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Solitary-wave propagation in superfluid ^{4}He films

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It is shown that in superfluid ^{4}He films, finite-amplitude localized density fluctuations can propagate as pulse-type solitary waves. These excitations are obtained by solving the nonlinear evolution equation for the superfluid order parameter associated with a microscopic pseudospin model of hard-core bosons with nearest-neighbor interactions. For a given film thickness both "hot" and "cold" solitary waves are possible, depending on the initial disturbance. The solitary-wave velocity is always less than that of the linear third-sound mode in the system. The width and amplitude of the solitary wave are determined as functions of its velocity and the microscopic parameters of the model.

I. INTRODUCTION

Several years ago, Atkins[1] predicted that when a film of ^{4}He is locally excited (say by a heat pulse), surface waves made up of only the superfluid component would be created, since the normal component would remain pinned to the substrate due to viscosity. This phenomenon of third sound is by now well established experimentally. Third-sound modes appear as oscillations in the film thickness with a velocity much less than that of the first and second-sound modes present in bulk helium.

High-precision measurements of third-sound velocity in thin ^{4}He films were carried out by Rutledge, McMillan, Mochel and Washburn.[2] They interpreted their results using a reformulation of Landau's bulk superfluid hydrodynamics in terms of surface parameters appropriate for ^{4}He films. Using a phenomenological approach, they derived the following nonlinear evolution for the superfluid order parameter ψ:

$$i\hbar(\partial\psi/\partial t) = -(\hbar^2/2m)\nabla^2\psi - A_R\psi/(a_R+|\psi|^2)^3 - \mu\psi - B\psi\nabla^2|\psi|^2 , \tag{1.1}$$

where m is the mass of a ^{4}He atom, μ is the chemical potential, A_R and a_R are phenomenological parameters which depend on the Van der Waals interaction binding the film to the substrate, and B depends on the surface tension of the film. Considering *small-amplitude* modes of the form

$$\psi=\psi_0+\chi_k\sin(\mathbf{k}\cdot\mathbf{r}-\omega_k t)+\eta_k\cos(\mathbf{k}\cdot\mathbf{r}-\omega_k t) , \tag{1.2}$$

Rutledge *et al.*[2] obtained the following Bogoliubov-like excitation spectrum[3] on using a linear approximation:

$$\omega_k^2 = c_3^2 k^2 + (k^4/4m^2)(\hbar^2+4Bm\rho_0) . \tag{1.3}$$

Here $\rho_0=|\psi_0|^2$ is the average superfluid surface density of the film. The third-sound velocity c_3 is given by

$$c_3^2 = 3A_R\rho_0/m(a_R+\rho_0)^4 . \tag{1.4}$$

Intriguing nonlinear effects such as an incipient shock behavior[4] (which develops as the wave amplitude changes) and the propagation of an undistorted pulse have been observed in experiments on third sound. These suggest the possibility of a mode with a velocity-dependent amplitude. Clearly, such observations cannot be explained by small amplitude modes such as (1.2). Huberman[5] was the first to recognize this. Starting with Eq. (1.1) and considering unidirectional propagation parallel to the substrate, he proceeded in a *heuristic* fashion to show that under certain approximations, the fluctuation f in the superfluid density (ρ_0-f) satisfies the Korteweg–de Vries (KdV) equation:[6]

$$\frac{\partial f}{\partial t}+\alpha_1 f\frac{\partial f}{\partial X}+\alpha_2\frac{\partial^3 f}{\partial X^3}=0 . \tag{1.5}$$

Here $X=(x+c_3,t)$, $\alpha_1=c_3$, and $\alpha_2=(\hbar^2+4Bm\rho_0)/8m^2c_3$. Equation (1.5) supports a soliton solution

$$f=A\,\mathrm{sech}^2[(x-vt)/\Delta] , \tag{1.6}$$

where A is arbitrary and the soliton width Δ is

$$\Delta=[3(\hbar^2+4Bm\rho_0)/2m^2c_3^2A]^{1/2} . \tag{1.7}$$

The soliton velocity v is

$$v=c_3(1-A/3), \quad or \quad A=3(c_3-v)/c_3 . \tag{1.8}$$

For *thin* films, Rutledge *et al.*[2] have shown that the quantity $(\hbar^2+4Bm\rho_0)$ which occurs in Eq. (1.3) must be positive. Thus for Δ in Eq. (1.7) to be real, A must be positive, i.e., $v<c_3$. Therefore, as v increases the amplitude decreases while the width increases. Now, $f>0$ corresponds to a *depletion* of the superfluid density. These modes are termed "hot" solitons. In the present case of thin films, this theory predicts that they are slower than the third-sound modes.

For thicker films, Huberman conjectured that one might have "cold" solitons (i.e., a crest wave, $f<0$) which would move faster than third sound. Subsequently, Biswas and Warke[7] applied the more systematic method of reductive perturbation theory to study *lowest-order* nonlinear effects in Eq. (1.1). They showed that below (above) a certain critical thickness, hot (cold) KdV solitons can exist. But *both* travel more slowly than the third-sound mode, contrary to Huberman's conjecture. Further, at the critical thickness, soliton modes are found to be absent according to this analysis.

Rutledge *et al.*[2] have shown experimentally that the phenomenological parameter B in Eq. (1.1), which is a measure of surface tension, becomes negligible in the thin-film limit. Kurihara[8] has solved Eq. (1.1) for $B=0$ to obtain a hot solitary wave. For weak nonlinearities, this wave is shown to reduce to a hot KdV soliton. However, beyond a certain thickness the KdV soliton becomes unstable. Hence for thicker films one must consider $B\neq 0$. Analytical solutions of Eq. (1.1) have not been found in this case. Numerically, Kurihara[9] has shown that the presence of B increases the width of the solitary wave. This is physically to be expected, since a nonvanishing surface tension makes it energetically unfavorable to create a narrow width pulse. For $B=0$, it has been shown (numerically) that the solitary waves are quite stable under collisions, suggesting that they could be (strict) solitons.[10]

The possible existence of solitons in a weakly excited superfluid film has also been suggested on the basis of Landau's two-fluid hydrodynamics by Nakajima *et al.*[11] under the assumption of zero temperature and weak nonlinearity. They used a method analogous to the classical derivation for waves on water in a shallow canal originally given by Korteweg and de Vries,[6] with the difference that the force of gravity is replaced by the Van der Waals interaction of the film with the substrate. They showed that thin films support hot solitons and thick films support cold solitons. The generalization of this work to finite temperatures, using Bergman's[12] (two-fluid) hydrodynamical model, has been carried out by Browne.[13] This approach (which considers only lowest-order nonlinearities) shows that thin films support hot KdV solitons which move faster than third sound, while thick films support cold solitons which are slower than the third-sound modes. This result contradicts both Huberman's conjecture for thick films as well as the results of Biswas and Warke[7] for thin films.

The models used so far for studying the possibility of solitary-wave propagation are *phenomenological* in nature. Besides, the classical two-fluid model is not very suitable for thin films wherein the quantum motion of ^{4}He atoms becomes significant. Furthermore, as discussed above, the results based on different models are in contradiction with each other. It is therefore desirable to study this problem using a quantum mechanical formalism based on a *microscopic* model of interacting ^{4}He atoms. This provides a motivation for this paper.

Recently, we have formulated a microscopic theory[14] for superfluid ^{4}He using a pseudospin model, which describes a system of hard-core bosons with nearest-neighbor interactions. We have derived a nonlinear evoultion equation for the superfluid order parameter in this model, and shown that it leads to a physically realistic description of vortices—a vortex core of finite thickness with a nonsingular vorticity is obtained. We have also shown that it takes into account depletion effects in a consistent fashion.[15] In the present work, it will be shown that the model supports pulse-type solitary-wave solutions for the superfluid density fluctuation. The layout of the paper is as follows: In Sec. II, we review the pseudospin model and present the evolution equation for the order parameter derived in earlier work.[14] We obtain the solitary wave solutions for the superfluid density fluctuation in Sec. III and show that their propagation in both thick films ($\sim 10^{-6}$ cm) and thin films ($\sim 10^{-7}$ cm) can be described in a unified fashion in this approach. For a given thickness, a hot or cold solitary wave propagates, depending on the initial disturbance. Both modes are found to be slower than the linear (third-sound) mode; the width of each mode increases as its velocity increases. Other conclusions are summarized in Sec. IV.

II. THE MODEL

It is possible to describe a system of interacting hard-core bosons with nearest-neighbor attractive interactions by using the pseudospin model proposed by Matsubara and Matsuda.[16] The details are summarized in Ref. 14. The salient feature of this lattice model is that the hard core is incorporated by demanding Fermi-like anticommutation relations for the field operators at the same site, while retaining Bose-like commutation relations for those at different sites. It can be shown that the field operators behave like $S=\frac{1}{2}$ spin-flip operators and that the system is effectively described by the following anisotropic Heisenberg spin Hamiltonian:

$$H=-\sum_{l}\left[(b-\mu)S_l^z+\frac{\hbar^2}{4ma^2}\sum_{\{\delta\}}(S_l^xS_{l+\delta}^x+S_l^yS_{l+\delta}^y)+v_0\sum_{\{\delta\}}S_l^zS_{l+\delta}^z\right], \tag{2.1}$$

where δ runs over the nearest neighbors of the lattice site l, and

$$b=D\left[\frac{\hbar^2}{ma^2}-v_0\right]. \tag{2.2}$$

D is the dimensionality of the lattice, and $-v_0$ $(v_0>0)$ is the attractive interaction between nearest-neighbor ^{4}He atoms separated by lattice spacing a. The evolution equation for the spin-flip operator S_l^+ is found by using

$$i\hbar\frac{\partial S_l^+}{\partial t}+[H,S_l^+]=0. \tag{2.3}$$

In our formalism, the superfluid order parameter is given by the spin coherent state[17] average $\langle S_l^+\rangle$:

$$\eta_l=\langle S_l^+\rangle=\tfrac{1}{2}\sin\theta_l\exp(i\phi_l), \tag{2.4}$$

where θ_l and ϕ_l are the polar and azimuthal angles of a classical spin vector at the site l. We also have

$$\langle S_l^z\rangle=\tfrac{1}{2}\cos\theta_l=\tfrac{1}{2}(1-4|\eta_l|^2)^{1/2}, \tag{2.5}$$

$$\rho_l=\tfrac{1}{2}-\langle S_l^z\rangle=\sin^2(\tfrac{1}{2}\theta_l)\ , \tag{2.6}$$

and

$$|\eta_l|^2=\rho_l(1-\rho_l)\ . \tag{2.7}$$

Finding the diagonal matrix elements of Eq. (2.3) in the spin-coherent representation [using Eqs. (2.4)–(2.7)] and going over to the continuum version appropriate for the description of a fluid, we get[14]

$$i\hbar(\partial\eta/\partial t)=\{b[1-(1-4|\eta|^2)^{1/2}]-\mu\}\eta-(\hbar^2/2m)(1-4|\eta|^2)^{1/2}\nabla^2\eta$$
$$-v_0a^2\eta[\nabla^2|\eta|^2+2(\nabla|\eta|^2)^2(1-4|\eta|^2)^{-1}](1-4|\eta|^2)^{-1/2}\ . \tag{2.8}$$

The dimensionality D of the system occurs in b. It is interesting to note that the parameters occurring in the phenomenological equation (1.1) used by Rutledge *et al.*[2] in the discussion of ^{4}He films can be written in terms of the parameters occurring in the microscopic Hamiltonian (2.1) as follows. Neglecting certain higher-order nonlinearities such as $(\nabla|\eta|^2)^2\eta$ in Eq. (2.8), expanding the nonpolynomial terms in powers of $|\eta|^2$ in both Eq. (2.8) and Eq. (1.1), and comparing the two equations we find that

$$\frac{3A_R}{2a_R^4}=b,\quad B=v_0a^2,\quad \text{and}\ \mu_R+\frac{A_R}{a_R^3}=\mu\ . \tag{2.9}$$

We note also that in Eq. (2.8), using small amplitude solutions as in Eq. (1.2) and linearizing gives[14]

$$c_3^2=(2b/m)\rho_0(1-\rho_0)\ . \tag{2.10}$$

Substituting $\eta=\frac{1}{2}\sin\theta\exp(i\phi)$ [see Eq. (2.4)] in Eq. (2.8) and equating real and imaginary parts yields

$$\frac{\partial\theta}{\partial t}=(-\hbar/2m)[2\cos\theta(\nabla\theta)\cdot(\nabla\phi)+\sin\theta\nabla^2\phi]\ , \tag{2.11}$$

and

$$\hbar\frac{\partial\phi}{\partial t}=\mu-b+b\cos\theta$$
$$+(\sin\theta)^{-1}[(\hbar^2/2m)\cos^2\theta+\tfrac{1}{2}v_0a^2\sin^2\theta]\nabla^2\theta$$
$$-(a^2b/2D)(\nabla\theta)^2\cos\theta-(\hbar^2/2m)(\nabla\phi)^2\cos\theta\ . \tag{2.12}$$

We must of course set $D=2$ in b [see Eq. (2.2)] in order to describe a ^{4}He film.

III. SOLITARY-WAVE SOLUTIONS

In this section, we specialize to unidirectional flows and show that the fluctuation in the superfluid density supports solitary-wave solutions. Considering $\theta=\theta(x,t)$ and $\phi=\phi(x,t)$, and denoting $\partial\theta/\partial t$ by θ_t, etc., Eqs. (2.11) and (2.12) yield

$$\theta_t\sin\theta=-(\hbar/2m)(\phi_x\sin^2\theta)_x \tag{3.1}$$

and

$$\hbar\phi_t\sin\theta=(\mu-b)\sin\theta+b\sin\theta\cos\theta+(\hbar^2/2m)\theta_{xx}$$
$$-\tfrac{1}{4}ba^2[\theta_{xx}\sin^2\theta+\sin\theta\cos\theta(\theta_x)^2]$$
$$-(\hbar^2/2m)\cos\theta\sin\theta(\phi_x)^2\ . \tag{3.2}$$

Using Eqs. (2.6) and (2.7), Eq. (3.1) becomes

$$\rho_t=-(\hbar/m)[\rho(1-\rho)\phi_x]_x\ . \tag{3.3}$$

We seek solutions of (3.2) and (3.3) of the form

$$\phi=\omega t+\phi(z)\ \text{and}\ \rho=\rho(z)\ , \tag{3.4}$$

where

$$z=(x-vt)/a\ .$$

Equation (3.3) can be readily solved to give

$$V\rho=\rho(1-\rho)\phi_z+c_1\ , \tag{3.5}$$

where c_1 is a constant of integration, and

$$V=mva/\hbar\ . \tag{3.6}$$

The boundary conditions $\rho\to\rho_0$ and $\phi_z\to 0$ as $|z|\to\infty$ give $c_1=V\rho_0$. Thus Eq. (3.5) reduces to

$$\phi_z=V(\rho-\rho_0)/[\rho(1-\rho)]$$
$$=4V(\sin^2\tfrac{1}{2}\theta-\rho_0)/\sin^2\theta\ . \tag{3.7}$$

Similarly using $\phi_t=\omega-(v/a)\phi_z$, Eq. (3.2) becomes

$$-V\sin\theta\phi_z=(E_1-E_2)\sin\theta+E_2\sin\theta\cos\theta+\tfrac{1}{2}\theta_{zz}$$
$$-\tfrac{1}{4}E_2\sin\theta(\theta_z\sin\theta)_z-\tfrac{1}{2}\cos\theta\sin\theta\phi_z^2\ , \tag{3.8}$$

where

$$E_1=ma^2(\mu-\hbar\omega)/\hbar^2 \tag{3.9a}$$

and

$$E_2=ma^2b/\hbar^2\ . \tag{3.9b}$$

Substituting Eq. (3.7) in Eq. (3.8) and simplifying we get

$$v^2[(2\rho_0-1)\tan(\tfrac{1}{2}\theta)\sec^2(\tfrac{1}{2}\theta)+8\rho_0^2(\csc^3\theta\csc\theta]$$
$$=(E_1-E_2)\sin\theta+E_2\sin\theta\cos\theta+\tfrac{1}{2}\theta_{zz}$$
$$-\tfrac{1}{4}E_2\sin\theta(\theta_z\sin\theta)_z\ . \tag{3.10}$$

Multiplying both sides of Eq. (3.10) by θ_z and integrating, we have

$$\tfrac{1}{4}(\theta_z)^2(1-\tfrac{1}{2}E_2\sin^2\theta) = V^2[(2\rho_0-1)\tan^2(\tfrac{1}{2}\theta)-(4\rho_0^2/\sin^2\theta)] + (E_1-E_2)\cos\theta - \tfrac{1}{2}E_2\sin^2\theta - C_2 \; , \tag{3.11}$$

where C_2 is the constant of integration. Multiplying both sides of Eq. (3.11) by $\frac{1}{4}\sin^2\theta$ and setting $\sin^2\frac{1}{2}\theta=\rho$, we get

$$\tfrac{1}{4}\rho_z^2[1-2E_2\rho(1-\rho)] = -\rho_0^2V^2+(E_1-E_2-C_2)\rho + [V^2(2\rho_0-1)-(3E_1-E_2-C_2)]\rho^2 + 2(E_1+E_2)\rho^3 - 2E_z\rho^4 \; . \tag{3.12}$$

Writing

$$\rho=\rho_0+f(z) \; , \tag{3.13}$$

with the boundary condition $f(z)\to 0$ as $|z|\to\infty$, we obtain for the superfluid density fluctuation $f(z)$ the following nonlinear equation:

$$\tfrac{1}{2}(df/dz)^2[\tfrac{1}{2}-E_2\rho_0(1-\rho_0)-E_2f(1-2\rho_0)+E_2f^2] = K_0+K_1f+K_2f^2+K_3f^3+K_4f^4 \; , \tag{3.14}$$

where

$$K_0=(E_1-E_2-C_2)\rho_0+(E_2+C_2-3E_1-2V^2)\rho_0^2 + 2(E_1+E_2+V^2)\rho_0^3-2E_2\rho_0^4 \; , \tag{3.15a}$$

$$K_1=(E_1-E_2-C_2)+2(E_2+C_2-3E_1-V^2)\rho_0 + 2(3E_1+3E_2+2V^2)\rho_0^2-8E_2\rho_0^3 \; , \tag{3.15b}$$

$$K_2=(E_2+C_2-3E_1-V^2) + 2(3E_1+3E_2+V^2)\rho_0-12E_2\rho_0^2 \; , \tag{3.15c}$$

$$K_3=2(E_1+E_2)-8E_2\rho_0 \; , \tag{3.15d}$$

and

$$K_4=-2E_2 \; . \tag{3.15e}$$

Neglecting nonlinear terms of the form $f(df/dz)^2$ and $f^2(df/dz)^2$ [a good approximation for slowly varying functions $f(z)$] and retaining all other nonlinearities, we get

$$\gamma^{-2}\left[\frac{df}{dz}\right]^2 = \sum_{n=0}^{4} K_nf^n \; , \tag{3.16}$$

where

$$\gamma^2=4[1-2E_2(\rho_0(1-\rho_0)]^{-1}=4[1-(ma/\hbar)^2c_3^2]^{-1} \; .$$

Equation (3.16) supports pulse-type solitary-wave solutions. These are obtained by choosing the integration constant C_2 and the frequency ω in the condensate phase ϕ [cf. Eq. (3.4)] appropriately—for instance, so as to have $K_0=K_1=0$. Imposing the latter conditions, we find that we have only to choose

$$E_1=2E_2\rho_0$$

and (3.17)

$$C_2=-2V^2\rho_0-E_2(1-2\rho_0+2\rho_0^2) \; .$$

In terms of the basic physical parameters, this means setting

$$\omega=(\mu-2b\rho_0)/\hbar$$

and (3.18)

$$C_2=-(ma^2/\hbar)[2mv^2\rho_0+b(1-2\rho_0+2\rho_0^2)] \; .$$

Using these values, the remaining constants K_2, K_3, and K_4 reduce, respectively, to

$$K_2=(ma/\hbar)^2(c_3^2-v^2) \; ,$$
$$K_3=(2ma^2b/\hbar^2)(1-2\rho_0) = (ma/\hbar)^2c_3^2(1-2\rho_0)/\rho_0(1-\rho_0) \; , \tag{3.19}$$
$$K_4=-2ma^2b/\hbar^2=-(ma/\hbar)^2c_3^2/\rho_0(1-\rho_0) \; .$$

We recall that c_3 is the third-sound velocity obtained from the linearized microscopic theory [see Eq. (2.10)].

Equation (3.16) yields

$$\tilde{z}=\pm\tfrac{1}{2}\int\frac{df}{f(K_2+K_3f+K_4f^2)^{1/2}}+\tilde{z}_0 \; , \tag{3.20}$$

where $\tilde{z}=\gamma z$ and $\tilde{z}_0$ is the integration constant. Hence

$$\pm\tanh[2(K_2)^{1/2}(\tilde{z}-\tilde{z}_0)]=2K_2^{1/2}(K_2+K_3f+K_4f^2)^{1/2}/(2K_2+K_3f) \; . \tag{3.21}$$

Solving for the density fluctuation f, we obtain

$$f^{(\pm)}(\tilde{z})=\frac{2K_2}{\pm(K_3^2-4K_2K_4)^{1/2}\cosh[2(K_2)^{1/2}(\tilde{z}-\tilde{z}_0)]-K_3} \; , \tag{3.22}$$

where K_2, K_3, and K_4 are as in Eqs. (3.19). Note that Eq. (3.22) represents two possible pulse-type solitary-wave solutions for the density fluctuation $f(z)$, provided $K_2>0$ or $v^2<c_3^2$. In this case $(K_3^2-4K_2K_4)$ is always positive. [For $K_2<0$ it is possible to obtain "nonlinear" periodic waves, with $v^2>c_3^2$, provided $|c_3^2-v^2| <(\hbar^2/8ma^2)(1-2\rho_0)^2$. These are not of interest to us in the present context.]

We summarize our results.

(i) For a pulse solitary wave to exist, we must have

$$K_2>0, \text{ i.e., } -c_3<v<c_3 \ . \tag{3.23}$$

Thus the solitary wave is always slower than the third-sound mode.

(ii) For given values of ρ_0, V, and E_2 (or equivalently, v_0), we can have *two* possible solitary-wave solutions with the same width but different amplitudes. They are

$$\rho^{(\pm)}(z)=\rho_0+f^{(\pm)}(z) \ , \tag{3.24}$$

with $f^{(+)}>0$ and $f^{(-)}<0$, i.e., cold and hot solitary waves. Since the amplitudes or maximum fluctuations (for a given v) are given by

$$f_m^{(\pm)}=2K_2/[(K_3^2+8E_2K_2)^{1/2}\mp K_3] \ , \tag{3.25}$$

we have $f_m^{(-)}<f_m^{(+)}$. This suggests that for a given film thickness, to create a cold solitary wave we need a cooling pulse which should be typically much stronger than the heat pulse required to create a hot solitary wave of the same width and velocity. A hot wave is not merely a sign-reversed cold wave.

(iii) From Eq. (3.22), the width Γ is found to be

$$\begin{aligned}\Gamma&=a/(2\gamma K_2^{1/2})\\ &=\hbar[1-(ma/\hbar)^2c_3^2]^{1/2}/4m(c_3^2-v^2)^{1/2} \ .\end{aligned} \tag{3.26}$$

Hence as the velocity v of the solitary wave increases from $v=0$, its width increases from the minimum value $(\hbar/4mc_3)[1-(ma/\hbar)^2c_3^2]^{1/2}$. Note that $c_3^2<(\hbar/ma)^2$. The velocity dependence of the amplitude may be obtained from Eq. (3.25).

(iv) The quantities K_2, K_3, and K_4 depend on c_3. Since c_3 in turn depends[2] on the film thickness, Eq. (3.22) shows that for a given thickness, it is possible to have the propagation of a hot or cold solitary wave, depending on the initial disturbance. [The applied pulse may be a heating (depleting) pulse or a cooling (thickness-enhancing) pulse.]

(v) In our model, Eq. (3.26) shows that the width Γ increases as c_3 decreases. And it is known[2] experimentally that as the thickness increases, c_3 decreases. (This can also be inferred from the expression $c_3^2=4[(\hbar^2/ma^2)-v_0]\rho_0(1-\rho_0)/m$: In a phenomenological interpretation, v_0 plays the role of the surface tension parameter B [see Eq. (2.9)] which is an increasing function of thickness.) Thus thicker films will support wider solitary waves. We have noted that in the model of Rutledge *et al.*[2] Kurihara[9] found numerically that the solitary-wave width for $B\neq 0$ (thick films) was greater than that for $B=0$. This behavior agrees with that of the analytical expression we have derived for Γ.

The question of whether these solitary waves are (strict) solitons[10] can be answered either by constructing a Lax pair for Eq. (2.8) and showing its complete integrability, or by performing a numerical experiment to see if the solitary wave retains its identity on collision with another solitary wave.

IV. CONCLUDING REMARKS

The suggestion that superfluid ^{4}He may be a good system to study the propagation of nonlinear waves experimentally was first made by Tsuzuki.[18] In that work, the reductive perturbation method for small amplitude solutions was applied to the Gross-Pitaevskii[19] (GP) equation for the condensate order parameter to obtain a KdV equation for the density fluctuation which was shown to support only hot solitons. As is well known, the GP model is not a very realistic one since it neglects the important hard-core interactions between ^{4}He atoms. Subsequently, the phenomenological model of Ref. 2 for ^{4}He films was used to study solitons. Huberman's heuristic analysis[5] showed that thin films can support hot KdV solitons which have a speed smaller than that of third sound, whereas thick films support cold solitons which move faster than third-sound modes. On the other hand, the more systematic analysis of the same model by Biswas and Warke[7] predicts that both types of solitons are slower than third sound. Browne's[13] results based on the phenomenological two-fluid model also predict hot and cold KdV solitons in thin and thick films, respectively. However, the hot soliton is found to move faster than third sound, contradicting the conclusions of both Refs. 5 and 7. It has also been suggested[13] that the two-fluid model analysis has perhaps neglected "half" of the possible soliton modes due to the approximations used, and that one might have *both* hot and cold solitons for a given film thickness.

Such contradictory results arise essentially because the underlying evolution equation for the superfluid order parameter has not been derived from first principles in any of the earlier treatments, leading to an inadequate incorporation of the inherent nonlinear effects in the system. The present treatment, on the other hand, is based on an evolution equation derived from a microscopic Hamiltonian that includes hard-core effects and nearest-neighbor interactions. We find that the evolution equation for the density fluctuation is not the KdV equation when all nonlinearities are retained, but two coupled nonlinear equations [Eqs. (2.11) and (2.12)]. Retaining all the important nonlinear terms in these leads to a pulse-type solution different from a KdV soliton. Our analysis shows unambiguously that for a given thickness, it is indeed possible to have both hot and cold solitary waves depending on the initial disturbance. We have shown that the solitary-wave velocity is always less than the third-sound velocity (which is $\sim 10^3$ cm/sec). Equation (3.26) shows the dependence of the width of the wave on its velocity and on the third-sound velocity which is expressed in terms of the microscopic parameters of the interacting hard-core boson model.

While our formalism (and the evolution equation derived from it) is a general one, we have analyzed in detail only unidirectional nonlinear waves on films in the foregoing. The possibility of having two-dimensional nonlinear excitations in a superfluid film has also been studied[20] within the framework of the model of Ref. 2 by using reductive perturbation theory for small-amplitude perturbations in thin films. It would be of interest to carry out an analysis of our model in the case of two-dimensional wave propagation. If one works at temperatures well below the Kosterlitz-Thouless transition temperature,[21] damping effects due to vortices on wave propagation will be negligible.[13,22] In spite of the existence of the various theories of soliton propagation discussed above, there are at present very few experimental studies[23] designed to verify theoretical predictions in a systematic fashion. Factors such as the smallness of the soliton width as compared to the macroscopic size of the detector could pose practical difficulties. An ideal experimental setup should have very sensitive detectors capable of detecting narrow width ($\sim 10^{-6}$ cm) solitary waves. Our results also show that it may be better to work with thicker films since they support wider excitations. We hope that the present work will provide further motivation for a careful experimental study of the nonlinear modes in superfluid ^{4}He.

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