Minimal renormalization without ϵ expansion: Critical behavior above and below T_c

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We present a field-theoretic renormalization-group treatment of the critical behavior of the φ^4 model within the minimal-subtraction scheme at fixed dimension $d<4$. We show that correlation functions above and below T_c can be calculated within this renormalization scheme without using the $\epsilon=4-d$ expansion. This is demonstrated for the specific heat, for the order parameter, and for the superfluid density (stiffness constant). Various asymptotic amplitudes and correction amplitudes are determined, and some of their universal ratios are calculated in three dimensions. Our result for a_c^-/a_{ρ_s} eliminates a previous error of the ϵ expansion regarding the sign of this ratio for $d=3$ and $n=2$.

I. INTRODUCTION

Because of its simplicity and elegance, the minimal-subtraction scheme[1] is the most popular renormalization procedure in the field-theoretic treatment of critical phenomena.[2] In numerous applications this scheme has been used in combination with an ϵ expansion around the upper or lower borderline dimension.[2] Recently we have shown[3] that the restriction to the ϵ expansion is unnecessary and that the minimal subtraction scheme can be used at fixed dimension $d<4$ provided that Symanzik's nonvanishing-mass shift[4] is properly taken into account.

Our recent work[3] was focused on the simplest case, i.e., on multiplicatively renormalizable quantities of the φ^4 model above T_c. Thus, we did not explicitly treat the specific heat which is the most favorable candidate for a quantitive comparison between theory and experiment. The concept of calculating a minimally renormalized specific heat in three dimensions above and below T_c without using the ϵ expansion was introduced previously by one of us[5] and was subsequently extended to the superfluid density.[6] In this paper we shall demonstrate in detail the validity of this concept and shall give further applications above and below T_c.

A particular advantage of the minimal-subtraction scheme is due to a natural and simple decomposition of renormalized correlation functions into exponential parts and amplitude functions. The exponential parts are determined entirely from pure pole terms $\sim(4-d)^{-n}$. These pole terms are identical above and below T_c and remain unchanged even at finite k, ω, and L (wave number, frequency, and system size). Thus, the exponential parts can be easily employed in extensions of the theory from $T>T_c$ to $T<T_c$ as well as in applications to critical dynamics,[7] to finite k, ω, and L,[8-11] and to surface critical phenomena.[12,13] In our earlier work[3,6] we have calculated these exponential parts (exponent functions and β function) to very good accuracy by means of the Borel resummation method[14] within the minimal-subtraction scheme.

Recently we have shown[15] that for the nonexponential parts (amplitude functions) the higher-order computations and Borel summations appear to be considerably less important, at least above T_c. The resummed higher-order corrections to low-order expressions of amplitude functions turned out to be quite small[15] within the minimal-renormalization scheme in three dimensions. A corresponding property is not known to be valid for other renormalization schemes, e.g., for that used in the $d=3$ theory of Bagnuls and Bervillier[16,17] which implies different decompositions between exponential and nonexponential parts.

Therefore, within our minimal-subtraction approach at $d=3$, it appears to be justified to extend the theory even to those cases where higher-order computations and Borel resummations are not (or not yet) possible, i.e., where the amplitude functions are known only to low order of perturbation theory. This includes the important case $n>1$ (n is the number of components of the order parameter) below T_c which was not treated by Bagnuls and Bervillier.[17] We argue that, within the minimal-renormalization approach, this case does not necessarily require new Borel resummations because the accurately known[3,6] exponential parts dominate the critical behavior whereas the amplitude functions constitute only smoothly varying prefactors which we expect to be well approximated by low-order calculations provided that the full dimensional dependence of the leading diagrams is appropriately taken into account.[3,5] This expectation is not only based on our accurate results *above* T_c (Ref. 15) but also on the good agreement between our previous theory *below* T_c (Refs. 5, 6, 8, 18, and 19) and the accurate data for the specific heat and superfluid density of ^{4}He.[20-22] We also mention the good agreement between the thermal conductivity data[23] and the amplitude function of this quantity calculated within the minimal-renormalization scheme in low order of perturbation theory.[24]

In Sec. II the specific heat of the n-vector model is treated above T_c at fixed dimension $d<4$ within the minimal-renormalization scheme. The theory is extended to $T<T_c$ in Sec. III for the examples of the order parameter, the specific heat, and the superfluid density (stiffness constant). Asymptotic amplitudes and correction amplitudes of these quantities are given in Sec. IV. Corresponding universal amplitude ratios for the universality classes $n=2$ and 3 in $d=3$ are calculated in Sec. V.

II. SPECIFIC HEAT ABOVE T_c AT FIXED $d<4$

We consider the statistical distribution $\sim \exp-(H_0+H_\varphi)$ with the usual Landau-Ginzburg-Wilson functional[2,25]

$$H_\varphi=\int d^dx[\tfrac{1}{2}r_0\varphi_0^2+\tfrac{1}{2}(\nabla\varphi_0)^2+u_0\varphi_0^4] , \tag{2.1}$$

$$r_0=r_{0c}+a_0t, \quad t=(T-T_c)/T_c \tag{2.2}$$

for an n-component order-parameter field $\varphi_0(x)$. The spatial variations of $\varphi_0(x)$ are confined to wave numbers less than a finite cutoff Λ. The leading critical behavior of the bare (physical) specific heat $\mathring{C}$ per unit volume (divided by Boltzmann's constant k_B) near T_c is described by

$$\mathring{C}=C_B+T_c^2V^{-1}\frac{\partial^2}{\partial T^2}\ln\int D\varphi_0\exp-H_\varphi \tag{2.3}$$

$$=C_B+\tfrac{1}{4}a_0^2\mathring{C}_\varphi , \tag{2.4}$$

$$\mathring{C}_\varphi=\int d^dx[\langle\varphi_0^2(x)\varphi_0^2(0)\rangle-\langle\varphi_0^2\rangle^2] . \tag{2.5}$$

In (2.3) and (2.4) the term $C_B>0$ represents a "background" contribution which comes from the background free energy H_0. In this section we confine ourselves to $T>T_c$. We shall use the notation of Ref. 3 and study $\mathring{C}_\varphi$ in terms of the vertex function

$$\mathring{\Gamma}^{(2,0)}(r_0-r_{0c},u_0,d)=-\tfrac{1}{4}\mathring{C}_\varphi . \tag{2.6}$$

By expressing r_0-r_{0c} as a function of the correlation length ξ above T_c we define[3]

$$\tilde{\Gamma}_0^{(2,0)}(\xi,u_0,d)=\mathring{\Gamma}^{(2,0)}(r_0-r_{0c},u_0,d) \tag{2.7}$$

$$=\xi^\epsilon\mathring{f}^{(2,0)}(u_0\xi^\epsilon,d) \tag{2.8}$$

$$\equiv-\tfrac{1}{4}\mathring{C}_\varphi(\xi,u_0,d) . \tag{2.9}$$

Working with $\tilde{\Gamma}_0^{(2,0)}$ rather than $\mathring{\Gamma}^{(2,0)}$ circumvents the problem of the nonperturbative nature of $r_{0c}(u_0)$.[3,4,16,17] At fixed $\xi<\infty$, $u_0<\infty$, $d<4$ the function $\tilde{\Gamma}_0^{(2,0)}$ remains finite in the limit $\Lambda\to\infty$. Throughout this paper we take this limit and use the prescriptions of dimensionless regularization.[1,2] The dimensional function $\mathring{f}^{(2,0)}(u_0\xi^\epsilon,d)$ has the form

$$\mathring{f}^{(2,0)}(z,d)=\frac{1}{\epsilon}\sum_{m=0}^{\infty}a_m^{(2,0)}(d)\left[\frac{z}{\epsilon}\right]^m \tag{2.10}$$

with coefficients $a_m^{(2,0)}(d)$ that are finite for $d\le 4$. This series differs in structure from those of ordinary vertex functions[3] by the extra pole $1/\epsilon$. Thus, at given order u_0^m, the strength of the leading pole $\sim\epsilon^{-m-1}$ (ultraviolet divergence for $\epsilon\to 0$) exceeds that of the infrared divergence $\sim z^m$ (for $\xi\to\infty$, u_0 fixed), therefore the previous analysis[3] does not apply directly. In the following we shall show that this property does not invalidate our conclusion regarding the applicability of the minimal-subtraction scheme at fixed $d<4$ without an ϵ expansion.

In this analysis we shall need the renormalized parameters

$$u=\mu^{-\epsilon}Z_u(u,\epsilon)^{-1}Z_\varphi(u,\epsilon)^2A_du_0 , \tag{2.11}$$

$$r=at=Z_r(u,\epsilon)^{-1}a_0t , \tag{2.12}$$

and the effective coupling $u(l)$ determined by the flow equation

$$l\frac{du(l)}{dl}=\beta_u(u(l),\epsilon) \tag{2.13}$$

with $u(1)=u$. The flow parameter l will be chosen as

$$l=(\mu\xi)^{-1} . \tag{2.14}$$

For a convenient choice of the geometrical factor A_d in (2.11) see Refs. 3 and 5 and Eq. (3.13). μ^{-1} is an arbitrary reference length.

A. Representation of $\mathring{C}$ via $\partial\mathring{C}/\partial t$

Among the various correlation functions the specific heat is known[2,25] to require a special discussion because of an additive ultraviolet divergence as reflected by the extra $1/\epsilon$ factor in (2.10). A possible way of circumventing this complication is to study the derivative $\partial\mathring{C}/\partial t$ or $\partial\tilde{\Gamma}_0^{(2,0)}/\partial r_0$ which is multiplicatively renormalizable.[25] The t dependence of $\mathring{C}$ can then be obtained, up to a finite additive background part, by integration *after* the treatment of $\partial\mathring{C}/\partial t$ via renormalized field theory. Thus, we consider

$$\frac{\partial}{\partial r_0}\tilde{\Gamma}_0^{(2,0)}(\xi,u_0,d)\bigg|_{u_0}=\tilde{\Gamma}_0^{(3,0)}(\xi,u_0,d) \tag{2.15}$$

$$=\xi^{2+\epsilon}\mathring{f}^{(3,0)}(u_0\xi^\epsilon,d) . \tag{2.16}$$

This vertex function has the usual form of a power series in $u_0\xi^\epsilon/\epsilon$ and is multiplicatively renormalized as[3]

$$\tilde{\Gamma}^{(3,0)}(\xi,u,\mu,d)=Z_r^3\tilde{\Gamma}_0^{(3,0)}(\xi,\mu^\epsilon Z_uZ_\varphi^{-2}A_d^{-1}u,d) \tag{2.17}$$

$$=\xi^{2+\epsilon}f^{(3,0)}(\mu\xi,u,d) . \tag{2.18}$$

The dimensionless function $\mathring{f}^{(3,0)}$ can be represented as[3]

$$\mathring{f}^{(3,0)}(u_0\xi^\epsilon,d)=Z_r(u,\epsilon)^{-3}f^{(3,0)}(1,u(l),d)\exp\int_1^l 3\zeta_r\frac{dl'}{l'} . \tag{2.19}$$

Our reasoning will be based on the fact that the function $f^{(3,0)}(1,u,d)$ is finite for fixed $d\le 4$ in the range $0\le u\le u^*$ and can be calculated without an ϵ expansion.[3] Furthermore, we shall make use of the function

$$\mathring{P}(u_0\xi^\epsilon,d)=\left[\frac{\partial r_0}{\partial\xi^{-2}}\right]_{u_0} \tag{2.20}$$

$$=Z_r(u,\epsilon)P(\mu\xi,u,d) \tag{2.21}$$

$$=Z_r(u,\epsilon)P(1,u(l),d)\exp\int_l^1\zeta_r(u(l'))\frac{dl'}{l'} \tag{2.22}$$

whose renormalized counterpart $P(1,u,d)$ is also finite in $d\le 4$ and calculable without an ϵ expansion.[3] In order to express $\tilde{\Gamma}_0^{(2,0)}$ in terms of $f^{(3,0)}$ and P, we rewrite (2.15) as

$$\frac{\partial}{\partial \xi^{-1}} \tilde{\Gamma}_0^{(2,0)}(\xi,u_0,d)\Bigg|_{u_0} = 2\xi^{-1}\mathring{P}(u_0\xi^{\epsilon},d)\tilde{\Gamma}_0^{(3,0)}(\xi,u_0,d) \ . \tag{2.23}$$

Substitution of (2.18)–(2.22) into the right-hand side (rhs) of (2.23) and using (2.14) in the form

$$\xi=\mu^{-1}\exp\int_l^1 \frac{dl'}{l'} \tag{2.24}$$

leads to

$$\frac{\partial}{\partial \xi^{-1}} \tilde{\Gamma}_0^{(2,0)}(\xi,u_0,d)\Bigg|_{u_0} = \xi\mu^{-\epsilon}\tilde{G}(u(l),u,d) \tag{2.25}$$

with the dimensionless function

$$\tilde{G}(u',u,d)=2Z_r(u,\epsilon)^{-2}P(1,u',d)f^{(3,0)}(1,u',d) \times \exp\int_u^{u'} \frac{2\zeta_r(u'')-\epsilon}{\beta_u(u'',\epsilon)}du'' \ . \tag{2.26}$$

We may integrate (2.25) from a noncritical value $\xi^{-1}=\mu$ up to $\xi^{-1}=\mu l$ to arrive at

$$\mathring{C}=C_B-a_0^2\tilde{\Gamma}_0^{(2,0)}(\xi,u_0,d) \ , \tag{2.27}$$

where

$$\tilde{\Gamma}_0^{(2,0)}(\xi,u_0,d)=\tilde{\Gamma}_0^{(2,0)}(\mu^{-1},u_0,d) +\mu^{-\epsilon}\int_1^l \tilde{G}(u(l'),u,d)\frac{dl'}{l'} \tag{2.28}$$

with $l=(\mu\xi)^{-1}$. The last term contains the critical temperature dependence and provides the desired representation of the critical contribution to $\mathring{C}$ in terms of the functions $f^{(3,0)}(1,u,d)$ and $P(1,u,d)$; thus, this contribution can be calculated within the minimally renormalized theory at fixed $d<4$ without using the ϵ expansion, as anticipated previously.[5] By means of the Borel resummation, the functions $P(1,u,3)$ and $f^{(3,0)}(1,u,3)$ have recently been calculated for $n=1$, 2, 3, to very good accuracy.[15]

Unlike the critical contribution, the noncritical term (first term) on the rhs of Eq. (2.28) is not meaningfully calculable within a dimensionally regularized theory because this term depends significantly on the finite cutoff. This term should be combined with C_B to yield the total noncritical contribution

$$C_B^{\text{tot}}=C_B+\tfrac{1}{4}a_0^2\mathring{C}_{\varphi}(\mu^{-1},u_0,d) \tag{2.29}$$

[for the definition of $\mathring{C}_{\varphi}(\xi,u_0,d)$ see (2.9)]. It is C_B^{tot}, rather than C_B and $\frac{1}{4}a_0^2\mathring{C}_{\varphi}(\mu^{-1},u_0,d)$ separately, which should be considered as the appropriate background parameter in a comparison with the specific heat of real systems.

B. Representation of $\mathring{C}$ via additive renormalization

The more conventional treatment of $\tilde{\Gamma}_0^{(2,0)}$ starts from a multiplicative and additive renormalization[2,25]

$$\tilde{\Gamma}^{(2,0)}(\xi,u,\mu,d)=Z_r^2\tilde{\Gamma}_0^{(2,0)}(\xi,\mu^{\epsilon}Z_uZ_{\varphi}^{-2}A_d^{-1}u,d) -\tfrac{1}{4}\mu^{-\epsilon}A_d A(u,\epsilon) \ , \tag{2.30}$$

where $A(u,\epsilon)$ absorbs just the leading poles of $\tilde{\Gamma}^{(2,0)}$ mentioned after (2.10). In the following we wish to derive the relationship between this additive renormalization and the representation of Sec. II A. For this purpose it is convenient to rewrite $\tilde{\Gamma}^{(2,0)}$ as

$$\tilde{\Gamma}^{(2,0)}(\xi,u,\mu,d)=-\tfrac{1}{4}\mu^{-\epsilon}A_dF_+(\mu\xi,u,d) \ . \tag{2.31}$$

The dimensionless function F_+ satisfies the renormalization-group equation (RGE)

$$(\mu\partial_\mu+\beta_u\partial_u+2\zeta_r-\epsilon)F_+(\mu\xi,u,d)=4B(u) \tag{2.32}$$

with the d-independent function

$$4B(u)=[2\zeta_r-\epsilon]A(u,\epsilon)+\beta_u(u,\epsilon)\frac{\partial A(u,\epsilon)}{\partial u} \ . \tag{2.33}$$

Integration of the RGE (2.32) leads to

$$F_+(\mu\xi,u,d)=\left[F_+(1,u(l),d))-4\int_1^l B(u(l'))\left[\exp\int_l^{l'}(2\zeta_r-\epsilon)\frac{dl''}{l''}\right]\frac{dl'}{l'}\right]\exp\int_1^l(2\zeta_r-\epsilon)\frac{dl'''}{l'''} \ . \tag{2.34}$$

The solution $A(u,\epsilon)$ of (2.33) in terms of $B(u)$ reads

$$A(u,\epsilon)=4\int_0^u \frac{B(u')}{\beta_u(u',\epsilon)}\left[\exp\int_u^{u'}\frac{2\zeta_r(u'')-\epsilon}{\beta_u(u'',\epsilon)}du''\right]du' \tag{2.35}$$

with $A(0,\epsilon)=-4B(0)/\epsilon=-2n/\epsilon$. Substituting Eqs. (2.35) and (2.34) into (2.30) and using (2.9) we arrive at the following representation of the correlation function (2.5):

$$\mathring{C}_{\varphi}(\xi,u_0,d)=\mu^{-\epsilon}A_dZ_r(u,\epsilon)^{-2}K_+(u(l),d) \times\exp\int_u^{u(l)}\frac{2\zeta_r(u')-\epsilon}{\beta_u(u',\epsilon)}du' \tag{2.36}$$

with

$$K_+(u,d)=F_+(1,u,d)-A(u,\epsilon) \ . \tag{2.37}$$

From (2.26)–(2.28), (2.36), and (2.37) the following relationship between $F_+(1,u,d)$, $P(1,u,d)$ and $f^{(3,0)}(1,u,d)$ can be derived (we drop the arguments of these functions):

$$8A_d^{-1}Pf^{(3,0)}=(\epsilon-2\zeta_r)F_++4B-\beta_u\partial F_+/\partial u \ . \tag{2.38}$$

We conclude that the function $F_+(1,u,d)$ is finite for $d\le 4$ and calculable at fixed d without using the ϵ expansion. We note that a distinction between F_+ and the pure pole term $A(u,\epsilon)$ in (2.37) becomes particularly useful in the extension of the specific heat to finite wave numbers k (Ref. 11) and to finite frequencies ω.[9] In these theories it is only F_+ that becomes explicitly k and ω dependent whereas $A(u(l),\epsilon)$ and the exponent functions $\zeta_r(u(l))$ and $\zeta_\varphi(u(l))$ remain unaltered and need not be calculated again. The latter quantities carry the dominant critical k and ω dependence through the flow parameter $l(\xi,k,\omega)$. Such a distinction between amplitude functions (like F_+) and RG functions (like A, ζ_r, and ζ_φ) is not made in the $d=3$ renormalization scheme with renormalization conditions.[16,17]

The function $F_+(\mu\xi,u,d)$ is identical with $F_+[u,r/\mu^2]$ defined in Eq. (4.6) of Ref. 5 if $r=r(\xi,u,d)$ is substituted into the latter. This relationship between r and ξ reads above T_c (Ref. 3)

$$r=at=\xi^{-2}Q(1,u(l),d)\exp\int_l^1 \zeta_r(u(l'))\frac{dl'}{l'} \tag{2.39}$$

with $l=(\mu\xi)^{-1}$. The amplitude functions $Q(1,u,3)$ and $F_+(1,u,3)$ have been recently calculated to very good accuracy by means of Borel resummation in three dimensions.[15]

We note that the representation (2.36) is slightly disadvantageous in that it does not exhibit the *additive* form of the noncritical contribution $\mathring{C}_\varphi(\mu^{-1},u_0,d)$ to the total specific heat

$$\mathring{C}=C_B^{\text{tot}}+\tfrac{1}{4}\mu^{-\epsilon}A_d a^2[F_+(\mu\xi,u,d)-F_+(1,u,d)] \ , \tag{2.40}$$

where C_B^{tot} is given by (2.29). The representation (2.40), with $F_+(\mu\xi,u,d)$ given by Eq. (2.34), has the advantage of producing more directly the dependence of the leading amplitudes A^+ and A^- on the critical exponent α, as given in (4.22) in Sec. IV B.

Finally, we note that Eq. (2.36) implies

$$\mathring{C}_\varphi(\mu^{-1},u_0,d)=Z_r^{-2}\mu^{-\epsilon}A_d K_+(u,d) \ . \tag{2.41}$$

As mentioned in Sec. II A, the cutoff dependence of $\mathring{C}_\varphi(\mu^{-1},u_0,d)$ is non-negligible if $\mu^{-1}\sim O(\xi_0)$: therefore, the renormalized expression on the rhs of (2.41) should not be considered as a computational prescription for determning $\mathring{C}_\varphi(\mu^{-1},u_0,d)$. This point will be taken up elsewhere in the discussion of a new amplitude ratio suggested by Bagnuls and Bervillier.[26]

C. Representation of $\mathring{C}$ via multiplicative renormalization

An alternative formulation of $\mathring{C}$ can be given in terms of the probability distribution $\sim\exp-H$ with the extended functional[5,27]

$$H=\int d^dx[\tfrac{1}{2}r_0\varphi_0^2+\tfrac{1}{2}(\nabla\varphi_0)^2+\tilde{u}_0\varphi_0^4 + \tfrac{1}{2}\chi_0^{-1}m_0^2+\gamma_0 m_0\varphi_0^2] \ . \tag{2.42}$$

Then we have

$$\mathring{C}_m\equiv\int d^dx[\langle m_0(x)m_0(0)\rangle-\langle m_0\rangle^2] \tag{2.43}$$

$$=\chi_0+\gamma_0^2\chi_0^2\mathring{C}_\varphi(\xi,u_0,d) \tag{2.44}$$

with

$$u_0=\tilde{u}_0-\tfrac{1}{2}\gamma_0^2\chi_0 \ . \tag{2.45}$$

Obviously we can make the identification

$$\mathring{C}_m=\mathring{C} \tag{2.46}$$

if we identify

$$\chi_0=C_B \tag{2.47}$$

and

$$\gamma_0^2\chi_0^2=a_0^2/4 \ . \tag{2.48}$$

Substitution of (2.36) into (2.44) leads to the form

$$\mathring{C}=Z_m(u,\gamma,\epsilon)\chi_0 C \tag{2.49}$$

with the multiplicatively renormalized specific heat

$$C=[1+\gamma(l)^2F_+(1,u(l),d))]\exp\int_l^1 4\gamma^2(l')B(u(l'))\frac{dl'}{l'} \ , \tag{2.50}$$

where $l=(\mu\xi)^{-1}$ and

$$Z_m(u,\gamma,\epsilon)^{-1}=1+\gamma^2A(u,\epsilon) \ , \tag{2.51}$$

$$\gamma^2=Z_m^{-1}Z_r^{-2}\mu^{-\epsilon}A_d\chi_0\gamma_0^2 \tag{2.52}$$

$$=\tfrac{1}{4}\mu^{-\epsilon}(\chi_0 Z_m)^{-1}A_d a^2 \ . \tag{2.53}$$

The effective coupling

$$\gamma(l)^2=\gamma^2\left[\exp\int_1^l(2\zeta_r-\epsilon)\frac{dl'}{l'}\right] \times\left[1-4\gamma^2\int_1^l B(u(l')) \times\left[\exp\int_1^{l'}(2\zeta_r-\epsilon)\frac{dl''}{l''}\right]\frac{dl'}{l'}\right]^{-1} \tag{2.54}$$

is the solution of the flow equation

$$l\frac{d\gamma(l)^2}{dl}=\gamma(l)^2[2\zeta_r(u(l))-\epsilon+4\gamma(l)^2B(u(l))] \tag{2.55}$$

with the initial condition $\gamma(1)^2=\gamma^2$. The introduction of $\gamma(l)$ is particularly useful for the extension of the theory to critical dynamics[9,24,27] and to finite k, ω, and L,[8] where the main k-ω-L dependence enters through the flow parameter l.

The formal origin of the multiplicative-renormalization factor Z_m^{-1} can be seen after substituting (2.44), (2.47), (2.48), (2.6)–(2.8), and (2.10) into (2.4). This yields

$$\mathring{C}/\chi_0=1-\frac{4\chi_0\gamma_0^2\xi^\epsilon}{\epsilon}\sum_{m=0}^{\infty}a_m^{(2,0)}(d)\left[\frac{u_0\xi^\epsilon}{\epsilon}\right]^m \tag{2.56}$$

which can be interpreted as a double series in powers of u_0 and $\chi_0\gamma_0^2$. The poles associated with the latter coupling can be absorbed in the usual way by means of a purely multiplicative renormalization in (2.49) and (2.52).

Like the representation (2.36), the representations (2.49)–(2.51) do not explicitly exhibit the additive form of the two noncritical contributions (2.29). In analogy to (2.29) one should treat the entire noncritical part

$$C_B^{\text{tot}} = Z_m(u,\gamma,\epsilon)\chi_0[1+\gamma^2 F_+(1,u,d)] \tag{2.57}$$

as an adjustable parameter, rather than χ_0 separately, because the non-negligible cutoff dependence of $\mathring{C}$ has not been taken into account in the minimally renormalized quantities of the expression (2.57). Equivalently, it would suffice to adjust $Z_m\chi_0$ as has been done previously.[6,19]

III. MINIMAL RENORMALIZATION BELOW T_c AT FIXED $d<4$

In this section we shall present the extension of our theory to the case $T<T_c$ at fixed $d<4$. First the bare vertex functions are considered as a function of $r_0-r_{0c}<0$ and u_0 in the limit $\Lambda\to\infty$ (with the prescriptions of dimensional regularization[1,2]). These functions are finite for $d<4$ (provided that a finite ordering field $\mathring{\mathcal{H}}$ is present in order to avoid Goldstone divergencies for $n\geq 2$). Similar to above T_c, one encounters the problem that these vertex functions are not expandable in integer powers of u_0 at fixed $d<4$. Therefore, an appropriate quantity ξ_- is needed which absorbs the nonanalytic u_0 dependence arising from r_{0c}, in analogy to the correlation length $\xi(r_0-r_{0c},u_0,d)$ above T_c. A common definition of ξ_- both for $n=1$ and $n\geq 2$ is not straightforward because of the Goldstone modes for $n\geq 2$. We go back to the function

$$r_0 = r_0(\xi,u_0,d), \quad r_0(\infty,u_0,d) = r_{0c} \ , \tag{3.1}$$

defined above T_c, see (4.2) of Ref. 3. We use this function to define $\xi_-=\xi_-(r_0,u_0,\Lambda=\infty,d)>0$ for $r_0<r_{0c}$ and $\mathring{\mathcal{H}}=0$ implicitly by

$$r_0 = r_0(\xi_-,u_0,d) - W_0(\xi_-,u_0,d) \ . \tag{3.2}$$

The function W_0 is to be constructed such that $\xi_-(r_0,u_0,\Lambda=\infty,d)$ is expandable, at fixed r_0 and $d\neq d_l$ (Ref. 3), in integer powers of u_0. This property will then ensure that the bare vertex functions, if considered as funtions of ξ_- rather than of r_0-r_{0c}, have a power series expansion in u_0 as well, and their renormalized counterparts are (presumably) Borel resummable. More specifically we require that (i) W_0 has no poles for $d<4$ and is expandable in integer powers of u_0, (ii) $\xi_-\sim\xi_0^-(-t)^{-\nu}$ for $-t\to 0$, and (iii) the ratio $(\xi/\xi_-)^2\sim(\xi_0/\xi_0^-)^2$ has the mean-field value 2 to lowest order in u_0. The requirement (ii) is met if W_0 is multiplicatively renormalized by Z_r,

$$W_0(\xi_-,\mu^\epsilon Z_u Z_\varphi^{-2} A_d^{-1} u,d) = Z_r(u,\epsilon)\mu^2 w(\mu\xi_-,u,d) \tag{3.3}$$

with $w(\mu\xi_-,u,d)$ being dimensionless and finite for $d\leq 4$, $\xi_-<\infty$. Similar as above T_c [see (4.21–(4.23) of Ref. 3] Eqs (3.1)–(3.3) lead to

$$r_0-r_{0c} = Z_r(u,\epsilon)\xi_-^{-2}[Q(1,u(l_-),d) - w(1,u(l_-),d)]\exp\int_{l_-}^1 \zeta_r \frac{dl'}{l'} \tag{3.4}$$

with the flow parameter l_- determined by

$$l_- = (\mu\xi_-)^{-1} \ . \tag{3.5}$$

The function $Q(1,u,d)$ is defined in (4.21) of Ref. 3 and is accurately known[15] for $d=3$ and $n=1,2,3$. Equation (3.4) indeed implies $\xi_-\sim\xi_0^-(-t)^{-\nu}$ with $\nu=(2-\zeta_r^*)^{-1}$. The requirement (iii) is satisfied by the simple choice

$$w(1,u,d) = \tfrac{3}{2} \tag{3.6}$$

which is in accord also with the requirement (i). Equations (3.4)–(3.6) may be considered as an implicit definition of ξ_- as a function of r_0-r_{0c}, $\xi_-=\xi_-(r_0-r_{0c},u_0,d)$. This definition is applicable both to the case $n=1$ and $n\geq 2$. Like ξ above T_c, ξ_- does not need an explicit renormalization. After substitution of (3.2) or (3.4) into the bare vertex functions they become expandable in integer powers of $u_0\xi_-^\epsilon/\epsilon$, in analogy to the case $T>T_c$. Owing to the minimal subtraction scheme, the corresponding renormalized quantities are introduced with the same Z factors (pure pole terms) as above T_c. The relationship between the reduced temperature $t=r/a<0$ and ξ_- reads, according to (3.4)–(3.6),

$$-2r = \xi_-^{-2} Q_-(1,u(l_-),d)\exp\int_{l_-}^1 \zeta_r(u(l'))\frac{dl'}{l'} \tag{3.7}$$

with

$$Q_-(1,u,d) = 1+2[1-Q(1,u,d)] = 1+O(u^2) \ . \tag{3.8}$$

In the remainder of this section we confine ourselves to the examples of the order parameter, the specific heat, and, in view of th application to ^{4}He, to the superfluid density. For these examples we set $\mathring{\mathcal{H}}=0$. As a general remark we note that in all amplitude functions of these quantities the effects due to a finite cutoff will be neglected.

A. Order parameter

We start from the thermodynamic potential (negative Helmholtz free energy divided by k_BT)[2,25]

$$\mathring{\Gamma} = \mathring{\Gamma}(\langle\varphi_0\rangle, r_0-r_{0c},u_0,d) \ ,$$

which determines the order parameter $\langle\varphi_0\rangle>0$ via the equation of state ($\mathring{\mathcal{H}}\to +0$)

$$\frac{\partial}{\partial\langle\varphi_0\rangle}\mathring{\Gamma}(\langle\varphi_0\rangle, r_0-r_{0c},u_0,d) = 0 \ . \tag{3.9}$$

After substitution of (3.2) or (3.4) into (3.9) we obtain

$$\langle\varphi_0\rangle = \langle\varphi_0\rangle(\xi_-,u_0,d) \ .$$

A more convenient quantity is $\langle\varphi_0\rangle^2$ which can be written as

$$\begin{aligned}\langle\varphi_0\rangle^2 &= \xi_-^{-2+\epsilon}\mathring{f}_\varphi(u_0\xi_-^\epsilon,d)\\ &= Z_\varphi\langle\varphi\rangle^2\\ &= Z_\varphi\xi_-^{-2+\epsilon} f_\varphi(\mu\xi_-,u,d) \ .\end{aligned} \tag{3.10}$$

By integrating the RGE for $\langle\varphi_0\rangle$ one obtains

$$f_\varphi(\mu\xi_-,u,d) = f_\varphi(1,u(l_-),d)\exp\int_{l_-}^1 \zeta_\varphi\frac{dl'}{l'} \tag{3.11}$$

with $l_- = (\mu\xi_-)^{-1}$. The amplitude function $f_\varphi(1,u,d)$ is finite for $d \le 4$, $u > 0$, and can be calculated at fixed d without using an ϵ expansion. A one-loop calculation yields (for general n)

$$f_\varphi(1,u,d) = A_d(8u)^{-1} + O(u) \quad (3.12)$$

without an $O(1)$ *term* which is canceled due to the convenient choice[3,5] of the geometrical factor

$$A_d = \frac{\Gamma(3-d/2)}{2^{d-2}\pi^{d/2}(d-2)} . \quad (3.13)$$

We expect that a Borel resummation of the higher-order terms of (3.12) will yield only a small correction to the leading term $A_d/8u$, owing to the choice (3.13), as suggested by the Borel resummation results for various amplitude functions above T_c.[15]

B. Specific heat below T_c

The analysis of the vertex function below T_c

$$\tilde{\Gamma}_{-0}^{(2,0)}(\xi_-, u_0, d) = -\tfrac{1}{4}\mathring{C}_\varphi \quad (3.14)$$

is parallel to that of $\tilde{\Gamma}_0^{(2,0)}$ in Sec. II. This function has an expansion of the form

$$\tilde{\Gamma}_{-0}^{(2,0)}(\xi_-, u_0, d) = -\frac{1}{8u_0} + \frac{\xi_-^\epsilon}{\epsilon} \sum_{m=0}^{\infty} a_{-m}^{(2,0)}(d) \left[\frac{u_0 \xi_-^\epsilon}{\epsilon}\right]^m \quad (3.15)$$

with the same poles (at $\epsilon=0$) as those of $\tilde{\Gamma}_0^{(2,0)}$ above T_c. Instead of (2.26)–(2.28), we obtain

$$\mathring{C} = C_B - a_0^2 \tilde{\Gamma}_{-0}^{(2,0)}(\xi_-, u_0, d) , \quad (3.16)$$

$$\tilde{\Gamma}_{-0}^{(2,0)}(\xi_-, u_0, d) = \tilde{\Gamma}_{-0}^{(2,0)}(\mu^{-1}, u_0, d) + \mu^{-\epsilon}\int_1^{l_-} \tilde{G}_-(u(l'), u, d)\frac{dl'}{l'} , \quad (3.17)$$

where $\tilde{G}_-(u',u,d)$ is given by (2.26) with $Pf^{(3,0)}$ replaced by $P_- f_-^{(3,0)}$. Correspondingly, Eqs. (2.36)–(2.38), (2.49), and (2.50) are replaced by formally identical equations with F_+, $f^{(3,0)}$, P, and l replaced by F_-, $f_-^{(3,0)}$, P_-, and l_-, respectively. P_- is defined by

$$-2(\partial r_0/\xi_-^{-2})_{u_0} = \mathring{P}_-(u_0\xi_-^\epsilon, d) = Z_r(u,\epsilon)P_-(\mu\xi_-, u, d) , \quad (3.18)$$

and is obtained from (3.1)–(3.8) as

$$P_-(1,u,d) = 1 + 2[1 - P(1,u,d)] - \tfrac{3}{2}\zeta_r(u) ; \quad (3.19)$$

compare Sec. 4 of Ref. 3. The dimensionless function

$$F_-(\mu\xi_-, u, d) = -4A_d^{-1}\mu^\epsilon \tilde{\Gamma}_-^{(2,0)}(\xi_-, u, \mu, d) \quad (3.20)$$

is defined in analogy to (2.31) and is identical with

$$F_-[u, r/\mu^2] \equiv F_-(\mu\xi_-, u, d) \quad (3.21)$$

of Ref. 5, with $r(\xi_-, u, d)$ determined by (3.7). Both $f_-^{(3,0)}(1,u,d)$ and $F_-(1,u,d)$ are finite for $d \le 4$ and are calculable at fixed d without an ϵ expansion, as anticipated previously.[5] As expected on general grounds and proven by Lawrie,[28] F_- and $f_-^{(3,0)}$ are also free of Goldstone singularities. From a calculation up to one-loop order we have[5] (for general n)

$$F_-(1,u,d) = (2u)^{-1} - 4 + O(u) \quad (3.22)$$

without a d dependence of the $O(1)$ term, due to the choice of the geometric factor (3.13) in the definition of u (2.11).

C. Superfluid density

In superfluid ^{4}He the complex order parameter ($n=2$) is not directly measurable. Instead, we consider the superfluid density ρ_s. In defining ρ_s we shall need the transverse susceptibility

$$\mathring{\chi}_T(k) = \int d^dk\, e^{ikx}\langle \varphi_{0T}(x)\varphi_{0T}(0)\rangle = Z_\varphi \chi_T(k) , \quad (3.23)$$

where $\varphi_{0T}(x)$ denotes the transverse part of the fluctuations of the two-component order parameter

$$\varphi_0(x) = [\langle\varphi_0\rangle + \varphi_{0L}(x), \varphi_{0T}(x)] .$$

By integrating the RGE for $\chi_T(k)$ and from dimensional arguments we obtain

$$\frac{\partial}{\partial k^2}[\chi_T(k)^{-1}]\bigg|_{k=0} = f_T(\mu\xi_-, u, d) \quad (3.24)$$

$$= f_T(1, u(l_-), d)\exp\int_1^{l_-} \zeta_\varphi \frac{dl'}{l'} \quad (3.25)$$

with a dimensionless function f_T. From Refs. 29–31 we adopt the definition of the superfluid density

$$\rho_s = (m_4/\hbar)^2 k_B T \langle\varphi_0\rangle^2 \frac{\partial}{\partial k^2}[\mathring{\chi}_T(k)^{-1}]\bigg|_{k=0} \quad (3.26)$$

which, according to (3.10), leads to

$$\rho_s = (m_4/\hbar)^2 k_B T \xi_-^{2-d} f_\varphi(1, u(l_-), d) f_T(1, u(l_-), d) . \quad (3.27)$$

In one-loop order we find

$$f_T(1,u,d) = 1 + 8u/d + O(u^2) . \quad (3.28)$$

This holds also for general n (where ρ_s corresponds to a stiffness constant). The function $G(u) \equiv G(1,u,3)$ of Ref. 6 is related to f_φ and f_T according to

$$G(1,u,d) = f_\varphi(1,u,d) f_T(1,u,d) A_d^{-1} \quad (3.29)$$

$$= (8u)^{-1} + d^{-1} + O(u) . \quad (3.30)$$

The results presented in Secs. II and III of this paper are applicable to the entire range of validity of the φ^4 model including the *nonasymptotic* region well away from criticality (except for cutoff effects). For corresponding analyses see, e.g., Ref. 8 and references therein. In the following sections we shall confine ourselves to the vicinity of the asymptotic critical region.

IV. ASYMPTOTIC AMPLITUDES AND CORRECTION AMPLITUDES

The renormalization-group description of the critical behavior is usually focused on the asymptotic power laws and power-law corrections.[32,33] In this section we shall give the expressions of the amplitudes appearing in these power laws in our framework of the minimally renormalized theory at fixed $d<4$. We shall use the asymptotic $(l\to 0)$ form of the effective coupling

$$u(l)=u^*+a_u l^{\omega}+O(l^{2\omega}) . \tag{4.1}$$

From the integral representation

$$l=\exp\int_u^{u(l)}\frac{du'}{\beta_u(u',\epsilon)} \tag{4.2}$$

of the flow equation (2.13) we obtain the Wegner exponent[34]

$$\omega=\frac{\partial}{\partial u}\beta_u(u,\epsilon)\bigg|_{u=u^*} \tag{4.3}$$

and the correction amplitude

$$a_u=(u-u^*)\exp\int_u^{u^*}\left[\frac{1}{u'-u^*}-\frac{\omega}{\beta_u(u',\epsilon)}\right]du' . \tag{4.4}$$

Note that no assumption has been made concerning the magnitude of $u-u^*$ and that, in general, a_u has a nonlinear u dependence according to (4.4). The conventional linearization around the fixed point yields

$$a_u=u-u^*+O((u-u^*)^2) . \tag{4.5}$$

A. Correlation length above T_c

The asymptotic form of the correlation length above T_c reads

$$\xi=\xi_0 t^{-\nu}(1+a_\xi t^{\Delta}+\cdots) . \tag{4.6}$$

From (2.39) we find the exponents

$$\nu=(2-\zeta_r^*)^{-1},\quad \Delta=\omega\nu \tag{4.7}$$

and the amplitudes

$$\xi_0=\mu^{2\nu-1}a^{-\nu}\left[Q^*\exp\int_u^{u^*}\frac{\zeta_r^*-\zeta_r(u')}{\beta_u(u',\epsilon)}du'\right]^{\nu} , \tag{4.8}$$

$$a_\xi=-\frac{\nu}{\omega}[\zeta_r'^*-\omega(\ln Q)'^*]a_u(\mu\xi_0)^{-\omega} , \tag{4.9}$$

where we have used the notation

$$f^*\equiv f(u^*),\quad f'^*\equiv\frac{\partial f}{\partial u}\bigg|_{u=u^*} ,\qquad (\ln f)'^*\equiv\frac{\partial}{\partial u}(\ln f)\bigg|_{u=u^*} , \tag{4.10}$$

for $f(u)=\zeta_r(u)$ or $f(u)=Q(1,u,d)$. Similar abbreviations will be used later.

B. Specific heat

The asymptotic form of the specific heat reads

$$\mathring{C}=B^{\pm}+\frac{A^{\pm}}{\alpha}|t|^{-\alpha}(1+a_c^{\pm}|t|^{\Delta}+\cdots) \tag{4.11}$$

with

$$\alpha=\frac{(\epsilon-2\zeta_r^*)}{(2-\zeta_r^*)} . \tag{4.12}$$

For $\alpha<0$, continuity of the finite specific heat at T_c requires $B^+=B^-$. From (2.26)–(2.28), (2.34), (2.40), (2.49), and (2.50) we obtain the representations (for $\alpha<0$)

$$B^+=C_B^{\rm tot}-2a^2\mu^{-\epsilon}\int_u^{u^*}\frac{f^{(3,0)}(1,u',d)P(1,u',d)}{\beta_u(u',\epsilon)}\left[\exp\int_u^{u'}\frac{2\zeta_r(u'')-\epsilon}{\beta_u(u'',\epsilon)}du''\right]du' \tag{4.13}$$

$$=C_B^{\rm tot}-\tfrac14\mu^{-\epsilon}A_d a^2\left[F_+(1,u,d)+4\int_u^{u^*}\frac{B(u')}{\beta_u(u',\epsilon)}\left[\exp\int_u^{u'}\frac{2\zeta_r(u'')-\epsilon}{\beta_u(u'',\epsilon)}du''\right]du'\right] \tag{4.14}$$

$$=Z_m(u,\gamma,\epsilon)\chi_0\exp\int_0^1 4\gamma(l')^2B(u(l'))\frac{dl'}{l'} . \tag{4.15}$$

In the following we use the notation $P_+f_+^{(3,0)}\equiv Pf^{(3,0)}$, $\xi_0^+\equiv\xi_0$, $Q_+^*\equiv Q^*$, and

$$f_+^*\equiv f_+^{(3,0)}(1,u^*,d),\quad f_-^*\equiv f_-^{(3,0)}(1,u^*,d) . \tag{4.16}$$

From (4.8) and from (A2) and (A3) of the Appendix we find the leading amplitudes

$$A^{\pm}=2(b_\pm)^2(\xi_0^{\pm})^{-d}\nu P_\pm^* f_\pm^* \tag{4.17}$$

$$=(b_\pm)^2(\xi_0^{\pm})^{-d}Q_\pm^* f_\pm^* \tag{4.18}$$

with

$$b_+=Q_+^*,\quad b_-=\tfrac12 Q_-^*,\quad \left[\frac{b_+}{b_-}\right]^{\nu}=\frac{\xi_0^+}{\xi_0^-} . \tag{4.19}$$

In (4.18) we have used

$$Q_{\pm}^{*}=2\nu P_{\pm}^{*} . \quad (4.20)$$

as follows from (4.24) of Ref. 3. Substituting (2.38) at the fixed point,

$$8A_d^{-1}P_{\pm}^{*}f_{\pm}^{*}=\frac{\alpha}{\nu}F_{\pm}^{*}+4B^{*} , \quad (4.21)$$

we obtain $A^{\pm}$ in the form

$$A^{\pm}=(b_{\pm})^2(\xi_0^{\pm})^{-d}A_d\tfrac{1}{4}(4\nu B^{*}+\alpha F_{\pm}^{*}) . \quad (4.22)$$

The correction amplitudes turn out to be proportional to α and can be expressed as (see the Appendix)

$$a_c^{\pm}=-a_u(\mu\xi_0^{\pm})^{-\omega}(\alpha/\omega)E_{\pm}^{*} \quad (4.23)$$

with

$$E_{\pm}^{*}=\zeta_r'^{*}-\omega(\ln Q_{\pm})'^{*} + (1-\alpha/\Delta)^{-1}[2\Delta^{-1}\zeta_r'^{*}+\nu^{-1}(\ln(f_{\pm}^{(3,0)}P_{\pm}))'^{*}] . \quad (4.24)$$

Equations (4.17)–(4.24) are valid for both $\alpha>0$ and $\alpha<0$.

C. Order parameter and superfluid density

For the asymptotic representation of the order parameter

$$\langle\varphi_0\rangle=A_M|t|^{\beta}(1+a_M|t|^{\Delta}+\cdots) , \quad (4.25)$$

we find from Sec. III A

$$2\beta=(d-2-\zeta_{\varphi}^{*})/(2-\zeta_r^{*})=\nu(d-2+\eta) , \quad (4.26)$$

$$A_M=Z_{\varphi}^{1/2}\mu^{-\eta/2}(\xi_0^{-})^{-\beta/\nu}f_{\varphi}^{*1/2}\exp\int_u^{u^*}\frac{\zeta_{\varphi}^{*}-\zeta_{\varphi}}{2\beta_u}du', \quad (4.27)$$

and

$$a_M=a_u(\mu\xi_0^{-})^{-\omega}[\beta\omega^{-1}(\zeta_r'^{*}-\omega(\ln Q_{-})'^{*}) -\tfrac{1}{2}(\zeta_{\varphi}'^{*}\omega^{-1}-(\ln f_{\varphi})'^{*})] . \quad (4.28)$$

The asymptotic result for the superfluid density reads (see Sec. III C)

$$\rho_s=A_{\rho_s}|t|^{(d-2)\nu}(1+a_{\rho_s}|t|^{\Delta}+\cdots) \quad (4.29)$$

with

$$A_{\rho_s}=\left[\frac{m_4}{\hbar}\right]^2 k_BT_{\lambda}(\xi_0^{-})^{2-d}A_dG^{*} \quad (4.30)$$

and

$$a_{\rho_s}=a_u(\mu\xi_0^{-})^{-\omega}[(d-2)\nu\omega^{-1}(\zeta_r'^{*}-\omega(\ln Q_{-})'^{*}) +(\ln G)'^{*}] . \quad (4.31)$$

In three dimensions the relation between ρ_s and an appropriately defined transverse correlation length[31] ξ_T is

$$\rho_s=\left[\frac{m_4}{\hbar}\right]^2 k_BT_{\lambda}\xi_T^{-1} , \quad (4.32)$$

where ξ_T has the asymptotic representation

$$\xi_T=\xi_0^T|t|^{-\nu}(1+a_{\xi_T}|t|^{-\Delta}+\cdots) \quad (4.33)$$

with $a_{\xi_T}=-a_{\rho_s}$ and

$$\xi_0^T=\xi_0^{-}(G^{*}A_3)^{-1} . \quad (4.34)$$

The expressions for the various amplitudes given in this section depend on the nonuniversal parameters of the renormalized theory. Thus, these parameters can be determined from the measured amplitudes of a specific system. Applications of this point will be given elsewhere.

V. UNIVERSAL AMPLITUDE RATIOS IN THREE DIMENSIONS

In the previous field–theoretic approach at $d=3$ of Ref. 17 only the case $n=1$ has been treated below T_c. Here we shall apply our $d=3$ field theory to also calculate a few of the universal amplitude ratios of the universality classes $n=2$ and 3. A more complete presentation including the case $n=1$ will be given elsewhere.

A. Leading amplitude ratios

We consider the ratios A^{+}/A^{-}, ξ_0/ξ_0^T, and

$$R_{\xi}^{T}=(A^{-})^{1/3}\xi_0^{T} . \quad (5.1)$$

From (4.22) and (4.34) we obtain

$$\frac{A^{+}}{A^{-}}=\left[\frac{b_{+}}{b_{-}}\right]^{-\alpha}\frac{4\nu B^{*}+\alpha F_{+}^{*}}{4\nu B^{*}+\alpha F_{-}^{*}} , \quad (5.2)$$

see also Eq. (4.33) of Ref. 5, and

$$\xi_0/\xi_0^T=\frac{1}{4\pi}\left[\frac{b_{+}}{b_{-}}\right]^{\nu}G^{*} . \quad (5.3)$$

$$R_{\xi}^{T}=(2\pi b_{-})^{2/3}(4\nu B^{*}+\alpha F_{-}^{*})^{1/3}G^{*-1} . \quad (5.4)$$

From the Borel resummation in three dimensions above T_c we have[15]

$$F_{+}^{*}=-n-2n(n+2)u^{*}(1+b_Fu^{*}) , \quad (5.5)$$

$$P^{*}=1-2(n+2)u^{*}(1+b_Pu^{*}) , \quad (5.6)$$

where

$$b_F=7.59,\quad b_P=0.606 \quad \text{for } n=2 , \quad (5.7)$$

$$b_F=10.3,\quad b_P=0.682 \quad \text{for } n=3 . \quad (5.8)$$

For B^{*} we take[5]

$$B^{*}=\frac{n}{2}+O(u^{*2}) , \quad (5.9)$$

where the correction is expected to be of $O(\eta)$. According to (3.20) and (3.30) we have (for general n)

$$F^*_- = (2u^*)^{-1} - 4 + O(u^*) \ , \tag{5.10}$$

$$G^* = (8u^*)^{-1} + \tfrac{1}{3} + O(u^*) \ , \tag{5.11}$$

where we expect that higher-order corrections, after Borel resummation, are comparable in magnitude with the (small) corrections found above T_c.[15] We approximate Q^*_- [see (3.8)] and Q'^*_- by

$$Q^*_- = 1 + O(u^{*2}), \quad Q'^*_- = 0 + O(u^*) \tag{5.12}$$

which is consistent with the approximations (5.10), and (5.11). The fixed-point values are[3]

$$u^* = 0.0362 \quad \text{for } n = 2, \quad u^* = 0.0328 \quad \text{for } n = 3 \ . \tag{5.13}$$

If we substitute the critical exponents[35]

$$\alpha = -0.013, \quad \nu = 0.671 \quad \text{for } n = 2 \tag{5.14}$$

and

$$\alpha = -0.13, \quad \nu = 0.710 \quad \text{for } n = 3 \tag{5.15}$$

and collect the results of (5.5)–(5.15), we finally obtain the values for A^+/A^-, ξ_0/ξ_0^T, and R_ξ^T listed in Table I.

We note that the deviation of A^+/A^- from 1 is of $O(\alpha)$ and is therefore sensitive to the uncertainties related to estimates of α. Within our theory it is natural to define

$$\tilde{P} = \alpha^{-1}(1 - (b_+/b_-)^\alpha A^+/A^-) \tag{5.16}$$

$$= \frac{F^*_- - F^*_+}{4\nu B^* + \alpha F^*_-} \tag{5.17}$$

which is the asymptotic value of Eq. (4.26) of Ref. 5. This quantity is insensitive to the value of α and is closely related to

$$P = \alpha^{-1}(1 - A^+/A^-) = \tilde{P}(1 + O(\alpha)) \ . \tag{5.18}$$

see, e.g., Ref. 36. From the values given above we obtain $P = 4.2$ for $n = 2$ and $P = 4.4$ for $n = 3$, in good agreement with the calculation of P by Chase and Kaufman[37] and by Bervillier.[38] The slight difference with our previously quoted[8,19] value of A^+/A^- ($n = 2$) is mainly due to a slightly different value of α ($= -0.01$) employed previously.

B. Correction amplitude ratios

We consider the universal ratios of the correction amplitudes a_c^+, a_c^-, and a_{ρ_s} in three dimensions. From (4.9), (4.23), (4.24), and (4.31) we obtain

$$a_c^+/a_c^- = \left[\frac{b_+}{b_-}\right]^{-\Delta} [1 - E^{*-1}_-(\nu - \alpha/\omega)^{-1} \times (\ln(P_- f_-^{(3,0)}/P_+ f_+^{(3,0)}))'^* - E^{*-1}_- \omega(\ln(Q_+/Q_-))'^*] \tag{5.19}$$

and

$$a_c^-/a_{\rho_s} = \frac{\alpha}{\nu} E^*_- [-\zeta_r'^* + \omega(\ln Q_-)'^* - (\omega/\nu)(\ln G)'^*]^{-1} \ . \tag{5.20}$$

In evaluating these quantities we use the following relations:

$$(\ln Q_+)'^* = Q'^*_+/Q^*_+ = [(\ln P_+)'^* + \nu\zeta_r'^*](1+\Delta)^{-1}, \quad (\ln Q_-)'^* = 0 \tag{5.21}$$

$$(\ln(f_\pm^{(3,0)} P_\pm))'^* = \Phi^*_\pm \ , \tag{5.22}$$

$$(\ln(P_- f_-^{(3,0)}/P_+ f_+^{(3,0)}))'^* = \Phi^*_- - \Phi^*_+ \ , \tag{5.23}$$

with

$$\Phi^*_\pm = [(\alpha - \Delta)F'^*_\pm - 2\nu\zeta_r'^* F^*_\pm + 4B'^*][\alpha F^*_\pm + 4\nu B^*]^{-1} \ . \tag{5.24}$$

The various fixed-point values can be obtained from our Borel resummation results[3,15] and from (3.22) and (3.30).

If we take the critical exponents (5.14) and (5.15) and[35] $\omega = 0.8$, 0.79 for $n = 2$, 3, respectively, we obtain the amplitude ratio $a_c^+/a_c^- = 0.79$, 0.84. We do not consider these values as quantitatively reliable because of the uncertainty related to the one-loop result for F'^*_-. In two-loop order we have found

$$F_-(1,u,3) = (2u)^{-1} - 4 + 8(10-n)u + O(u^2) \ , \tag{5.25}$$

which yields considerably larger values of F'^*_-. Using these values we obtain a_c^+/a_c^- as given in Table I for $n = 2,3$. The presently available experimental estimates[22] of a_c^+/a_c^- for $n = 2$ are not inconsistent with our result but they also have an uncertainty of a factor of about 2. We note that our expression (5.19) for a_c^+/a_c^- does not depend sensitively on the value of α, therefore the α variation of $D_c/D_c' \equiv a_c^+/a_c^-$ in Table IV of Ref. 22 is not reproduced by (5.19).

By contrast, our result (5.20) shows the amplitude ratio a_c^-/a_{ρ_s} to be proportional to α. This confirms a conjecture by Singsaas and Ahlers[22] which was based on an argument related to the $\alpha \to 0$ limit of Eq. (4.11). This was

TABLE I. Universal ratios of leading and correction amplitudes calculated from the $d = 3$ field theory within the minimal-subtraction scheme.

	A^+/A^-	P	ξ_0/ξ_0^T	R_ξ^T	a_c^+/a_c^-	a_c^-/a_{ρ_s}
$n=2$	1.05	4.2	0.50	0.78	1.6	−0.045
$n=3$	1.58	4.4	0.56	0.73	1.4	−0.69

not apparent from previous ϵ expansion results which gave[39,40]

$$a_c^- / a_{\rho_s} = \tfrac{1}{3}(1 - \tfrac{1}{2}\epsilon) + O(\epsilon^2) \quad \text{for } n = 2 \tag{5.26}$$

and

$$a_c^- / a_{\rho_s} = \tfrac{1}{6}(1 - \tfrac{529}{220}\epsilon) + O(\epsilon^2) \quad \text{for } n = 3 \;, \tag{5.27}$$

thus $\frac{1}{6}$ and $-\frac{103}{440}$, respectively, in three dimensions. The ϵ expansion result disagrees qualitatively with the experimental result[22] for $n = 2$

$$(a_c^- / a_{\rho_s})^{\text{expt}} \simeq 4\alpha < 0 \;. \tag{5.28}$$

Part of the failure of (5.26) and (5.27) can be traced back to the ϵ expansion result

$$\alpha = \tfrac{1}{10}\epsilon + O(\epsilon^2) \quad \text{for } n = 2 \;, \tag{5.29}$$

$$\alpha = \tfrac{1}{22}\epsilon + O(\epsilon^2) \quad \text{for } n = 3 \;, \tag{5.30}$$

which yields the wrong sign of α at $\epsilon = 1$. Although the $O(\epsilon^2)$ corrections to (5.29) and (5.30) would yield a significant improvement, this $O(\epsilon^2)$ correction term is not adequately taken into account in the strict ϵ expansion results (5.26) and (5.27). Our $d = 3$ theory avoids the ϵ expansion and yields, as a general result, the correct structure of the amplitude ratios (5.2)–(5.4) and (5.19) and (5.20) in terms of α, ν, ω, and Δ. This implies a *negative* a_c^- / a_{ρ_s} for $n \geq 2$ and eliminates a qualitative discrepancy between the previous theoretical estimate[39,40] and experimental observation[22] of a_c^- / a_{ρ_s}.

Because of the smallness and the uncertainty of the value of α for $n = 2$, $d = 3$ the result (5.20) suggests to write (5.20) as

$$a_c^- / a_{\rho_s} = \alpha R_{c\rho} \tag{5.31}$$

and to compare the experimental and theoretical results for the universal quantity $R_{c\rho}$. Our theory predicts $R_{c\rho}$ to be rather insensitive to the precise value of α. This prediction agrees with the experimental values of $R_{c\rho} \equiv D_c' / (\alpha D_\rho)$ obtained from Table IV of Ref. 22 where three different estimates of α were used:

$$R_{c\rho}^{\text{expt}} = 4.0, 4.3, 4.0 \;, \tag{5.32}$$

for

$$\alpha = -0.007, -0.016, -0.025 \;, \tag{5.33}$$

respectively, for ^{4}He ($n = 2$, $d = 3$). From (5.20) we obtain for these values of α [and corresponding values of $\nu = (2-\alpha)/3$]

$$R_{c\rho}^{\text{theor}} = 3.4, 3.4, 3.5 \;, \tag{5.34}$$

in good agreement with the experimental result (5.32). In (5.34) we have used the two-loop expression for $F_-'^*$. If the one-loop result for $F_-'^*$ had been used $R_{c\rho}^{\text{theor}}$ would be by a factor of 2 larger, thus, the agreement between (5.34) and (5.32) is somewhat fortuitous. In Table I the ratio a_c^- / a_{ρ_s} is given for the choice of α and ν according to (5.14) and (5.15) and with $F_-'^*$ calculated from the two-loop expression (5.25).

Finally, as a test of the results given above, we consider the universal ratio a_c^+ / a_{ρ_s} which is also proportional to α. It has the advantage of being independent of F_- which is the main source of the theoretical inaccuracy. For $\alpha = -0.013$, $\nu = 0.671$, and $\omega = 0.8$ we find, from (4.23) and (4.31), the theoretical value

$$a_c^+ / a_{\rho_s} = -0.072 \quad \text{for } n = 2 \tag{5.35}$$

in good agreement with the experimental value -0.070 obtained from Table IV of Ref. 22 (for a slightly different $\alpha = -0.016$).

APPENDIX

In this appendix we sketch the derivation of the expressions (4.23) and (4.24) for the correction amplitudes of the specific heat. For simplicity we assume $\alpha < 0$ in the following but the results also remain valid for $\alpha > 0$. From the asymptotic representation (4.11) we have

$$\frac{\partial \ln(\mathring{C} - B^+)}{\partial \ln|t|} = -\alpha + a_c^{\pm} \Delta |t|^{\Delta} + \cdots \;. \tag{A1}$$

We shall determine $a_c^{\pm}$ by comparing (A1) with the logarithmic derivative of our theoretical expression

$$\mathring{C} - B^+ = -2a^2 \mu^{-\epsilon} \int_0^{l_\pm} R_\pm(l') \left\{ \exp \int_1^{l'} (2\zeta_r - \epsilon) \frac{dl''}{l''} \right\} \frac{dl'}{l'} \;, \tag{A2}$$

$$R_\pm(l) \equiv f_\pm^{(3,0)}(1, u(l), d) P_\pm(1, u(l), d) \;. \tag{A3}$$

Equations (A2) and (A3) follow from (2.26)–(2.29) and (4.13). We shall calculate (A1) in the form

$$\frac{\partial \ln(\mathring{C} - B^+)}{\partial \ln|t|} = \frac{\partial \ln l_\pm}{\partial \ln|t|} \frac{\partial \ln(\mathring{C} - B^+)}{\partial \ln l_\pm} \;, \tag{A4}$$

where, according to (2.39) and (3.7),

$$\frac{\partial \ln l_\pm}{\partial \ln|t|} = [2 - \zeta_r(\bar{u}) + \beta_u(\bar{u}, \epsilon) \partial_{\bar{u}} \ln Q_\pm(1, \bar{u}, d)]^{-1} \tag{A5}$$

with $\bar{u} \equiv u(l_\pm)$. From (A2) we obtain

$$\frac{\partial \ln(\mathring{C} - B^+)}{\partial \ln l_\pm} = R_\pm(l_\pm) / I_\pm(l_\pm) \;, \tag{A6}$$

$$I_\pm(l) = \int_0^l R_\pm(l') \left\{ \exp \int_l^{l'} (2\zeta_r - \epsilon) \frac{dl''}{l''} \right\} \frac{dl'}{l'} \;. \tag{A7}$$

We expand $I_\pm(l)$ around $l = 0$,

$$I_\pm(l) = I_\pm^* + I_\pm'^* [u(l) - u^*] + \cdots \tag{A8}$$

with

$$I_\pm^* \equiv I_\pm(0) \;, \tag{A9}$$

$$I_\pm'^* \equiv [\partial I_\pm(l) / \partial u(l)]|_{u = u^*} \;, \tag{A10}$$

and

$$u(l_\pm) - u^* = a_u (\mu \xi_0^{\pm})^{-\omega} |t|^{\Delta} + \cdots \;. \tag{A11}$$

From

$$l\partial I_{\pm}(l)/\partial l = R_{\pm}(l) - [2\zeta_r(u(l)) - \epsilon]I_{\pm}(l) \quad \text{(A12)}$$

we obtain

$$I^{*}_{\pm} = -(\alpha/\nu)R^{*}_{\pm} \quad \text{(A13)}$$

and

$$I'^{*}_{\pm} = (R'^{*}_{\pm} - 2\zeta'^{*}_r I^{*}_{\pm})(\omega - \alpha/\nu)^{-1} . \quad \text{(A14)}$$

Together with a corresponding expansion of (A5) these results lead to (4.23) and (4.24).

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