Phase transition and scaling properties of the energy spectrum for a hierarchical quantum Ising model

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We study a one-dimensional quantum Ising model where the exchange couplings and the transverse fields are hierarchically distributed. Exact analytical results for the critical line, energy gap, and dispersion relation of the low-energy excitations are obtained. It is shown that only in the case of $R_1=R_2=R>R_c$, where R_1 and R_2 are the hierarchical parameters for the exchange couplings and the transverse fields, does the system preserve the logarithmic singularity in the specific heat. Using renormalization-group techniques and an iteration map for the transfer matrix, we have presented the analytical results for the scaling properties for the energy spectra of the models with $R_1=R_2$ and $R_2=1$. From knowledge of the scaling properties of the energy spectra, we have exhibited a close relationship between the disappearance of the logarithmic singularity in the specific heat and the anomalous scaling behavior at the lower band edges.

I. INTRODUCTION

It is known that hierarchical structures appear in different physical contexts ranging from molecular diffusion on complex macromolecules[1] to anomalous relaxation in spin glasses[2,3] and computing architectures.[4] Recently, several authors have studied the problem of transport[5-8] and the electronic and vibrational properties[9-13] of hierarchical systems. In the case of diffusion, it has been shown that a hierarchical arrangement of energy barriers can give rise to anomalous behavior. Furthermore, a dynamical phase transition from ordinary to anomalous diffusion is found as the hierarchical parameter R is varied. In the electronic or vibrational problems, the eigenspectrum is found to be a zero-measure Cantor set and to possess eigenfunctions which are self-similar and critical. In this paper, we investigate the quantum Ising model (QIM) where both the exchange couplings and the transverse fields are distributed in a hierarchical way, as shown schematically in Fig. 1. The quantum-mechanical phase transition of this system is supposed to be equivalent to the critical behavior of some two-dimensional classical layered Ising model.[14] Thus we may expect this model to be of direct importance for magnetic superlattices constructed in hierarchical fashion.

The paper is organized as follows. In Sec. II we characterize and specify the model. Exact analytical results for the critical line, energy gap, and dispersion relation for the low-energy excitations are obtained for the general case of the m-furcating[8] hierarchical QIM with hierarchical parameters R_1 and R_2 for the exchange couplings and the transverse fields, respectively. We find that the usual logarithmic singularity (LOGS) in the ground-state energy and the specific heat appears only in the case of $R_1=R_2>R_c$ with R_c dependent on the furcating number m. In this case the correlation length exponent ν equals 1 and the model belongs to the Ising universality class. In Sec. III we treat the special case with $R_1=R_2$ following a renormalization-group (RG) approach through an exact decimation procedure. The fixed points and the corresponding eigenvalues for the linearized RG transformation matrix are discussed and analytical results for the scaling properties of the energy spectrum are presented. It is shown that at criticality

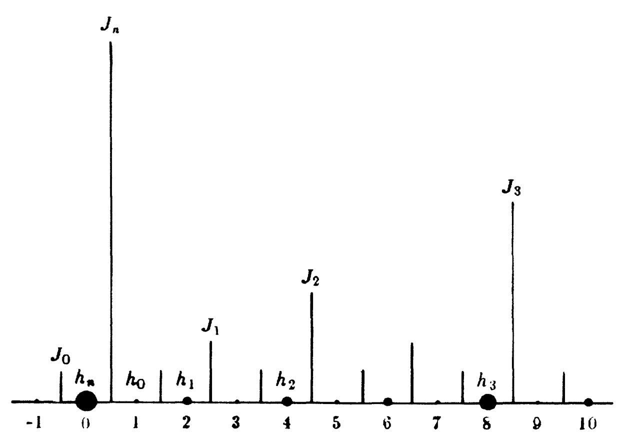

FIG. 1. Schematic representation of the hierarchy in both the exchange couplings and the transverse fields. The lengths of the vertical segment represent the strengths of the exchange couplings, and the diameters of the dots those of the transverse fields. Note that the drawing is for $R_{1,2}>1$ for convenience.

this model is equivalent to the vibrational problem with a hierarchy of spring constants. In Sec. IV, we analyze the problem in terms of a trace map for the transfer matrices for two cases with $R_1=R_2$ and $R_2=1$, which allows the discussion of the relationship between the scaling properties of the energy spectrum and LOGS in the specific heat. Finally in Sec. V we summarize our main results.

II. LOGARITHMIC SINGULARITY IN THE SPECIFIC HEAT

The model is given by the Hamiltonian

$$H=-\sum_i J(i)\sigma_i^x\sigma_{i+1}^x-\sum_i h(i)\sigma_i^z\ , \tag{2.1}$$

where σ_i^x and σ_i^z are the Pauli matrices at site i and the exchange couplings $J(i)$ and the transverse fields $h(i)$ are given by (in bifurcating hierarchical way, see Fig. 1)

$$J(i)=J_0R_1(i)=J_n=\begin{cases}J_0, & i=2l+1\\ J_0R_1^n, & i=2^n(2l+1)\end{cases} \tag{2.2}$$

and

$$h(i)=R_2(i)=h_n=\begin{cases}1, & i=2l+1\\ R_2^n, & i=2^n(2l+1)\ .\end{cases} \tag{2.3}$$

Here R_1 and R_2 are hierarchical parameters chosen to be positive for the ferromagnetic QIM and lie in the interval [0,1] for simplicity. Note that $R_2=1$ corresponds to the case in the uniform transverse field and when $R_2=R_2=1$ one recovers the periodic QIM.[15,16]

To solve (2.1) we proceed in the well-known Jordan-Wigner transformation[17] and rewrite (2.1) as

$$H=c^\dagger Ac+\tfrac{1}{2}(c^\dagger Bc^\dagger+\text{H.c.})\ , \tag{2.4}$$

where $c=(c_1,c_2,\ldots,c_{2^N})$ and the c_i's are anticommuting fermionic operators. The matrices A and B are given by

$$\begin{aligned}A_{i,j}&=-J(j)\delta_{i,j+1}-2h(j)\delta_{i,j}-J(i)\delta_{i,j-1}\ ,\\ B_{i,j}&=J(j)\delta_{i,j+1}-J(i)\delta_{i,j-1}\ .\end{aligned} \tag{2.5}$$

Since we are interested in properties of the infinite system, in this paper we work in so-called "c-cycle" problem[17,18] with periodic boundary condition for (2.4). In particular, in this section the system of period $p_N=2^N$ is obtained by setting all couplings $J(i)$ and transverse fields $h(i)$ in (2.2) and (2.3) with $n>N-1$ to be of the following form:

$$\begin{aligned}J(i)&=J_0R_1^{N+1}\ ,\\ h(i)&=R_2^{N+1}\ ,\end{aligned}\quad i=2^n(2l+1)\ \text{ with } n>N-1\ . \tag{2.6}$$

Thus the real hierarchical system corresponds to $N\to\infty$. The model displays long-range magnetic order above a certain critical coupling J_{0c}. The quantum-mechanical transition in the ground state is known to be driven by the soft mode of the Hamiltonian (2.4), which corresponds to the vanishing of the gap at the onset of criticality and is given by $(A_c-B_c)\varphi_0=0$ and $(A_c+B_c)\psi_0=0$, where A_c and B_c are the matrices A and B calculated at the critical line $J_{0c}=J_{0c}(R_1,R_2)$. The solution to these equations are easily obtained as follows:

$$\begin{aligned}\varphi_{0,j}&=(-1)^{j-1}\varphi_1\sum_{i=1}^{j-1}[J_c(i)/h(i+1)]\ ,\\ \psi_{0,j}&=(-1)^{j-1}\psi_1\prod_{i=1}^{j-1}[h(i)/J_c(i)]\ ,\end{aligned} \tag{2.7}$$

where φ_1 and ψ_1 are normalized constants and $J_c(i)$ is given by (2.2) with $J_0=J_{0c}$. The periodic boundary condition requires $\prod_{i=1}^{2^N}[h(i)/J_c(i)]=1$, which gives the critical line

$$J_{0c}=(R_2/R_1)\equiv\xi\ . \tag{2.8}$$

Now we turn to the calculation of the dispersion relation ΔE_k of low-energy excitations close to the critical point. The model with the Hamiltonian of a general bilinear fermionic form like (2.4) has been completely investigated by Ceccatto.[18] The energy gap between the first excited and ground-state energies is given by

$$\Delta E=2(|\varphi_0H'\psi_0|/|\varphi_0||\psi_0|)\tau+O(\tau^{3/2})\ , \tag{2.9}$$

where $\tau=|J_0-J_{0c}|/J_{0c}$ and the matrix $H'_{i,j}=J_c(i)\delta_{i,j-1}$. If the prefactor in the first term of (2.9) $\eta\equiv|\varphi_0H'\psi_0|/|\varphi_0||\psi_0|\neq0$, the dispersion relation for the low-energy excitations is

$$\Delta E_k=2\eta(\tau^2+k^2)^{1/2}\ , \tag{2.10}$$

where k is the pseudo-wave-number. Integration of (2.10) produces the usual LOGS in the ground-state energy and the specific heat at criticality. The singularity may disappear by the vanishing of the prefactor η.[18] To evaluate η, we have to compute

$$\begin{aligned}|\varphi_0|^2&=\sum_{j=1}^{2^N}(\varphi_{0,j})^2\ ,\\ |\psi_0|^2&=\sum_{j=1}^{2^N}(\psi_{0,j})^2\ .\end{aligned}$$

Representing the number j in the binary system and paying attention to (2.8) (see Appendix for details), we obtain

$$\begin{aligned}|\varphi_0|^2&=\varphi_1^2(1+\xi^2)^ND_2^2\ ,\\ |\psi_0|^2&=\psi_1^2(1+\xi^{-2})^N\ ,\end{aligned} \tag{2.11}$$

where $\xi=R_2/R_1$ and

$$D_2^2=R_1^2[1-(s_2R_1^2)^{-N}]/(s_2R_1^2-1)+1/[(\xi R_1)^2(s_2R_1^2)^N]\ , \tag{2.12}$$

with $s_2=1+\xi^2$. With these results and $|\varphi_0H'\psi_0|=2^N\varphi_1\psi_1$, we have

$$\eta=D_2^{-1}[2/(\xi+\xi^{-1})]^N\ . \tag{2.13}$$

Thus when $\xi\neq1$, we note that η vanishes for the real hierarchical system corresponding to $N\to\infty$, which will wash out LOGS in the ground-state energy and the specific heat. On the other hand, for $\xi=1$, with $\xi+\xi^{-1}=2$ and $R_1=R_2=R$, (2.13) gives

$$\eta=\begin{cases}0, & R^2<\frac{1}{2}\ ;\\ (2R^2-1)^{1/2}/R, & R^2>\frac{1}{2}\ .\end{cases} \tag{2.14}$$

Thus we observe a critical value for the hierarchical parameter $R_c=1/\sqrt{2}$. In the case of $R>R_c$, (2.10) and (2.14) imply a correlation length exponent $\nu=1$ and the specific heat has an LOGS. The transition belongs to the periodic Ising universality class[15,16] at least for the thermal sector. In contrast, for $R<R_c$, with the vanishing of η, LOGS in the specific heat disappears.[18] The transition may fall into another universality class. The situation is quite different from the Fibonacci quasiperiodic QIM, where the Ising-like phase transition is preserved independent of the "dilution factor" J_1/J_2.[18–20] For ferromagnetic quasiperiodic QIM, it is believed that only the change of quasiperiodicity may give rise to the disappearance of LOGS in the specific heat.[21]

Up to now, our discussion has been concentrated on the bifurcating hierarchical system, i.e., the system with regular uniformly bifurcating hierarchical arrays of exchange couplings and transverse fields. For a general m-furcating hierarchical structure,[8] the system with a period $p_N=m^N$ is obtained similarly by setting all exchange couplings $J(i)$ and transverse fields $h(i)$ of the following forms:

$$J(i)=\begin{cases}J_0R_1^n, & i=m^n(ml+q),\ q=1,2,\ldots,m-1 \text{ with } n<N\ ;\\ J_0R_1^{N+1/(m-1)}, & i=m^n(ml+q),\ q=1,2,\ldots,m-1 \text{ with } n\ge N\ ;\end{cases} \tag{2.15}$$

$$h(i)=\begin{cases}R_2^n, & i=m^n(ml+q),\ q=1,2,\ldots,m-1 \text{ with } n<N\ ;\\ R_2^{N+1/(m-1)}, & i=m^n(ml+q),\ q=1,2,\ldots,m-1 \text{ with } n\ge N\ ,\end{cases} \tag{2.16}$$

and the real hierarchical system is approached with $N\to\infty$.

In a similar way, one has the critical line

$$J_{0c}=\xi^{1/(m-1)} \tag{2.17}$$

and the prefactor

$$\eta=(m\xi/s_m)^N D_m^{-1}\ , \tag{2.18}$$

where

$$s_m=\sum_{n=0}^{m-1}\xi^{2n/(m-1)}\ , \tag{2.19}$$

$$D_m^2=(s_m-\xi^2)R_1^2[1-(s_mR_1^2)^{-N}]/(s_mR_1^2-1)+1/[(\xi R_1)^{2/(m-1)}(s_mR_1^2)^N]\ . \tag{2.20}$$

Similarly to the bifurcating hierarchical system, we note that LOGS in the specific heat appears only in the case of $R_1=R_2=R>R_c=1/\sqrt{m}$. In this case, the dispersion relation for the low-energy excitations is

$$\Delta E_k=2\eta(\tau^2+k^2)^{1/2}\ , \tag{2.21}$$

with

$$2\eta=2[(mR^2-1)/(m-1)]^{1/2}/R \tag{2.22}$$

being the sound velocity for the m-furcating hierarchical QIM. So the low-energy excitations will produce a transition in the same universality class as in the periodic case.[15,16]

III. SCALING PROPERTIES OF THE ENERGY SPECTRUM—RG APPROACH

We now proceed to study the scaling properties of the energy spectrum of the Hamiltonian (2.4) with $R_1=R_2=R$ at criticality $J_0=J_{0c}=1$. Note that the problem involves a hierarchy of both the exchange couplings and the transverse fields and has a self-similar structure; one may expect the RG approach to be useful. To do so we start from the eigenvalue equation $(A+B)(A-B)\varphi=\Lambda^2\varphi$. With $J_0=1$ and $R_1=R_2$, we have $h(i)=J(i)$. The eigenvalue equation reduces to

$$\varphi_{i-1}J^2(i-1)+\varphi_i[J^2(i-1)+J^2(i)]+\varphi_{i+1}J^2(i) =(\Lambda^2/4)\varphi_i\ . \tag{3.1}$$

Now we choose to eliminate all φ_i where $i=4j+2$ or $i=4j+3$ in terms of the remaining variables. For sites $0,1,\ldots,4$ (see Fig. 1) we have

$$\begin{aligned}(\varepsilon-J_0^2-J_1^2)\varphi_2&=J_1^2\varphi_3+J_0^2\varphi_1\ ,\\ (\varepsilon-J_0^2-J_1^2)\varphi_3&=J_0^2\varphi_4+J_1^2\varphi_2\ ,\end{aligned} \tag{3.2}$$

with $\varepsilon\equiv\Lambda^2/4$. Thus our equation for φ_1 becomes

$$(\varepsilon-J_n^2-J_0^2)\varphi_1 = J_n^2\varphi_0+J_0^2\varphi_2 = J_n^2\varphi_0+[J_0^2(\varepsilon-J_0^2-J_1^2)\varphi_1+J_0^4J_1^2\varphi_4]/\Delta \; , \tag{3.3}$$

where $\Delta=(\varepsilon-J_0^2-J_1^2)^2-J_1^4$. After recasting this equation in a form so as to make the renormalized system resemble the original one, we have the renormalization equations

$$\begin{aligned} J_0' &= J_0 \; , \\ J_{n-1}'^2 &= J_n^2\Delta/(J_0^2J_1^2) \; , \\ \varepsilon' &= [(\varepsilon-J_0^2)\Delta+(2J_1^2+J_0^2-\varepsilon)J_0^4]/(J_0^2J_1^2) \; . \end{aligned} \tag{3.4}$$

Generalizing the definition of the exchange couplings in (2.2) at criticality $J_0=1$ by

$$J_n=\begin{cases} 1, & n=0 \; ; \\ K^{1/2}R^{n-1}, & \text{otherwise} \; , \end{cases} \tag{3.5}$$

we obtain the two parameter-renormalization equations

$$\begin{aligned} K' &= R^2\Delta \; , \\ \varepsilon' &= [(\varepsilon-1)\Delta+(1+2K-\varepsilon)]/K \; , \end{aligned} \tag{3.6}$$

where $\Delta=(1+K-\varepsilon)^2-K^2$ and other renormalized exchange couplings are $J_0'=1$ and $J_n'=K'^{1/2}R^{n-1}$. Note that the two-parameter map (3.6) is identical to that for the one-dimensional (1D) vibrational problem with a hierarchy of spring constants,[12] except that in the recursion relation of K (3.6) we have R^2 instead of R in the vibrational problem.[12] So the energy spectrum of the model at criticality is the same as the vibrational spectrum.[12] The total bandwidth B_N of the period p_N system behaves exactly as $B_N=4R^{2N}$. The energy spectrum of the real hierarchical QIM forms a zero Lebesgue measure Cantor set with the fractal dimension $D_0=\ln 2/\ln(2/R^2)$. There are two scaling parameters that describe the energy spectrum. Numerical investigation[12] reveals that these two scaling parameters are governed by two fixed points

$$K_1^*=\begin{cases} R^2/(1-2R^2), & 0<R^2<\frac{1}{2}, \quad \varepsilon_1^*=0 \; , \\ +\infty, & \frac{1}{2}<R^2<1, \quad \varepsilon_1^*=0 \; , \end{cases} \tag{3.7}$$

$$K_2^*=R^2/(1-R^4), \quad \varepsilon_2^*=(2-R^2)/(1-R^2) \; , \tag{3.8}$$

with the corresponding eigenvalues of the matrix $\partial(K',\varepsilon')/\partial(K,\varepsilon)$

$$\lambda_{1,2}^{(1)}=\begin{cases} 2R^{\mp 2}, & 0<R^2<\frac{1}{2} \; , \\ 4, 2R^2, & \frac{1}{2}<R^2<1 \; , \end{cases} \tag{3.9}$$

$$\lambda_{1,2}^{(2)}=[T\pm(T^2-8R^6)^{1/2}]/(2R^2) \; , \tag{3.10}$$

where $T=2+2R^2+R^4$. It can be shown[12] that $\lambda_{\max}^{(1)}$ should be a relevant scaling parameter for the lower band edges and $\lambda_{\max}^{(2)}$ that for the upper band edges. The integrated density of states $N(\varepsilon)$ scales as

$$N(\varepsilon)\sim\varepsilon^{\beta} \quad \text{as } \varepsilon\to 0 \; , \tag{3.11}$$

or as a function of energy Λ:

$$N(\Lambda)\sim\Lambda^{2\beta} \quad \text{as } \Lambda\to 0 \; , \tag{3.12}$$

with

$$\beta=\ln 2/\ln\lambda_{\max}^{(1)}=\begin{cases} \ln 2/\ln(2/R^2), & 0<R^2<\frac{1}{2} \; ; \\ \frac{1}{2}, & \frac{1}{2}<R^2<1 \; . \end{cases} \tag{3.13}$$

So we observe a critical value of R. When $R^2>R_c^2=\frac{1}{2}$, the integrated density of states $N(\Lambda)$ obeys the usual scaling law for the periodic case when $\Lambda\to 0$. On the other hand, for $R<R_c$, we obtain an anomalous scaling exponent depending explicitly on R. We tend to believe that it is this anomalous scaling behavior that washes out the usual LOGS in the specific heat at the critical point. This will be made more transparent in Sec. IV.

IV. TRACE-MAP ANALYSIS OF THE CASES $R_1=R_2$ AND $R_2=1$

In Sec. III, we studied the scaling properties of the spectrum for the QIM at criticality $J_0=\xi=1$. For $J_0\neq 1$, it is convenient to analyze the problem in terms of an iterative map for the transfer matrix. In doing so we cast the eigenvalue equations $(A-B)\varphi=\Lambda\psi$ and $(A+B)\psi=\Lambda\varphi$ in the form

$$\begin{pmatrix} \psi_{j+1} \\ \varphi_{j+1} \end{pmatrix} = \begin{pmatrix} -h(j)/J(j) & \omega/J(j) \\ -\omega h(j)/[J(j)h(j+1)] & [\omega^2-J^2(j)]/[J(j)h(j+1)] \end{pmatrix} \begin{pmatrix} \psi_j \\ \varphi_j \end{pmatrix} , \tag{4.1}$$

with $\omega=\Lambda/2$. Iterating (4.1) we have

$$\begin{pmatrix} \psi_{2^N+1} \\ \varphi_{2^N+1} \end{pmatrix} = M_N \begin{pmatrix} \psi_1 \\ \varphi_1 \end{pmatrix} , \tag{4.2}$$

with

$$M_N=\prod_{i=2^N}^{1} Z_i \tag{4.3}$$

and

$$Z_i=\begin{pmatrix} -h(i)/J(i) & \omega/[J(i)h(i)] \\ -\omega h(i)/J(i) & [\omega^2-J^2(i)]/[J(i)h(i)] \end{pmatrix} , \tag{4.4}$$

where we have made use of $h(1)=h(2^N+1)=1$. Taking into account the fact that the two subchains of length 2^N composing the chain of length 2^{N+1} are equal except for the last exchange coupling and transverse field, after some algebra we get

$$M_{N+1}=V_N M_N^2 , \tag{4.5}$$

where

$$\begin{gathered} V_N=\xi I_0+[(1-\xi^2)/\xi]I_1-[(1-\xi^2)\omega/\xi]I_2+Q_N I_3 , \\ Q_{N+1}=Q_N/R^2 , \end{gathered} \tag{4.6}$$

with I_0 being the 2×2 unit matrix and

$$I_1=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} , \quad I_2=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} , \quad I_3=\begin{bmatrix} 1 & -1/\omega \\ \omega & -1 \end{bmatrix} . \tag{4.7}$$

The above recursion relation for the transfer matrix is very complicated. In order to shed some light on the relationship between the scaling properties of the energy spectrum and LOGS in the specific heat, we choose two special cases where $R_1=R_2$ and $R_2=1$ for discussion.

For $R_1=R_2=R$, with $\xi=1$, V_N reduces to

$$V_N=I_0+Q_N I_3 , \tag{4.8}$$

and the initial conditions become

$$M_0=Z_1$$

and

$$Q_0=[\omega^2(1-R^2)]/[J^2R^2] .$$

Equation (4.5) can be rewritten in the following form:

$$M_{N+1}=(I_0+Q_N I_3)M_N(M_N+M_N^{-1})-(I_0+Q_N I_3) . \tag{4.9}$$

By exploiting the knowledge of the equality $M_N+(M_N)^{-1}=\mathrm{tr}(M_N)I_0$ and defining $x_N=\frac{1}{2}\mathrm{tr}(M_N)$ and $y_N=x_N Q_N \mathrm{tr}(I_3 M_N)$, we obtain the autonomous two-dimensional (2D) map

$$\begin{gathered} x_{N+1}=2x_N^2-1+y_N , \\ y_{N+1}=(2/R^2)x_{N+1}y_N , \end{gathered} \tag{4.10}$$

or by eliminating y_N we have a single two-step relation involving x alone:

$$x_{N+1}=2(1+R^{-2})x_N^2+(2/R^2)x_N -(4/R^2)x_N x_{N-1}^2-1 , \tag{4.11}$$

with the initial conditions $x_0=(\omega^2-J_0^2-1)/(2J_0)$, $y_0=J_0Q_0x_0$, and $x_1=2x_0^2-1+y_0$. The energy spectrum of the period-p_N system can be determined by the condition $|x_N|<1$. For the real hierarchical system, we need to investigate the behavior of this iteration as $N\to\infty$. We do so by examining the fixed points on the 2D map (4.10), which are easily identified and listed in Table I together with the corresponding eigenvalue of the matrix $\partial(x',y')/\partial(x,y)$. The fixed point no. 1 is unphysical because it gives rise to an imaginary $(M_N)_{11}$. The other two fixed points are relevant to the problem. Numerical investigation reveals that eigenvalues near the lower band edges of the spectrum are attracted towards fixed point no. 2 (cf. Refs. 12 and 13). Thus the relevant scaling value governing the lower band edges should be

$$\beta_2=\ln 2/\ln\lambda_{\max}^{(2)}=\begin{cases} \ln 2/\ln(2/R^2), & 0<R^2<\frac{1}{2} ; \\ \frac{1}{2}, & \frac{1}{2}<R^2<1 . \end{cases} \tag{4.12}$$

Note the exponent β_2 characterizes the scaling behavior both at and away from the critical point, while β in (3.13) governs the scaling behavior at criticality. They are naturally the same. From (4.12) we obtain again an anomalous exponent dependent on the hierarchical parameter R when $R<R_c=1/\sqrt{2}$, which leads to the vanishing of LOGS in the specific heat. When $R>R_c$ the lower band edges obey the usual scaling laws for the periodic case, the QIM has an LOGS in the specific heat.

In order to get a better understanding of the relationship between the scaling properties at the lower band edge and LOGS in the specific heat, in the following we study the second special case where $R_1=r\neq 1$ and $R_2=1$. With $\xi=1/r$, it is possible to derive

$$M_{N+1}=[I_0/r+(r-r^{-1})(I_1-\omega I_2)]M_N^2 . \tag{4.13}$$

Defining $a_{N+1}=\mathrm{tr}(M_N)$ and $b_N=-\mathrm{tr}[(I_1-\omega I_2)M_N]$, we can write the following recursion relations:

$$\begin{gathered} a_{N+2}=[a_{N+1}^2-(r^2-1)a_{N+1}b_N-(r^2+1)]/r , \\ b_{N+1}=ra_{N+1}b_N+r , \end{gathered} \tag{4.14}$$

with the initial conditions $a_1=[\omega^2-(J_0^2+1)]/$

TABLE I. Fixed points and the corresponding eigenvalues of trace map (4.10). $T_1=R^2/2+1+R^{-2}$.

Fixed point no.	x^*	y^*	λ_1,λ_2
1	$-\frac{1}{2}$	0	$-2,-R^{-2}$
2	1	0	$4,2/R^2$
3	$R^2/2$	$R^2/2+1-R^4/2$	$T_1\pm(T_1^2-2R^2)^{1/2}$

TABLE II. Fixed points and the corresponding eigenvalues of trace map (4.14). $T_{2,3}=\frac{1}{2}\pm r^{-1}\pm r$.

Fixed point no.	a^*	b^*	λ_1,λ_2
1	1	$r/(1-r)$	$T_2\pm(T_2^2-2)^{1/2}$
2	-1	$r/(1+r)$	$T_3\pm(T_3^2-2)^{1/2}$
3	$r+r^{-1}$	$-r^{-1}$	$2,(r+r^{-1})^2$

$J_0, b_0=J_0$. Note that (4.14) can be equivalently cast in terms of a single two-step relation

$$a_{N+2}=(r+r^{-1})a_{N+1}^2+(2-a_N^2)a_{N+1}-(r+r^{-1}) \ . \tag{4.15}$$

The problem is equivalent to the quantum Schrödinger problem where the hierarchy is in the transition matrix elements and the site energies are taken as a constant.[10] The energy spectrum has been found to be a self-similar zero-measure Cantor set with the fractal dimension $D_0=\ln 2/\ln(2/r)$. The total bandwidth for the period $p_N=2^N$ system scales with N like $B_N\sim p_N^{-\delta}$ with $\delta=-\ln r/\ln 2$. The fixed points of the recursion relations (4.14) can be evaluated and listed together with the corresponding eigenvalues in Table II. Fixed point no. 3 is clearly irrelevant owing to the fact $|a_3^*|>2$. Only the first two fixed points are relevant to the scaling properties of the spectrum with both of the two scaling values $\ln 2/\ln|\lambda_{\max}^{(1,2)}|$ dependent on r (see from Table II) and independent of energy. Thus the anomalous scaling behavior is obtained for any value of $r\neq 1$, which corresponds to the disappearance of LOGS in the specific heat. We conjecture that it is a general phenomenon that the anomalous scaling behavior at the lower band edges will wash out LOGS in the specific heat of aperiodic QIM. The current conjecture gets its further evidence from the case of the Fibonacci quasiperiodic QIM. In that model the scaling properties of the lower band edges are governed by the exponent $\alpha_{\rm edge}=\ln\sigma_G^2/\ln K_2$, where $\sigma_G=(1+\sqrt{5})/2$ and K_2 is given in Refs. 18 and 22. The "dilution factor" $r=J_1/J_2$ dependence of $\alpha_{\rm edge}$ appears in K_2 through the constant of motion[18] $I=\omega^2(r-r^{-1})^2$. At the onset of criticality, with the vanishing of the energy gap 2ω, the system is expected to scale as in the periodic case with the scaling value independent of r,[20] which interprets the fact that the quasiperiodic QIM has the usual LOGS in the specific heat.

V. SUMMARY

In this paper, we have considered a quantum Ising chain with both the exchange couplings and the transverse fields arranged in a hierarchical way. Exact analytical results for the critical line and the dispersion relation for the low-energy excitations are obtained. It is shown that only in the case where $R_1=R_2=R>R_c$, the system preserves an LOGS in the specific heat. In this case, the model belongs to the Ising universality class. Employing renormalization-group analyses, we have studied the scaling properties of the spectrum at the onset of the criticality in the case of $R_1=R_2=R$. The scaling exponent governing the lower band edges is found to undergo a phase transition as the hierarchical parameter R is varied. For $R>R_c$, the usual scaling law for the periodic case is obtained, while for $R<R_c$ one has the anomalous scaling exponent explicitly dependent on the parameter R. The critical value of R_c is believed to be relevant to the existence of LOGS in the specific heat. To clarify the close relationship furthermore, we have reanalyzed the scaling properties of the energy spectrum in two cases where $R_1=R_2$ and $R_2=1$ in terms of an interative trace map for the transfer matrix. Analyses on both cases, together with the brief discussion on the Fibonacci quasiperiodic QIM, also show the close relationship between the absence of the usual LOGS in the specific heat and the anomalous scaling behavior at the lower band edges. We conjecture that it is a general phenomenon that the ferromagnetic aperiodic QIM has the usual LOGS in the specific heat only when the lower band edges obey the usual scaling law for the periodic case.

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APPENDIX

In this appendix we present the detailed derivation of (2.11). Defining

$$\xi(i)=R_2(i)/R_1(i)=\begin{cases}1, & i=2l+1 \ ;\\ \xi^n, & i=2^n(2l+1) \ ,\end{cases} \tag{A1}$$

and paying attention to (2.8), we have

$$\begin{aligned}|\varphi_0|^2&=\sum_{j=1}^{2^N}\varphi_1^2\prod_{i=1}^{j-1}J_c^2(i)/h^2(i+1)\\ &=\varphi_1^2\sum_{j=0}^{2^N-1}Q_j \ ,\end{aligned} \tag{A2}$$

where $Q_j=[1/R_2(j+1)]^2\prod_{i=1}^{j}\xi^2/\xi^2(i)$.

By representing the number j in the binary system

$$X_NX_{N-1},\ldots,X_2X_1 \tag{A3}$$

where X_i may be either 1 or 0, we obtain

$$\prod_{i=1}^{j}\xi^2/\xi^2(i)=\xi^{2m} \tag{A4}$$

with m being the number of 1's in the set $\{X_i|i=1,2,\ldots,N\}$. When j takes the form (A3) with $X_1=X_2=\cdots=X_{n-1}=1$, $X_n=0$ and X_i $(i>n)$ being either 0 or 1, with the use of $R_2(j+1)=R_2^{n-1}$, we have

$$Q_j=\xi^{2m}/R_2^{2(n-1)}=\xi^{2j_n}/R_1^{2(n-1)}$$

where $j_n=m-n+1$ is the number of 1's in the subset $\{X_i|i=n+1,n+2,\ldots,N\}$.

Thus

$$|\varphi_0|^2=\varphi_1^2\left[\sum_{n=1}^{N}\sum_{j'=0}^{N-n}C_{j'}^{N-n}\xi^{2j'}/R_1^{2(n-1)}+1/(R_2^2R_1^{2N})\right], \tag{A5}$$

where $C_q^p=p!/[q!(p-q)!]$ and the last term in the right hand side is Q_{2^N-1}. After some algebra, one has the first equation of (2.11) from (A5). In a similar and simpler way, we have

$$|\psi_0|^2=\psi_1^2\sum_{j=0}^{2^N-1}\prod_{i=0}^{j}\xi^2(i)/\xi^2$$

$$=\psi_1^2\sum_{m=0}^{N}C_m^N\xi^{-2m}=\psi_1^2(1+\xi^{-2})^N . \tag{A6}$$

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