequivalent in the sense that their difference is not of the form $i\beta([\ ,\ ])$, define distinct central extensions.

Once again, it must be kept in mind that the correspondence between extensions of G and Lie G may not always be one-to-one.

Though contained in the work of Bargmann,[5] it is useful to exhibit the central extensions of Lie $E(2)$, essential for what follows, and to see explicitly that Lie $E(n)$ has no nontrivial central extensions for $n>2$. The Lie brackets for $E(2)$ are

$$[P_i,P_j]=0,\quad [J,P_i]=i\epsilon_{ij}P_j \tag{3}$$

and Eqs. (2) are easily solved:

$$\begin{aligned}&\gamma(P_1,P_2)=-\gamma(P_2,P_1)=i\alpha\ ,\\ &\gamma(J,P_i)=i\epsilon_{ij}\delta_j\ ,\end{aligned} \tag{4}$$

α,δ_i arbitrary real numbers. Obviously $\gamma(P_1,P_2)$ cannot be a linear function of $[P_1,P_2](=0)$ except for $\alpha=0$. But, defining

$$\beta(P_i)=-i\epsilon_{ij}\beta([J,P_j])=-\delta_i\ , \tag{5}$$

we see that δ_i can be absorbed in P_i without changing the structure of the Lie algebra; in other words, an extension defined by an arbitrary δ_i is a trivial extension. Hence, extensions of Lie $E(2)$ are parametrized by one real number α, $-\infty < \alpha < \infty$, and these are all obtained from extensions of the translation subalgebra

$$[P_1, P_2] = i\alpha \ . \tag{6}$$

For every value of $\alpha \neq 0$, the momenta satisfy the Heisenberg commutation relations.

To illustrate what happens in higher dimensions, it is sufficient to look at $n=3$. $E(3)$ has Lie brackets

$$\begin{aligned} &[P_i, P_j] = 0 \ , \\ &[J_i, J_j] = i\epsilon_{ijk} J_k \ , \\ &[J_i, P_j] = i\epsilon_{ijk} P_k \end{aligned} \tag{7}$$

and the substitutions $X=J_i$, $Y=P_j$, $Z=P_k$; $X=J_i$, $Y=J_j$, $Z=P_k$; and $X=J_i$, $Y=J_j$, $Z=J_k$ lead, respectively, to the vanishing of $\gamma(P_j,P_j)$, $\gamma(J_i,P_j)$, and $\gamma(J_i,J_j)$ (a special case of a general theorem valid for the Lie algebra of the semidirect product of a semisimple group and an Abelian group).

Finally, the passage from the Lie algebra to the group also works differently for $E(2)$ and $E(3)$. Even though Lie SO(3) [=Lie SU(2)] has no nontrivial central extension, the group SO(3) has one, whose UR's are the half-integral spin PUR's of SO(3). In contrast, the group SO(2) has no nontrivial central extension as follows from its being compact and Abelian[5]—two-dimensional angular momentum is integral.

The upshot of all this is that a translation and Euclidean invariant system of particles in two dimensions has a state space containing sectors V_α parametrized by an arbitrary real number α. Each V_α is a vector space carrying a UR of the α extension $\tilde{E}_\alpha(2)$ of $E(2)$ [and of the corresponding Heisenberg extension $\tilde{R}^2_\alpha$, defined by Eq. (6), of the translation group R^2] or, equivalently, carrying an α PUR of $E(2)$ (and of R^2). Since it is a general result[7] that inequivalent PUR spaces of the group of symmetries are actually superselection sectors, each V_α is a superselection sector. A sector with $\alpha \neq 0$ is a direct sum of identical ∞-dimensional subspaces (the Heisenberg group has a unique ∞-dimensional irreducible UR). For $\alpha=0$, the situation is degenerate: V_0 is an infinite direct sum of inequivalent one-dimensional UR's of R^2, the familiar plane-wave representations. That the translation group has nontrivial PUR's (in fact, for all $n \geq 2$) is an old result.[4,8] Nevertheless, it is necessary to stress that $P_{\alpha i}$, the self-adjoint operators representing P_i in the α sector *are* the momentum operators even though they do not commute—operating on states in V_α, they generate translations.

The fact that α is a superselection charge is important: it implies that time evolution cannot take the system from one sector to another. Consequently, there cannot be one Hamiltonian valid for all sectors, but rather a family $\{H_\alpha\}$. For H_α to respect translation (or Euclidean) symmetry in V_α, H_α must be invariant not under R^2 or $E(2)$ but under $\tilde{R}^2_\alpha$ and $\tilde{E}_\alpha(2)$ because only the latter are unitarily represented on V_α. Obviously, $P^2_{\alpha 1}+P^2_{\alpha 2}$ is not invariant under $\tilde{R}^2_\alpha$. More generally, if Q is a polynomial function of $P_{\alpha i}$, and invariant under $\tilde{R}^2_\alpha$,

$$[Q, P_{\alpha i}] = i\epsilon_{ij} \frac{\partial Q}{\partial P_{\alpha j}} = 0 \ ,$$

then Q is a constant, showing that there is no nontrivial Hamiltonian that is a function of the momenta alone.

In spite of this, V_α can be realized as the Hilbert space of conventional wave functions, i.e., functions on the configuration space with

$$P_{\alpha i} = -i\frac{\partial}{\partial x_i} + \phi_i \ , \tag{8}$$

where ϕ is a potential satisfying

$$\frac{\partial \phi_1}{\partial x_2} - \frac{\partial \phi_2}{\partial x_1} = \alpha \ , \tag{9}$$

i.e., a connection on a U(1) bundle on R^2 whose curvature is constant and equal to α. The $P_{\alpha i}$ satisfy the canonical commutation relations $[P_{\alpha i}, x_j] = -i\delta_{ij}$ as expected of momentum operators. One easily verifies now that the only operators linear in $P_{\alpha i}$ (to preserve rotation symmetry) and commuting with $P_{\alpha i}$ are

$$K_{\alpha i} = P_{\alpha i} - \alpha\epsilon_{ij} x_j \tag{10}$$

and that the unique invariant operator quadratic in momenta is, up to a constant,

$$H_\alpha = K^2_{\alpha 1} + K^2_{\alpha 2} \ . \tag{11}$$

This is the closest we can get to a free Hamiltonian.

Strictly speaking, H_α even for a fixed α is a family of operators, on account of the gauge freedom in choosing the "kinematic" vector potential ϕ. But the uniqueness theorem for UR's of the Heisenberg group ensures that gauge transforms of $P_{\alpha i}$ and H_α are unitarily equivalent. In the gauge

$$\phi_i = \tfrac{1}{2}\alpha\epsilon_{ij} x_j \ ,$$

Eqs. (10) and (11) give

$$H_\alpha = -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} - i\alpha\epsilon_{ij} x_i \frac{\partial}{\partial x_j} + \frac{\alpha^2}{4} x_i x_i \ . \tag{12}$$

Note that in this gauge, $J = -i\epsilon_{ij} x_i \partial/\partial x_j$. This is the Hamiltonian for a "charged" particle in a constant transverse "magnetic field." Whether the particle is endowed with charge α and the magnetic field with unit strength or vice versa is a matter of convenience; for definiteness we adopt the former picture. Also, $K_{\alpha i}$ are not covariant derivatives: $K_{\alpha i} \neq -i\partial/\partial x_i + \text{const}\ \phi_i$ in an arbitrary gauge for ϕ_i.

Equation (12) shows that the Hartree-Fock approximation of Laughlin[3] for a free gas of particles obeying fractional statistics is the exact description of particles obeying normal statistics but in a nontrivial superselection sector of the translation and Euclidean groups. The N-particle Hamiltonian

$$H_\alpha^N = \sum_{r=1}^{N} (K_{\alpha 1}^{(r)2} + K_{\alpha 2}^{(r)2}) ,$$

$$K_{\alpha i}^{(r)} = -i\frac{\partial}{\partial x_i^{(r)}} - \tfrac{1}{2}\epsilon_{ij}\alpha x_j^{(r)} ,$$

operating on (anti)symmetrized wave functions $\psi(x^{(1)}, \ldots, x^{(n)})$, is readily shown to commute with the total momentum $P_{N\alpha}$. However, the modification of plane-wave quantum mechanics peculiar to $\alpha \neq 0$ sectors is not a statistical or many-particle effect, nor is it a boundary effect.

I conclude with two remarks.

(1) Unusual physics in the $\alpha \neq 0$ sectors depends not so much on the dimension of space as on the invariance group. The discussion above holds unchanged for particles in three dimensions provided there is translation invariance at least in planes and rotation invariance at most about the normal perpendicular to the planes, e.g., for electrons in layered materials.

(2) Full Galilean invariance (or for that matter Poincaré invariance) destroys the superselection structure described in the main part of this paper. The Lie algebra of the Galilei group in two dimensions, $\mathcal{G}(2)$, is obtained from that of $E(2)$ by adjoining the generators of velocity transformations (M_i) and time translation (H). Equations (2) for M_i, H, and P_j and the Lie brackets

$$[M_i, H] = iP_i, \quad [M_i, P_j] = [H, P_j] = 0 ,$$

imply directly that $\gamma(P_i, P_j) = 0$; in other words, the functions of γ which satisfy Eq. (2) on all of $\mathcal{G}(2)$ necessarily vanish on the translation subgroup. [As in the well known case of $\mathcal{G}(3)$, $\mathcal{G}(2)$ does have a one-parameter family of extensions, corresponding to γ which is nontrivial on the pair (M_i, P_i), $\gamma(M_i, P_j) = i\delta_{ij}\epsilon$, $-\infty < \epsilon < \infty$; ϵ is related to the total mass and is not relevant in the present context.] What this means is that, for a strictly isolated system with no external forces, $\alpha \neq 0$ sectors are absent. On the other hand, systems (electron gas in a material) in which the exotic phenomena mentioned in the beginning occur are not isolated and the Galilean invariance is restored by the motion of the environment. The external magnetic field that simulates the lack of commutativity of momenta is a signal of the breaking of Galilean invariance. In a real system, the magnetic field is presumably physical, and reflects the influence of the (neglected) environment on the electrons.

After completing this work, I have seen a copy of work by Chen *et al.*[9] which discusses a number of conceptual issues connected with the properties of a system of particles with fractional statistics, including Laughlin's Hartree-Fock wave function. The notion of a spontaneous breaking of the commutativity of translations plays an important role in this paper. While a comparison of the paper of Chen *et al.* and the present work cannot be made here, it should be stated that it is perhaps more appropriate to think of translation invariance as anomalously implemented in $\alpha \neq 0$ sectors, through the extra (anomalous) terms in the Hamiltonian.

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[1]F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 957 (1982); F. Wilczek and A. Zee, *ibid.* **51**, 2250 (1983).

[2]R. B. Laughlin, Science **242**, 525 (1988), provides a general review of the mechanisms that may be involved as well as their relation to exotic two-dimensional quantum phenomena. References, far too numerous to be cited individually here, can be traced from this review.

[3]R. B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988); Science **242**, 525 (1988).

[4]H. Weyl, *The Theory of Groups and Quantum Mechanics,* (Dover, New York, 1931).

[5]V. Bargmann, Ann. Math. **59**, 1 (1954).

[6]See V. S. Varadarajan, *Geometry of Quantum Theory* (Van Nostrand Reinhold, New York, 1970), Vol. II, for a detailed exposition of the material. A pedagogical account of the mathematical material as well as applications to different physical systems is given by P. P. Divakaran, Tata Institute of Fundamental Research Report (unpublished).

[7]P. P. Divakaran, Tata Institute of Fundamental Research Report No. TIFR/TH/83-36, 1983 (unpublished); National Laboratory for High Energy Physics Report No. KEK-TH 95, 1984 (unpublished); Tata Institute of Fundamental Research Report No. TIFR/TH/90-6 (unpublished).

[8]E. P. Wigner, Ann. Math. **40**, 49 (1939).

[9]Y-H. Chen, F. Wilczek, E. Witten, and B. Halperin, Int. J. Mod. Phys. B **3**, 1001 (1989).