Anyons on a torus

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We show that the states of an anyon system on a torus are not completely determined by the positions of the anyons. There are q states for each fixed anyon configuration if the statistics of the anyons is given by $\theta=p\pi/q$. We explicitly construct the lattice Hamiltonian for the anyon system on the torus. The Hamiltonian is shown, both analytically and numerically, to respect the translation symmetries and the rotation symmetries. The flux of the anyon system is found to be quantized in units of $2\pi/q$, without any shift. We also write down the effective Hamiltonian for the holons (with $\theta=\pi/2$) in the chiral spin state.

I. INTRODUCTION

Recent studies on high-T_c superconductors[1,2] arouse considerable interest on anyon systems.[3–5] It has been shown that the charged quasiparticles in some time-reversal symmetry and parity-breaking spin-liquid states may have fractional statistics, and a charged anyon system is probably a superconductor at low temperatures.

Because of the topological character of anyons, it is necessary to give special attention to the boundary conditions. Recent numerical work has focused on systems with geometries such as a cylinder and a sphere.[4,5] In this paper we will consider the case of toroidal boundary conditions. In the course of our discussion an interesting topological structure, due to the nontrivial connectivity of the torus, will be made apparent.

There is a naive argument indicating that one can only put $nq+1$ anyons on a torus if the anyons have a statistics $\theta=\pi p/q$. The argument goes as follows. Let us put N anyons on a torus. Consider the problem of the Nth anyon moving in the background of other $N-1$ anyons. Because each anyon behaves like a 2θ flux tube to another anyon, the Nth anyon sees $2\theta(N-1)$ flux. Due to the Dirac quantization condition, the wave function of the Nth anyon can be consistently defined on the torus only when the flux is quantized, i.e.,

$$2\theta(N-1)=2\pi\times\text{integer}\ . \tag{1.1}$$

Equation (1.1) implies that $N-1$ must be a multiple of q. In particular, the above argument suggests that one can only consistently put an odd number of semions ($\theta=\pi/2$) on a torus.

On the other hand, we also know that a doped chiral spin state on a torus with an even number of sites only allows an even number of holons (assuming there is no spinon). Since the holons are semions, we have a consistent microscopic theory that contains an even number of semions on a torus. This example suggests that the above argument is incorrect.

In this paper, we resolve this puzzle by showing a correct way to put anyons on a torus. We explicitly construct a Hamiltonian describing an anyon system on the torus. Our Hamiltonian respects the translation symmetry and 180° rotation symmetry (or 90° rotation symmetry if the lattice is a square). The consistency of the Hamiltonian is checked numerically. We also derive the effective Hamiltonian for the holons in the chiral spin state.

Numerical calculations on a cylinder[4] have suggested that there is an asymmetry in the dependence of the ground-state energy on the magnetic flux going through the cylinder. In this paper we find that on a torus there is no such asymmetry. Thus we conclude that the possible asymmetry may result from a particular choice of the boundary conditions and the definition of the magnetic flux in the anyon Hamiltonian. Recent numerical calculations on the sphere did not yield any asymmetry either.[5]

The paper is organized as follows. In Sec. II we discuss the structure of the Hilbert space of anyons on a torus. We argue that there must be (at least) q states for each fixed anyon configuration if the statistics of the anyons is given by $\theta=\pi p/q$. In Sec. III we study an explicit anyon model on a torus.[6] We show explicitly that each anyon configuration contains q states. In Sec. IV we construct the explicit anyon hopping Hamiltonian on the torus and demonstrate the consistency of the Hamiltonian. In Sec. V the anyon hopping Hamiltonian is shown explicitly to respect translation and rotation symmetry. The operators generating the translation and 90° rotation are constructed. In Sec. VI we numerically test the anyon hopping Hamiltonian to further demonstrate the consistency of the Hamiltonian. In Sec. VII we discuss some issues associated with anyons on a torus with nonzero magnetic field and write down the effective Hamiltonian for the holons (or spinons) in the chiral spin state. In Sec. VIII we discuss the flux quantization of the anyon system on a torus concluding the paper.

II. HILBERT SPACE OF ANYONS ON A TORUS

The crucial step to resolve the above-described puzzle is to realize that the ground state of a Hamiltonian on a torus must be degenerate if the Hamiltonian supports anyonic excitations.[7,8] In general we may assume that the Hamiltonian has a finite energy gap. The energy gap for the anyon quasiparticle is Δ_p and for the anyon quasihole is Δ_h. (Δ_p and Δ_h are measured relative to a "zero" chemical potential.) When the chemical potential μ satisfies $-\Delta_h < \mu < \Delta_p$ the system contains no anyons. Even in this case we will show that the ground states must be (at least) q-fold degenerate (assuming $\theta = \pi p/q$). When $\mu > \Delta_p$ (or $\mu < -\Delta_h$) there is a finite density of quasiparticles (or quasiholes). In this case we can show that there are q states for each fixed anyon configuration.

To understand these results let us first assume the anyon density to be zero and consider the following tunneling process. A pair of an anyon and an antianyon is created at a certain time. The anyon propagates in $\hat{\mathbf{x}}$ direction all the way around the torus and then annihilates with the antianyon (Fig. 1). Such a tunneling process induces a transition between ground states. The transition can be represented by a unitary operator T_1:

$$|\Psi_0'\rangle = T_1 |\Psi_0\rangle \ . \tag{2.1}$$

If we let the antianyon go all the way around the torus in the $\hat{\mathbf{x}}$ direction we will obtain a different transition operator $\overline{T}_1$. However, $\overline{T}_1$ must be equal to T_1^{-1} because the two tunneling processes must cancel each other. Similarly, we can obtain another transition operator T_2 by letting the anyon propagate in the $\hat{\mathbf{y}}$ direction.

Now let us consider a sequence for four tunneling processes described by T_1, T_2, T_1^{-1}, and T_2^{-1} (Fig. 2). Notice that the four tunneling paths can be deformed into two linked loops (Fig. 3) that give rise to a pure phase $e^{-i2\theta}$ as implied by the fractional statistics of the anyons. Therefore we have

$$T_2^{-1} T_1^{-1} T_2 T_1 = e^{-i2\theta} = e^{-i2\pi(p/q)} \ . \tag{2.2}$$

The ground states form a representation of the algebra (2.2). Because the algebra (2.2) has only one q-dimensional irreducible representation, the ground states must be (at least) q-fold degenerate. Haldane[9] has noticed that a similar degeneracy occurs in the theory of the fractional quantum Hall effect on a torus. An analogous construction is also found in topological field theory.[10,11]

Now let us consider N anyons on a torus. The above argument also implies that the state of the system is not uniquely determined by the positions of the anyons. For each position configuration $\{(x_i,y_i)\}$ there must be q different states that can be labeled by α:

$$|\{(x_i,y_i)\};\alpha\rangle, \quad \alpha = 1, \ldots, q \ , \tag{2.3}$$

where x_i and y_i are integers describing the position of the ith anyon. To understand this result let us notice that, as illustrated in Fig. 4, moving an anyon all the way around the torus is equivalent to the anyon-antianyon tunneling discussed above. Therefore moving an anyon around the torus in the $\hat{\mathbf{x}}$ ($\hat{\mathbf{y}}$) direction produces the transition operators T_1 (T_2). The operator T_1 and T_2 act on the states labeled by α with fixed anyon positions. T_1 and T_2 satisfy the algebra (2.2). Therefore the states for each fixed anyon configuration $\{(x_i,y_i)\}$ must form a representation of (2.2) and span a q-dimensional Hilbert space.

Strictly speaking, the above arguments are based on the following assumptions. (i) If the anyon tunneling paths are restricted to a finite region, the ground states cannot be changed and the induced transition operator T

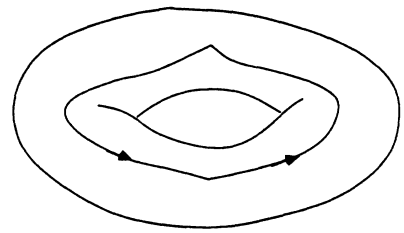

FIG. 1. A tunneling process of a pair of anyon and antianyon hopping all the way around the torus.

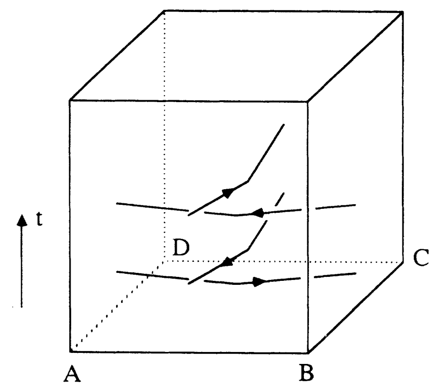

FIG. 2. The tunneling paths of the four tunneling processes T_1, T_2, T_1^{-1}, and T_2^{-1} are presented in space-time.

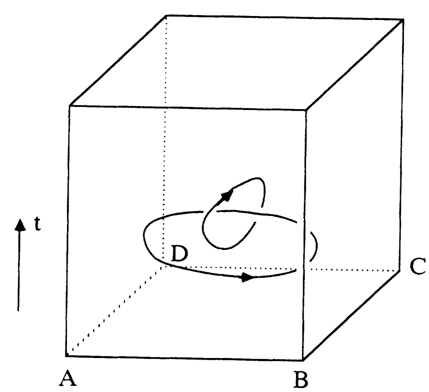

FIG. 3. The tunneling paths in Fig. 2 can be deformed into two linked loops.

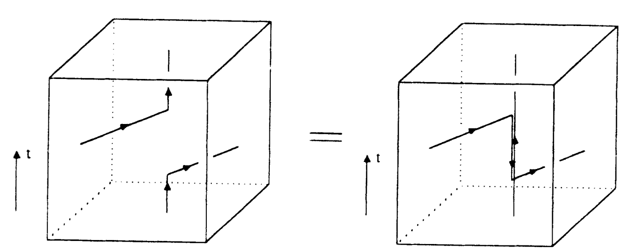

FIG. 4. Moving an anyon all the way around the torus is equivalent to the anyon-antianyon tunneling process described in Fig. 1.

is the identity operator multiplied by a phase factor. We call such tunneling processes local fluctuations. The ground states cannot be changed by local fluctuations. The phase factor is determined by the fractional statistics in the usual way. (ii) Two anyon tunneling paths induce the same transition operator (with the same phase factor) if the two tunneling paths can be continuously deformed into each other. Those assumptions are automatically satisfied by anyon systems.

From the above discussion we see that the Hilbert space of an anyon system on a torus is not given by the positions of the anyons. The Hilbert space is spanned by the states given in Eq. (2.3). The lattice hopping Hamiltonian should act on those states. Notice that the states in Eq. (2.3) are defined to be invariant under permutations between anyons, e.g.,

$$|\{(x_1,y_1),(x_2,y_2)\};\alpha\rangle = |\{(x_2,y_2),(x_1,y_1)\};\alpha\rangle \ , \tag{2.4}$$

and two anyons can never occupy the same site: $(x_i,y_i)\neq(x_j,y_j)$.

The results in this section can be summarized as follows. In order for a Hamiltonian to describe the anyon system, the Hamiltonian must satisfy the following conditions.

(a) An anyon hopping around a plaquette induces no phases. This is equivalent to saying that there is no external magnetic field.

(b) An anyon going around another anyon induces a phase $e^{\pm i2\theta}$. The plus-minus sign depends on the orientation of the loop.

(c) Interchanging two anyons induces a phase $e^{\pm i\theta}$.

(d) The anyon Hamiltonian acts on the Hilbert space given by (2.4). Moving an anyon around the torus in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions induces operators T_1 and T_2 that must satisfy the algebra (2.2).

(b) and (c) are local requirements due to the fractional statistics. Actually (c) implies (b). (d) is a global requirement for the torus geometry. In Sec. IV we will see that the conditions (a)–(d) determine the anyon hopping Hamiltonian completely. In the next section we will study an anyon model on a torus and construct explicitly the Hilbert space of the anyon system on it.

III. ANYONS ON A TORUS AND THE CHERN-SIMONS GAUGE THEORY

In Sec. II we discussed some crucial general properties of the Hilbert space of a system of anyons. In particular we argued that the location of the anyons themselves is not sufficient to label the states, since, on general grounds, one expects the states to exhibit a degeneracy determined entirely by the topology of space. In this section we make these ideas more concrete by considering a model that captures the general features of the problem at hand. The model is simply a system of "free" anyons on a square lattice with the topology of a torus. These anyons are free insofar as the Hamiltonian will only contain a nearest-neighbor hopping term. However, these anyons will be assumed to have *hard cores*. This last requirement is essential to the whole construction. As a matter of fact, only in the presence of hard cores it is possible to meaningfully define fractional statistics in two dimensions. In other words, we will demand that the world lines of the anyons should never intersect with each other. This is also a natural requirement given that the anyon states should be representations of the braid group. No braids are possible if the world lines are allowed to intersect. Anyons on a lattice have been discussed recently by several groups.[4–6,12]

In a recent paper, one of us[6] showed that the problem of a gas of N_a anyons with hard cores on a square lattice is equivalent to a gas of $N_f=N_a$ fermions, on the square lattice, coupled to a Chern-Simons gauge field defined on the links of that lattice. To be more precise, let $\hat{a}^\dagger(\mathbf{x})$ and $\hat{a}(\mathbf{x})$ be a set of anyon creation and annihilation operators defined on the sites $\{\mathbf{x}\}$ of the square lattice, which satisfy the generalized equal-time commutation relations

$$\hat{a}^\dagger(\mathbf{x})\hat{a}(\mathbf{y})=\delta_{\mathbf{x},\mathbf{y}}-e^{i\theta}\hat{a}(\mathbf{y})\hat{a}^\dagger(\mathbf{x}) \ . \tag{3.1}$$

The angle θ indicates that we are dealing with fractional statistics. The choice of sign is such that for $\theta=0$ we have fermions. The hard-core condition implies that, when acting on physical states, these operators obey

$$\hat{a}^\dagger(\mathbf{x})\hat{a}^\dagger(\mathbf{x})=\hat{a}(\mathbf{x})\hat{a}(\mathbf{x})=0 \ . \tag{3.2}$$

The second quantized Hamiltonian is simply given by

$$H=\sum_{\langle \mathbf{x},\mathbf{y}\rangle}\hat{a}^\dagger(\mathbf{x})\hat{a}(\mathbf{y})+\text{H.c.} \ , \tag{3.3}$$

where $\langle \mathbf{x},\mathbf{y}\rangle$ are nearest-neighboring sites on the square lattice. Consider now a set of *fermion* creation and annihilation operators $\hat{c}^\dagger(\mathbf{x})$ and $\hat{c}(\mathbf{x})$ on the same square lattice. Let $\hat{A}_j(\mathbf{x})$ be a set of *boson* operators defined on the links of the lattice $\{(\mathbf{x},\mathbf{x}+\hat{e}_j)\}$ (with $j=1,2$) representing statistical gauge fields that satisfy the equal-time commutation relations

$$[\hat{A}_j(\mathbf{x}),\hat{A}_k(\mathbf{y})]=i2\theta\epsilon_{jk}\delta_{\mathbf{x},\mathbf{y}} \ , \tag{3.4}$$

where ϵ_{jk} is the Levi-Civita tensor. The dynamics of the system is governed by the Hamiltonian

$$H_f=\sum_{\mathbf{x},j}\hat{c}^\dagger(\mathbf{x})\exp[i\hat{A}_j(\mathbf{x})]\hat{c}(\mathbf{x}+\hat{e}_j)+\text{H.c.} \tag{3.5}$$

and the physical states $\{|\psi\rangle\}$ are required to satisfy a local constraint between the fermion density $\rho(\mathbf{x})$ and the local magnetic flux $\hat{B}(\mathbf{x})$ of the statistical gauge fields

$$\left[\rho(\mathbf{x})-\frac{1}{2\theta}\hat{B}(\mathbf{x})\right]|\psi\rangle=0 \ . \tag{3.6}$$

This constraint implies that a fluxoid of strength 2θ is attached to each particle at the level of the lattice scale. The local statistical flux $\hat{B}(\mathbf{x})$ is given by the usual formula

$$\hat{B}(\mathbf{x})=\Delta_1\hat{A}_2(\mathbf{x})-\Delta_2\hat{A}_1(\mathbf{x}) \ , \tag{3.7}$$

where Δ_j is the finite difference operator on direction j. The flux thus defined effectively exists only on the dual lattice. This formulation has the additional advantage that the particles are not allowed to get "inside" the flux. The Hamiltonian of Eq. (3.6), together with the constraint Eq. (3.5) and the commutation relations Eq. (3.4), follow from canonical quantization, in the gauge $A_0=0$, of the Lagrangian density $\mathcal{L}$,

$$\mathcal{L}=c^{\dagger}(x)(i\partial_0+A_0+\mu)c(x)-\mathcal{H}(c^{\dagger},c,\mathbf{A})-\mathcal{L}_{\rm CS} . \quad (3.8)$$

Here $\mathcal{H}$ is the Hamiltonian per site, μ the chemical potential, $x=(\mathbf{x},t)$ and $\mathcal{L}_{\rm CS}$ is the Chern-Simons Lagrangian density, which, in terms of the vector potential A_μ and the field strength tensor $F_{\mu\nu}$ suitably defined on a lattice,[6] has the form

$$\mathcal{L}_{\rm CS}=\frac{1}{8\theta}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} . \quad (3.9)$$

The equivalence between the anyon Hamiltonian, Eq. (3.3), and the Chern-Simons gauge theory coupled to fermions, Eq. (3.5) and Eq. (3.6), is established by solving the constraint of Eq. (3.6), which relates the local flux to the local density. This can be accomplished by fixing a gauge, such as the Coulomb gauge $\nabla\cdot\mathbf{A}(\mathbf{x})=0$, as was done in Ref. 6. The statistical vector potential $\mathbf{A}(\mathbf{x})$, which is the solution of the constraint in this gauge, is an explicit function only of the local particle density. Thus it may appear that there are no gauge degrees of freedom left. This, however, is not the case in general. Whether or not there are any gauge degrees of freedom left depends on the boundary conditions, i.e., on the topology of the configurations of gauge fields. We are now going to solve the constraint equation Eq. (3.5) on lattice with the topology of a torus. Let L_1 and L_2 be the linear dimensions of the lattice along directions 1 and 2, respectively.

It is impossible to eliminate all gauge degrees of freedom by solving Eq. (3.5) no matter what gauge is chosen unless *large* gauge transformations, which wrap around the torus along directions 1 or 2, are included. Consider the circulation of the statistical vector potential on a *noncontractible* closed loop wrapping around the torus along one of its large circles $\mathcal{C}_j$ $(j=1,2)$. Any *local* time-independent gauge transformation shifts the spacial components of the vector potential A_k by the gradient of a smooth function Λ of the coordinates $A_k(\mathbf{x},t)\to A_k(\mathbf{x},t)+\Delta_k\Lambda(\mathbf{x})$. Thus, the circulation Γ_j, with $\Gamma_j=\oint_{\mathcal{C}_j}d\mathbf{x}\cdot\mathbf{A}(\mathbf{x})$, is unchanged, since Λ is a smooth and single-valued function of $\mathbf{x}$. Notice that this is the case even in the absence of fermions. Thus, the circulations Γ_j, or *nonintegrable phases,* are *global* degrees of freedom of the gauge field. A consistent treatment of this problem must take into account their dynamics.

There is a simple way to take care of both global and local gauge degrees of freedom. The local gauge degrees of freedom are nonlocal functions of the local particle density $\rho(\mathbf{x},t)$ given by the solution of the local constraint equation in some particular gauge. The global degrees of freedom are the nonintegrable phases Γ_j. To make any further progress it is necessary to fix the gauge. At the level of the functional integral, we first observe that the component A_0 of the statistical gauge field can always be integrated out giving rise to the local constraint Eq. (3.4) at all times. We next write the spatial components of the statistical vector potential A_j in the form

$$A_j(x)=\mathcal{A}_j(x)+\overline{A}_j(x) , \quad (3.10)$$

where $\mathcal{A}_j$ is a particular solution of the constraint equation, and $\overline{A}_j$ generate the nonintegrable phases and are solutions to the homogeneous constraint equation (i.e., without fermions). We can completely determine all of these fields by choosing a particular gauge. The fields $\mathcal{A}_j$ can be represented in terms of "Dirac strings," a path γ beginning at an arbitrary (but fixed) plaquette and ending at the plaquette "southwest" of the anyon. $\mathcal{A}_i$ on a link is given by 2θ times the number of times that the Dirac strings cross that link. $\overline{A}_i$ are given by

$$\overline{A}_1(\mathbf{x},t)=\Gamma_1(t)/L_1 , \quad (3.11)$$

$$\overline{A}_2(\mathbf{x},t)=\Gamma_1(t)/L_2 . \quad (3.12)$$

These expressions can now be substituted back into the Lagrangian Eqs. (3.8) and (3.9). The formalism of canonical quantization yields the Hamiltonian of Eq. (3.5) with the operators A_j given by Eq. (3.10). Notice that these operators contain information about both local and global degrees of freedom. The contribution of the global degrees of freedom was *chosen* to be nonzero only on the boundaries. By carrying out the canonical formalism to completion, it is easy to check that the nonintegrable phases obey the commutation relations

$$[\Gamma_1,\Gamma_2]=i2\theta . \quad (3.13)$$

It is easy now to check that the operators $\exp(i\Gamma_j)$ satisfy the algebra

$$\exp(i\Gamma_1)\exp(i\Gamma_2)=e^{-2i\theta}\exp(i\Gamma_2)\exp(i\Gamma_1) . \quad (3.14)$$

Thus, the operators $\exp(i\Gamma_j)$ can be identified with the operators T_j of Sec. II. These operators will given an extra phase whenever the strings attached to two anyons cross as the anyons move around each other. Furthermore, since Γ_1 and Γ_2 do not commute, the *eigenstates* of the Hamiltonian are only functions of either variable but not of both at the same time. Also, both Γ_1 and Γ_2 enter only through the exponential operators T_j. Thus we can always choose, say, Γ_1 to be an angle with a range $[0,2\pi]$. Hence $(1/2\theta)\Gamma_2$ is an angular-momentum-like operator whose spectrum is the set of integers. In all cases of physical interest, the statistical angle θ can only take the restricted set of values $\theta=\pi(p/q)$. The algebra of the operators T_j then implies that only distinct quantum numbers are the integers modulo q. Thus, the Hilbert space is decomposed into classes each labeled by a quantum number α with $\alpha=1,\ldots,q$ as anticipated in Sec. II. In particular, the ground states form the q-dimensional irreducible representation of algebra (3.14). This happens because both $e^{i\Gamma_1}$ and $e^{i\Gamma_2}$ commute with the Hamiltonian. One of us[8] has stressed in a recent paper that such topological degeneracies occur quite generally in spin-liquid states and other topologically ordered states.

In this section we have used the Chern-Simons gauge theory coupled to fermions to yield a second quantized Hamiltonian for anyons coupled to global degrees of freedom. The eigenstates of this Hamiltonian are thus given in terms of both the coordinates of the anyons and an extra label that represents the degeneracy required by the global degrees of freedom Γ_j. In the next section we will discuss a first quantized version of this problem in the space of states with a fixed total number of particles equal to N.

IV. AN EXPLICIT ANYON HOPPING HAMILTONIAN ON A TORUS

In the following we will construct an explicit anyon hopping Hamiltonian based on the conditions (a)–(d). Let us first discuss how to construct a Hamiltonian satisfying (a), (b), and (c). We attach a string to each anyon in the way described in Fig. 5. The phases of the anyon hopping amplitude are given according to the following rules. When an anyon hops across a string from left to right (from right to left), the amplitude obtains a phase $e^{i\theta}$ ($e^{-i\theta}$). When the string of the hopping anyon crosses an anyon from left to right (from right to left), the hopping amplitude obtains a phase $e^{-i\theta}$ ($e^{i\theta}$). We will call the strength of such a string $e^{i\theta}$. The hopping amplitude has zero phases otherwise. According to these rules one can easily check that an anyon hopping around a loop can induce a nonzero phase only when the loop encloses some other anyons. If the loop encloses only one anyon, the induced phase is $e^{i2\theta}$; one $e^{i\theta}$ comes from anyon 1 crossing string 2 and another $e^{i\theta}$ comes from string 1 crossing anyon 2 (Fig. 6). One can also easily check that interchanging two anyons induces a phase $e^{i\theta}$ according to these rules. Thus conditions (a)–(c) are satisfied.

In order to write down the anyon hopping Hamiltonian on a torus, let us first regard the torus as a continuous space. Condition (d) implies that the anyons must satisfy a non-Abelian boundary condition that can be imposed as follows. First we choose a loop in the x direction and a loop in the y direction (Fig. 7) that we will call the "boundary." As an anyon hops across the boundary in the x direction (y direction), it induces a matrix $\tilde{T}_2$ ($\tilde{T}_1$) acting on the states (2.3). $\tilde{T}_1$ and $\tilde{T}_2$ are $q\times q$ matrices that act on the α indices [see (2.3)]. $\tilde{T}_1$ and $\tilde{T}_2$ also satisfy the algebra (2.2). On a compactified space, like the torus the strings of the anyons, should end somewhere. A consistent assignment of the strings attached to the anyons is given in Fig. 8. The solid lines represent the strings of strength $e^{i\theta}$. The bold solid lines represent strings of double strength, i.e., $e^{2i\theta}$. To check that such an assignment is really consistent, let us first consider anyon 1 hopping around the turning point A of the string of anyon 2 (Fig. 9). As anyon 1 is hopping around the point A, it crosses the $e^{2i\theta}$ string once and the $e^{i\theta}$ string once inducing a phase $e^{2i\theta}e^{-i\theta}$. The string of anyon 1 also crossed anyon 2 once (Fig. 9), which induces a phase $e^{-i\theta}$. Notice that anyon 1 crosses the non-Abelian cut twice inducing two matrices, $\tilde{T}_2$ and $\tilde{T}_2^{-1}$. The total phase of anyon 1 hopping around A is given by

$$e^{2i\theta}e^{-i\theta}e^{-i\theta}\tilde{T}_2\tilde{T}_2^{-1}=1\ .$$

Therefore there is no singularity at point A, and A is equivalent to any other point on the torus. The situation becomes more clear in the lattice version described

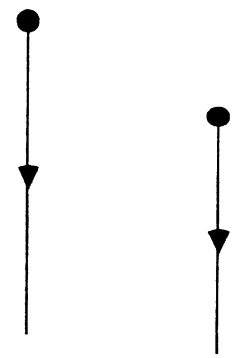

FIG. 5. Each anyon is attached to a string in the $-\hat{\mathbf{y}}$ direction.

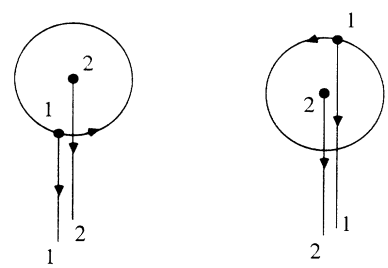

FIG. 6. A phase factor $e^{i2\theta}$ is induced as anyon 1 goes around anyon 2. One $e^{i\theta}$ comes from anyon 1 crossing string 2; another $e^{i\theta}$ comes from string 1 crossing anyon 2.

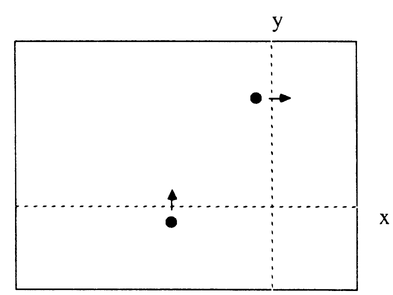

FIG. 7. An anyon hopping across the "boundary" in the x direction (y direction) induces a non-Abelian phase (matrix) $\tilde{T}_2$ ($\tilde{T}_1$).

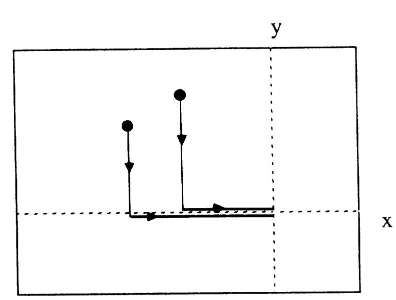

FIG. 8. The strings on the torus. The solid lines represent the strings of strength $e^{i\theta}$. The heavy solid lines represent strings of double strength, i.e., $e^{2i\theta}$.

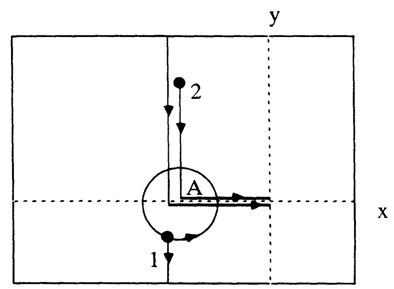

FIG. 9. An anyon hopping around the point A induces no phase.

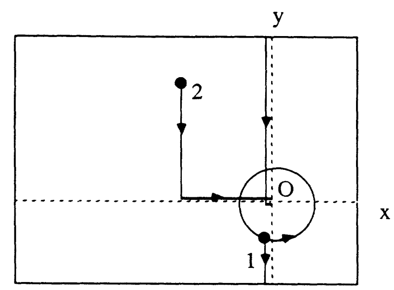

FIG. 10. An anyon hopping around the point O also induces no phase if the total number of the anyons is a multiple of q.

below, where A is at the center of a plaquette providing a natural regularization to the problem.

Now let us consider an anyon hopping around point O (Fig. 10) where the two "boundaries" met. The anyon crosses the $e^{i2\theta}$ strings of the rest of the $N-1$ anyons (assuming there are N anyons on the torus), which induces a phase $e^{-i(N-1)2\theta}$. The anyon also crossed the non-Abelian cuts in the x direction and y direction twice. This induces a phase $\tilde{T}_2^{-1}\tilde{T}_1^{-1}\tilde{T}_2\tilde{T}_1$. The total phase of the anyon hopping around point O is given by

$$e^{-i(N-1)2\theta}\tilde{T}_2^{-1}\tilde{T}_1^{-1}\tilde{T}_2\tilde{T}_1=e^{-i2N\theta} .$$

Therefore there is no singularity at point O if and only if $e^{-i2N\theta}=e^{-i2\pi(Np/q)}=1$, i.e., N is a multiple of q. Thus we have shown that one can consistently put N anyons on a continuous torus if N is a multiple of q. Another simple way to arrive to this condition is as follows. We know that an anyon carries a flux 2θ. Then, N anyons on a torus are equivalent to N solenoids of flux 2θ each coming from infinity and penetrating into the torus. Imagine now that the solenoids merge inside the torus and that this large solenoid of flux $2\theta N$ cuts again the surface of the torus leaving the system. For this final cut to be unobservable it must be satisfied that

$$2\theta N=2\pi n$$

(where n is an integer) or equivalently $N=nq$ if $\theta=\pi/q$ is used.

Guided by the picture we obtained on the continuous torus, we find that the lattice Hamiltonian satisfying (a)–(d) is given by (on a lattice of size $L\times L$)

$$H=-t\sum_{(x,y),\tau}H(x,y;\tau) , \tag{4.1}$$

where $H(x,y;\tau)$ moves an anyon from (x,y) to $(x+\tau_x,y+\tau_y)$ and $\tau=(\pm1,0),(0,\pm1)$ are vectors connecting nearest-neighbor sites of the lattice. $H(x,y;\tau)$ formally satisfies the following.

(i) $H(x,y;\tau)|\{(x_i,y_i)\};\alpha\rangle=0$, if $(x,y)\neq(x_i,y_i)\forall i$ or if $(x+\tau_x,y+\tau_y)=(x_i,y_i)$, for some i.

(ii) If $(x,y)=(x_{i_0},y_{i_0})$,

$$H(x_{i_0},y_{i_0};\tau)|\{(x_i,y_i)\};\alpha\rangle=h_{\alpha\beta}(x_{i_0},y_{i_0};\{(x_i,y_i)\};\tau)|\{(x_1,y_1),\dots,(x_{i_0}+\tau_x,y_{i_0}+\tau_y),\dots,(x_N,y_N)\};\beta\rangle , \tag{4.2}$$

where $h_{\alpha\beta}$ is given by

(1) $\tau=\hat{x}$,

$$h_{\alpha\beta}=\delta_{\alpha\beta}f_{\hat{x}}[1-\delta^p(x_{i_0}-L)]+\delta^p(x_{i_0}-L)\tilde{T}_{1\alpha\beta}f_{\hat{x}} , \tag{4.3a}$$

$$f_{\hat{x}}=\exp\left[+i\theta\sum_{i\neq i_0}\delta^p(x_{i_0}-x_i+1)\theta(y_i-y_{i_0}-\tfrac{1}{2})\right]\exp\left[-i\theta\sum_{i\neq i_0}\delta^p(x_{i_0}-x_i)\theta(y_{i_0}-y_i-\tfrac{1}{2})\right] , \tag{4.3b}$$

(2) $\tau=-\hat{x}$,

$$h_{\alpha\beta}=\delta_{\alpha\beta}f_{-\hat{x}}[1-\delta^p(x_{i_0}-1)]+\delta^p(x_{i_0}-1)\tilde{T}^{-1}_{1\alpha\beta}f_{-\hat{x}} , \tag{4.4a}$$

$$f_{-\hat{x}}=\exp\left[-i\theta\sum_{i\neq i_0}\delta^p(x_{i_0}-x_i)\theta(y_i-y_{i_0}-\tfrac{1}{2})\right]\exp\left[i\theta\sum_{i\neq i_0}\delta^p(x_{i_0}-x_i-1)\theta(y_{i_0}-y_i-\tfrac{1}{2})\right] , \tag{4.4b}$$

(3) $\tau=\hat{y}$,

$$h_{\alpha\beta}=\delta_{\alpha\beta}[1-\delta^p(y_{i_0}-L)]+\delta^p(y_{i_0}-L)\tilde{T}_{2\alpha\beta}g_{\hat{y}}\ , \tag{4.5a}$$

$$g_{\hat{y}}=\exp\left\{ i\theta \sum_{i\neq i_0}[\theta(x_{i_0}-x_i+\tfrac{1}{2})+\theta(x_{i_0}-x_i-\tfrac{1}{2})]\right\}\ , \tag{4.5b}$$

(4) $\tau=-\hat{y}$,

$$h_{\alpha\beta}=\delta_{\alpha\beta}[1-\delta^p(y_{i_0}-1)]+\delta^p(y_{i_0}-1)\tilde{T}^{-1}_{2\alpha\beta}g_{-\hat{y}}\ , \tag{4.6a}$$

$$g_{-\hat{y}}=g_{\hat{y}}^{*}\ . \tag{4.6b}$$

In Eqs. (4.3)–(4.6), $\tilde{T}_1$ and $\tilde{T}_2$ are $q\times q$ matrices satisfying the algebra (2.2), $\theta(x)$ satisfies

$$\theta(x)=\begin{cases}1, & x>0\\ 0, & x<0\end{cases}\ , \tag{4.7}$$

and $\delta^p(x)$ is a periodic δ function

$$\delta^p(x)=\begin{cases}1, & x=0 \bmod L\\ 0, & x\neq 0 \bmod L\end{cases}\ . \tag{4.8}$$

Let us explain in simple words the Hamiltonian defined in Eqs. (4.1)–(4.6).

(1) When (x_{i_0},y_{i_0}) lies in the interior of the lattice, the terms containing T_1 and T_2 in (4.3a)–(4.6a) can be ignored, and $h_{\alpha\beta}$ is diagonal. The hopping Hamiltonian reduces to the expression we would have used in an open manifold. The rules discussed at the beginning of this section are satisfied through the phase factor $f_{\pm\hat{x}}$. The first exponentials in (4.3b) and (4.4b) come from the i_0th anyon (which is the hopping anyon) crossing the strings of other anyons. The second exponentials in (4.3b) and (4.4b) come from the string of the i_0th anyon crossing other anyons. On the lattice the strings are attached to the anyons in the way described in Fig. 11.

(2) The hopping of an anyon around the torus along a given path is described by the operator

$$T_1=\prod_{x_{i_0}=L}^{1} h((x_{i_0},y_{i_0}),\{(x_i,y_i)\};\hat{\mathbf{x}}) =e^{i\varphi_1}\tilde{T}_1 \tag{4.9}$$

if the hopping is in the $\hat{x}$ direction. For the hopping in the $\hat{\mathbf{y}}$ direction we have

$$T_2=\prod_{y_{i_0}=L}^{1} h((x_{i_0},y_{i_0}),\{(x_i,y_i)\};\hat{\mathbf{y}}) =e^{i\varphi_2}\tilde{T}_2\ . \tag{4.10}$$

φ_1 and φ_2 in Eqs. (4.9) and (4.10) are two phases that may depend on the hopping path. When restricted to the q-dimensional subspace of a fixed anyon position, T_1 and T_2 given by (4.9) and (4.10) satisfies the algebra (2.2). Therefore (d) is satisfied by the Hamiltonian (4.1).

(3) It is very easy to check that our Hamiltonian satisfies the condition (a) except for a row of plaquettes

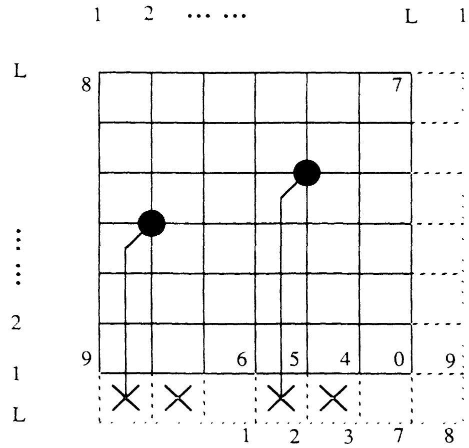

FIG. 11. Anyons and their strings on a periodic lattice.

lying between the row 1 and the row L. In what follows we will demonstrate that (a) is satisfied even for those plaquettes. Consider the plaquette (1256) and (2345) in Fig. 11. An anyon hopping from 5 to 6 induces a phase $e^{-i\theta}$ because the anyon hops across the string of anyon A. Similarly, an anyon hopping from 2 to 3 induces a phase $e^{-i\theta}$ because the string of the hopping anyon crosses the anyon A. To compensate the phases on the horizontal links, we introduce nontrivial phases $g_{\pm\hat{y}}$ on the vertical links between the row 1 and the row L. The hopping amplitude from 2 to 5 contains one more factor $e^{i\theta}$ compared to the hopping amplitude from 1 to 6. Similarly the hopping from 3 to 4 contains one more factor $e^{i\theta}$ compared to the hopping from 2 to 5. These results can be directly derived from the expressions of $g_{\pm\hat{y}}$ [(4.5b) and (4.6b)]. The phase factor $g_{\pm g}$ corresponds to the $e^{i2\theta}$ strings discussed before. Therefore as an anyon hops around the plaquette, say (1256), it obtains a "phase":

$$(e^{i\varphi_{25}}T_2)(e^{-i\theta})(e^{-i\varphi_{16}}T_2^{-1})=e^{i(\varphi_{25}-\varphi_{16}-\theta)}=1\ .$$

Therefore (a) is satisfied for the plaquette (1256). In general, the vertical link on the right-hand side of a crossed plaquette contains one more factor $e^{i\theta}$ compared to the vertical link on the left-hand side of the plaquette, while the two vertical links on the two sides of an uncrossed plaquette have the same phase. The end of each $e^{i\theta}$ string produces two neighboring crossed plaquettes as demonstrated in Fig. 11, thus producing a horizontal string of strength $e^{i2\theta}$.

(4) A real nontrivial test for our Hamiltonian is to check whether condition (a) is satisfied by the plaquette (7890) or not, which naively seems to be a singular point in our Hamiltonian. The phase on the link (89) is given by $\varphi_{89}=\theta N_1$ [see (4.5b)], where N_1 is the number of fixed anyons in column 1. By "fixed anyons" we mean the other $N-1$ anyons except the i_0th anyon. The i_0th anyon is hopping around the plaquette (7890). The phase on link (70) is given by $\varphi_{70}=2\theta N_0+\theta N_L$ [see (4.5b)], where $N_0=N-1-N_L$ is the number of fixed anyons in column 1 to $L-1$, and N_L is the number of fixed anyons in

column L. The phase on the link (90) is $\varphi_{90}=-\theta N_1$ because the i_0th anyon crosses the strings of the anyons on the column L, and the phase on the link (78) is $\varphi_{78}=-\theta N_L$ because the string of the i_0th anyon crosses the anyons in column L. The total "phase" of the anyon hopping around (7890) is given by

$$(e^{-i\varphi_{70}}\tilde{T}_2^{-1})(e^{i\varphi_{90}}\tilde{T}_1^{-1})(e^{i\varphi_{89}}\tilde{T}_2)(e^{i\varphi_{78}}\tilde{T}_1) = e^{-i2\theta(N-1)}\tilde{T}_2^{-1}\tilde{T}_1^{-1}\tilde{T}_2\tilde{T}_1 = e^{-i2\theta N} . \quad (4.11)$$

Condition (a) is satisfied if $e^{-i2\theta N}=1$. From (4.11) we see that an anyon system can be consistently defined on a torus (with translation symmetries) when and only when the number of anyons is a multiple of q (remember $\theta=p\pi/q$). [The quantization condition (4.11) is modified in the presence of magnetic field. See Sec. VII.]

We also checked that conditions (a)–(d) are satisfied for many other anyon hoppings. To further demonstrate the consistency of our Hamiltonian, we will show that the anyon system described by Eq. (4.1) is translational invariant.

V. SYMMETRIES OF THE ANYON HOPPING HAMILTONIAN

In the discussions above we see that the cuts (boundary) and the strings of the anyons are all unobservable and hence unphysical. Then, we expect the "physics" described by the Hamiltonian (4.1) to respect the translation symmetries.

Let us first consider the translation in the x direction t_x: $(x_i,y_i)\to(x_i+1,y_i)$. For $\tau=\hat{\mathbf{x}}$, $h_{\alpha\beta}(\hat{\mathbf{x}})$ changes to $h_{\alpha\beta}^{(1)}(\hat{\mathbf{x}})$ under such a translation:

$$h_{\alpha\beta}(\hat{\mathbf{x}})\to h_{\alpha\beta}^{(1)}(\hat{\mathbf{x}})=\delta_{\alpha\beta}f_{\hat{\mathbf{x}}}^{(1)}[1-\delta^p(x_{i_0}+1-L)] + \delta^p(x_{i_0}+1-L)\tilde{T}_{1\alpha\beta}f_{\hat{\mathbf{x}}}^{(1)} , \quad (5.1a)$$

$$f_{\hat{\mathbf{x}}}\to f_{\hat{\mathbf{x}}}^{(1)}=f_{\hat{\mathbf{x}}} . \quad (5.1b)$$

Equation (5.1b) exists because $f_{\hat{\mathbf{x}}}$ only depends on $(x_{i_0}-x_i)$. Making a "gauge" transformation

$$|\{(x_i,y_i)\};\alpha\rangle\to(U_x)_{\alpha\beta}|\{(x_i,y_i)\};\beta\rangle ,$$

$$U_x=(\tilde{T}_1)^{\mathcal{S}} , \quad (5.2)$$

where $\mathcal{S}=\sum_i\delta^p(x_i-L)$, we transform $h_{\alpha\beta}^{(1)}(\hat{\mathbf{x}})$ to

$$h_{\alpha\beta}^{(2)}(\hat{\mathbf{x}})=(U_x^{-1}(x_{i_0}+1,y_{i_0})h^{(1)}(\hat{\mathbf{x}})U_x(x_{i_0},y_{i_0}))_{\alpha\beta} .$$

One can easily check that $h_{\alpha\beta}^{(2)}(\hat{\mathbf{x}})$ is equal to $h_{\alpha\beta}(\hat{\mathbf{x}})$ in (4.3a). Therefore $\sum_{(x,y)}H(x,y;\hat{\mathbf{x}})$ is invariant [up to a gauge transformation (5.2)] under the translation t_x. Similarly one can show $\sum_{(x,y)}H(x,y;-\hat{\mathbf{x}})$, is invariant under translations in $\hat{\mathbf{x}}$ direction.

For $\tau=\hat{\mathbf{y}}$, we must be careful about the transformation of the Θ function in $g_{\hat{\mathbf{y}}}$. Equation (4.5b) is valid only when x_i and y_i lie within the range $[1,L]$. For x_i and y_i outside the range $[1,L]$, $g_{\hat{\mathbf{y}}}$ is defined by periodic extension. Notice that x_i+1 may be larger than L if $x_i=L$. In this case we should replace (x_i+1) by 1 instead of $(L+1)$. Keeping this subtlety in mind, we find that $h_{\alpha\beta}(\hat{\mathbf{y}})$ transforms to

$$h_{\alpha\beta}(\hat{\mathbf{y}})\to h_{\alpha\beta}^{(1)}(\hat{\mathbf{y}})=\delta_{\alpha\beta}[1-\delta^p(y_{i_0}-L)]+\delta^p(y_{i_0}-L)\tilde{T}_{2\alpha\beta}g_{\hat{\mathbf{y}}}^{(1)} ,$$

$$g_{\hat{\mathbf{y}}}\to g_{\hat{\mathbf{y}}}^{(1)}=\exp\left[i\theta\sum_{i\neq i_0}[\Theta(x_{i_0}-x_i+\tfrac{1}{2})+\Theta(x_{i_0}-x_i-\tfrac{1}{2})][1-\delta^p(x_{i_0}-L)][1-\delta^p(x_i-L)]\right] \quad (5.3a)$$

$$\times\exp\left[i\theta\sum_{i\neq i_0}\{2[1-\delta^p(x_{i_0}-L)]\delta^p(x_i-L)+\delta^p(x_{i_0}-L)\delta^p(x_i-L)\}\right]$$

$$=\exp\left[i\theta\sum_{i\neq i_0}[\Theta(x_{i_0}-x_i+\tfrac{1}{2})+\Theta(x_{i_0}-x_i-\tfrac{1}{2})]\right]\exp\left[-i\theta\sum_{i\neq i_0}2[\delta^p(x_{i_0}-L)-\delta^p(x_i-L)]\right] . \quad (5.3b)$$

Notice that

$$\exp\left[-i\theta\sum_{i\neq i_0}2[\delta^p(x_{i_0}-L)-\delta^p(x_i-L)]\right]=\exp\{-2i\theta[(N-1)\delta^p(x_{i_0}-L)-N_L]\}$$

$$=\exp[2i\theta\delta^p(x_{i_0}-L)]\exp(+2i\theta N_L) , \quad (5.4)$$

where we have used the fact that $e^{-2i\theta N}=1$. Under the gauge transformation (5.2), $h_{\alpha\beta}^{(1)}(\hat{\mathbf{y}})$ changes to

$$h_{\alpha\beta}^{(1)}(\hat{\mathbf{y}})\to h_{\alpha\beta}^{(2)}(\hat{\mathbf{y}})=\delta_{\alpha\beta}[1-\delta^p(y_{i_0}-L)]+\delta^p(y_{i_0}-L)\tilde{T}_{2\alpha\beta}g_{\hat{\mathbf{y}}}^{(2)} ,$$

$$g_{\hat{\mathbf{y}}}^{(1)}\to g_{\hat{\mathbf{y}}}^{(2)}=g_{\hat{\mathbf{y}}}^{(1)}\tilde{T}_2^{-1}U_x^{-1}\tilde{T}_2U_x \quad (5.5a)$$

$$=g_{\hat{\mathbf{y}}}^{(1)}\tilde{T}_2^{-1}\tilde{T}_1^{-N_L-\delta^p(x_{i_0}-L)}\tilde{T}_2\tilde{T}_1^{N_L+\delta^p(x_{i_0}-L)}$$

$$=\exp\left[i\theta\sum_{i\neq i_0}[\Theta(x_{i_0}-x_i+\tfrac{1}{2})+\Theta(x_{i_0}-x_i-\tfrac{1}{2})]\right] . \quad (5.5b)$$

Comparing Eqs. (5.5b) and (4.5b) we see that the translation $(x_i,y_i)\to(x_i+1,y_i)$ plus the gauge transformation (5.2) leaves $\sum_{(x,y)}H(x,y;\hat{\mathbf{y}})$ invariant. One can also show that $\sum_{(x,y)}H(x,y;-\hat{\mathbf{y}})$, and hence the total Hamiltonian (4.1), are invariant under the translation in the $\hat{\mathbf{x}}$ direction [up to the gauge transformation (5.2)].

Now let us consider a translation in the $\hat{\mathbf{y}}$ direction t_y: $(x_i,y_i)\to(x_i,y_i+1)$. For $\tau=\hat{\mathbf{y}}$, $h_{\alpha\beta}(\hat{\mathbf{y}})$ changes to

$$h_{\alpha\beta}(\hat{\mathbf{y}})\to h^{(1)}_{\alpha\beta}(\hat{\mathbf{y}})=\delta_{\alpha\beta}[1-\delta^p(y_{i_0}+1-L)]+\delta^p(y_{i_0}+1-L)\tilde{T}_{2\alpha\beta}g^{(1)}_{\hat{\mathbf{y}}} , \tag{5.6a}$$

$$g_{\hat{\mathbf{y}}}\to g^{(1)}_{\hat{\mathbf{y}}}=\exp\left[i\theta\sum_{i\neq i_0}[\Theta(x_{i_0}-x_i+\tfrac{1}{2})+\Theta(x_{i_0}-x_i-\tfrac{1}{2})]\right] \tag{5.6b}$$

under the $\hat{\mathbf{y}}$ translation. Again we need a gauge transformation to change (5.6) to (4.5). The proper gauge transformation is given by

$$\begin{aligned}&|\{(x_i,y_i)\};\alpha\rangle\to(U_y)_{\alpha\beta}|\{(x_i,y_j)\};\beta\rangle ,\\ &U_y=(\tilde{T}_2)^{\mathcal{S}_1}\exp\left[i\theta\sum_{i\neq j}\delta^p(y_i-L)[\Theta(x_i-x_j+\tfrac{1}{2})+\Theta(x_i-x_j-\tfrac{1}{2})]\right] ,\end{aligned} \tag{5.7}$$

where $\mathcal{S}_1=\sum_i\delta^p(y_i-L)$. Under U_y, $h^{(1)}_{\alpha\beta}(\hat{\mathbf{y}})$ in (5.6) is transformed to $h^{(2)}_{\alpha\beta}(\hat{\mathbf{y}})$:

$$\begin{aligned}h^{(1)}(\hat{\mathbf{y}})\to h^{(2)}&=U_y^{-1}(x_{i_0},y_{i_0}+1)h^{(1)}(\hat{\mathbf{y}})U_y(x_{i_0},y_{i_0})\\ &=U_y^{-1}(x_{i_0},y_{i_0}+1)U_y(x_{i_0},y_{i_0})h^{(1)}(\hat{\mathbf{y}})\\ &=\exp\left[i\theta\sum_{i\neq i_0}[\delta^p(y_{i_0}-L)-\delta^p(y_{i_0}+1-L)][\Theta(x_{i_0}-x_i+\tfrac{1}{2})+\Theta(x_{i_0}-x_i-\tfrac{1}{2})]\right]\\ &\quad\times\tilde{T}_2^{[\delta^p(y_{i_0}-L)-\delta^p(y_{i_0}+1-L)]}h^{(1)}(\hat{\mathbf{y}})\\ &=h(\hat{\mathbf{y}}) .\end{aligned} \tag{5.8}$$

Therefore $\sum_{(x,y)}H(x,y;\hat{\mathbf{y}})$ is invariant under the translation by $\hat{\mathbf{y}}$ plus the gauge transformation U_y. For $\tau=\hat{\mathbf{x}}$, we have

$$\begin{aligned}&h_{\alpha\beta}(\hat{\mathbf{x}})\to h^{(1)}_{\alpha\beta}(x)=\delta_{\alpha\beta}f^{(1)}_{\hat{\mathbf{x}}}[1-\delta^p(x_{i_0}-L)]+\delta^p(x_{i_0}-L)\tilde{T}_{1\alpha\beta}f^{(1)}_{\hat{\mathbf{x}}} ,\\ &f_{\hat{\mathbf{x}}}\to f^{(1)}_{\hat{\mathbf{x}}}=\exp\left[i\theta\sum_{i\neq i_0}[\delta^p(x_{i_0}-x_i+1)\Theta(y_i-y_{i_0}-\tfrac{1}{2})\right.\\ &\qquad\qquad\left.-\delta^p(x_{i_0}-x_i)\Theta(y_{i_0}-y_i-\tfrac{1}{2})][1-\delta^p(y_i-L)][1-\delta^p(y_{i_0}-L)]\right]\end{aligned} \tag{5.9a}$$

$$\begin{aligned}&\qquad\times\exp\left[i\theta\sum_{i\neq i_0}\{\delta^p(x_{i_0}-x_i+1)[1-\delta^p(y_i-L)]\delta^p(y_{i_0}-L)-\delta^p(x_{i_0}-x_i)[1-\delta^p(y_{i_0}-L)]\delta^p(y_i-L)\}\right]\\ &\quad=f_{\hat{\mathbf{x}}}\exp\left[i\theta\sum_{i\neq i_0}[\delta^p(x_{i_0}-x_i+1)+\delta^p(x_{i_0}-x_i)][\delta^p(y_{i_0}-L)-\delta^p(y_i-L)]\right] .\end{aligned} \tag{5.9b}$$

The gauge transformation changes $h^{(1)}_{\alpha\beta}(\hat{\mathbf{x}})$ to $h^{(2)}_{\alpha\beta}(\hat{\mathbf{x}})$:

$$h^{(1)}_{\alpha\beta}(\hat{\mathbf{x}})\to h^{(2)}_{\alpha\beta}(x)=\delta_{\alpha\beta}f^{(2)}_{\hat{\mathbf{x}}}[1-\delta^p(x_{i_0}-L)]+\delta^p(x_{i_0}-L)\tilde{T}_{1\alpha\beta}f^{(2)}_{\hat{\mathbf{x}}} , \tag{5.10a}$$

$$\begin{aligned}f^{(1)}_{\hat{\mathbf{x}}}\to f^{(2)}_{\hat{\mathbf{x}}}&=U_y^{-1}(x_{i_0}+1,y_{i_0})f^{(1)}_{\hat{\mathbf{x}}}U_y(x_{i_0},y_{i_0})\\ &=f^{(1)}_{\hat{\mathbf{x}}}\exp\left[i\theta\sum_{i\neq i_0}\delta^p(y_i-L)[\delta^p(x_i-x_{i_0})+\delta^p(x_i-x_{i_0}-1)]\right]\\ &\quad\times\exp\left[i\theta\sum_{i\neq i_0}\delta^p(y_{i_0}-L)[\delta^p(x_{i_0}-x_i+1)+\delta^p(x_{i_0}-x_i)]\right]\\ &=f^{(1)}_{\hat{\mathbf{x}}}\exp\left[i\theta\sum_{i\neq i_0}[\delta^p(y_i-L)-\delta^p(y_{i_0}-L)][\delta^p(x_{i_0}-x_i)+\delta^p(x_{i_0}-x_i+1)]\right]\\ &=f_{\hat{\mathbf{x}}} .\end{aligned} \tag{5.10b}$$

Therefore the translation t_y and the gauge transformation U_y leave $\sum_{(x,y)} H(x,y;\hat{\mathbf{x}})$ invariant. We show that the total Hamiltonian

$$H=\sum_{(x,y)}[H(x,y;\hat{\mathbf{x}})+H(x;y;\hat{\mathbf{y}})+\mathrm{H.c.}]$$

respects translational symmetry in the $\hat{\mathbf{y}}$ direction.

To show that the Hamiltonian (4.1) is invariant under a 90° rotation, let us first make a gauge transformation, which changes the vertical strings into horizontal strings (Fig. 12). The gauge transformation is given by W:

$$|\{(x_i,y_i)\};\alpha\rangle \to W|\{(x_i,y_i)\};\alpha\rangle ,$$
$$W=\exp\left[+i\theta\sum_{i\neq j}\Theta(x_i-x_j+\tfrac{1}{2})\Theta(y_j-y_i-\tfrac{1}{2})\right] . \tag{5.11}$$

Under this gauge transformation

$$h_{\alpha\beta}(\tau)\to\tilde{h}_{\alpha\beta}(\tau)=W^{-1}(x_{i_0}+\tau_x,y_{i_0}+\tau_y)h_{\alpha\beta}(\tau)W(x_{i_0},y_{i_0}) . \tag{5.12}$$

For $\tau=\hat{\mathbf{x}}$, (5.12) becomes

$$\tilde{h}_{\alpha\beta}(\hat{\mathbf{x}})=\delta_{\alpha\beta}\tilde{f}'_x[1-\delta^p(x_{i_0}-L)]+\delta^p(x_{i_0}-L)\tilde{T}_1\tilde{g}_{\hat{\mathbf{x}}} ,$$

$$\begin{aligned}\tilde{f}'_{\hat{\mathbf{x}}}=\exp&\left[-i\theta\sum_{i\neq i_0}\{[\Theta(x_{i_0}+1-x_i+\tfrac{1}{2})-\Theta(x_{i_0}-x_i+\tfrac{1}{2})]\Theta(y_i-y_{i_0}-\tfrac{1}{2})\right.\\
&\left.\qquad+[\Theta(x_i-(x_{i_0}+1)+\tfrac{1}{2})-\Theta(x_i-x_{i_0}+\tfrac{1}{2})]\Theta(y_{i_0}-y_i-\tfrac{1}{2})\}\right]\\
\times\exp&\left[i\theta\sum_{i\neq i_0}[\delta^p(x_{i_0}-x_i+1)\Theta(y_i-y_{i_0}-\tfrac{1}{2})-\delta^p(x_{i_0}-x_i)\Theta(y_{i_0}-y_i-\tfrac{1}{2})]\right]\\
=1 ,&\end{aligned} \tag{5.13a}$$

$$\begin{aligned}\tilde{g}_{\hat{\mathbf{x}}}&=W^{-1}(x_{i_0}=1,y_{i_0})f_{\hat{\mathbf{x}}}W(x_{i_0}=L,y_{i_0})\\
&=\exp\left[-i\theta\sum_{i\neq i_0}\{[\Theta(1-x_i+\tfrac{1}{2})-\Theta(L-x_i+\tfrac{1}{2})]\Theta(y_i-y_{i_0}-\tfrac{1}{2})+[\Theta(x_i-1+\tfrac{1}{2})-\Theta(x_i-L+\tfrac{1}{2})]\Theta(y_{i_0}-y_i-\tfrac{1}{2})\}\right]\\
&\quad\times\exp\left[i\theta\sum_{i\neq i_0}[\delta^p(L-x_i+1)\Theta(y_i-y_{i_0}-\tfrac{1}{2})-\delta^p(L-x_i)\Theta(y_{i_0}-y_i-\tfrac{1}{2})]\right]\\
&=\exp[i\theta(N-1)]\exp\left[-i\theta\sum_{i\neq i_0}[\Theta(y_{i_0}-y_i+\tfrac{1}{2})+\Theta(y_{i_0}-y_i-\tfrac{1}{2})]\right] .\end{aligned} \tag{5.13b}$$

For $\tau=\hat{\mathbf{y}}$, (5.12) becomes

$$\tilde{h}_{\alpha\beta}(\hat{\mathbf{y}})=\delta_{\alpha\beta}\tilde{f}_{\hat{\mathbf{y}}}[1-\delta^p(y_{i_0}-L)]+\delta^p(y_{i_0}-L)\tilde{T}_2\tilde{g}'_{\hat{\mathbf{y}}} , \tag{5.14a}$$

$$\begin{aligned}\tilde{f}_{\hat{\mathbf{y}}}&=W^{-1}(x_{i_0},y_{i_0}+1)W(x_{i_0},y_{i_0})\\
&=\exp\left[-i\theta\sum_{i\neq i_0}\{\Theta(x_{i_0}-x_i+\tfrac{1}{2})[\Theta(y_i-y_{i_0}-1-\tfrac{1}{2})-\Theta(y_i-y_{i_0}-\tfrac{1}{2})]\right.\\
&\qquad\left.+\Theta(x_i-x_{i_0}+\tfrac{1}{2})[\Theta(y_{i_0}+1-y_i-\tfrac{1}{2})-\Theta(y_{i_0}-y_i-\tfrac{1}{2})]\}\right]\\
&=\exp\left[i\theta\sum_{i\neq i_0}[\delta^p(y_{i_0}-y_i+1)\theta(x_{i_0}-x_i+\tfrac{1}{2})-\delta^p(y_{i_0}-y_i)\Theta(x_i-x_{i_0}+\tfrac{1}{2})]\right] ,\end{aligned} \tag{5.14b}$$

$$\begin{aligned}\tilde{g}'_{\hat{\mathbf{y}}}&=W^{-1}(x_{i_0},y_{i_0}=1)g_{\hat{\mathbf{y}}}W(x_{i_0},y_{i_0}=L)\\
&=\exp[i\theta(N-1)]\exp\left[i\theta\sum_{i\neq i_0}[\delta^p(L+1-y_i)\Theta(x_{i_0}-x_i+\tfrac{1}{2})-\delta^p(y_i-L)\Theta(x_i-x_{i_0}+\tfrac{1}{2})]\right]\\
&=\exp[i\theta(N-1)]\tilde{f}_{\hat{\mathbf{y}}}(y_{i_0}=L) .\end{aligned} \tag{5.14c}$$

One can see that (5.13) and (5.14) are very similar to (4.3)–(4.6). We can show explicitly that (5.13) and (5.14) are identical to (4.6) and (4.3) after a 90° rotation,

$$x_i \to L - y_i + 1 \ , \qquad y_i \to x_i \ , \tag{5.15}$$

and a redefinition of $\tilde{T}_1$ and $\tilde{T}_2$,

$$\tilde{T}_1 \to e^{-i\theta(N-1)}\tilde{T}_2^{-1} \ , \qquad \tilde{T}_2 \to e^{-i\theta(N-1)}\tilde{T}_1 \ . \tag{5.16}$$

Now (5.13) changes to

$$\tilde{h}_{\alpha\beta}(\hat{\mathbf{x}}) \to \tilde{h}^{(1)}_{\alpha\beta}(-\hat{\mathbf{y}}) = \delta_{\alpha\beta}[1-\delta^p(y_{i_0}-1)] + \delta^p(y_{i_0}-1)\tilde{T}_2^{-1}\tilde{g}^{(1)}_{-\hat{\mathbf{y}}} \ , \tag{5.17a}$$

$$\begin{aligned}\tilde{g}_{\hat{\mathbf{x}}} \to \tilde{g}^{(1)}_{-\hat{\mathbf{y}}} &= \exp\left\{ i\theta \sum_{i\neq i_0} [\Theta(x_{i_0}-x_i+\tfrac{1}{2}) + \Theta(x_{i_0}-x_i-\tfrac{1}{2})] \right\} \\ &= g_{-\hat{\mathbf{y}}} \ ,\end{aligned} \tag{5.17b}$$

and (5.14) changes to

$$\tilde{h}_{\alpha\beta}(\hat{\mathbf{y}}) \to \tilde{h}^{(1)}_{\alpha\beta}(\hat{\mathbf{x}}) = \delta_{\alpha\beta}[1-\delta^p(x_{i_0}-L)]\tilde{f}^{(1)}_{\hat{\mathbf{x}}} + \delta^p(x_{i_0}-L)\tilde{T}_1\tilde{f}^{(1)}_{\hat{\mathbf{x}}} \ , \tag{5.18a}$$

$$\begin{aligned}\tilde{f}_{\hat{\mathbf{y}}} \to \tilde{f}^{(1)}_{\hat{\mathbf{x}}} &= \exp\left\{ i\theta \sum_{i\neq i_0} [\delta^p(x_{i_0}-x_i+1)\Theta(y_i-y_{i_0}+\tfrac{1}{2}) - \delta^p(x_{i_0}-x_i)\Theta(y_{i_0}-y_i+\tfrac{1}{2})] \right\} \\ &= \exp\left\{ i\theta \sum_{i\neq i_0} [\delta^p(x_{i_0}-x_i+1)\Theta(y_i-y_{i_0}-\tfrac{1}{2}) - \delta^p(x_{i_0}-x_i)\Theta(y_{i_0}-y_i-\tfrac{1}{2})] \right\} \\ &= f_{\hat{\mathbf{x}}} \ .\end{aligned} \tag{5.18b}$$

The second equality in (5.18b) is because $\tilde{f}^{(1)}_{\hat{\mathbf{x}}}$ is defined only for (x_{i_0},y_{i_0}), satisfying $(x_{i_0},y_{i_0})\neq(x_i,y_i)$ and $(x_{i_0}+1,y_{i_0})\neq(x_i,y_i)$ for any $i\neq i_0$. We see that Eqs. (5.18) and (5.17) are identical to Eqs. (4.3) and (4.6). Because $h_{\alpha\beta}(-\hat{\mathbf{x}})$ and $h_{\alpha\beta}(\hat{\mathbf{y}})$ are simply the complex conjugates of $h_{\alpha\beta}(\hat{\mathbf{x}})$ and $h_{\alpha\beta}(-\hat{\mathbf{y}})$, our result implies that the total Hamiltonian (4.1) is invariant [up to a gauge transformation (5.11)] under a 90° rotation.

We have shown that the translations t_x and t_y when combined with the gauge transformation U_x and U_y leave H in (4.1) invariant:

$$U_x^{-1}t_x^{-1}Ht_xU_x = U_y^{-1}t_y^{-1}Ht_yU_y = H \ . \tag{5.19}$$

However, this does not imply $T_x = t_xU_x$ and $T_y = t_yU_y$ commute with each other. In the following we are going to show explicitly that T_x and T_y indeed commute and generate the usual translation algebra. First let us show that

$$U_xU_y = t_x^{-1}U_yt_xU_x \ . \tag{5.20}$$

Notice that

$$\begin{aligned} t_x^{-1}U_y(x_i,y_i)t_x &= U_y(x_i+1,y_i) \\ &= \tilde{T}_2^{\mathcal{S}_2}\exp\left\{ i\theta\sum_{i\neq j}\delta^p(y_i-L)[\Theta(x_i-x_j+\tfrac{1}{2})+\Theta(x_i-x_j-\tfrac{1}{2})][1-\delta^p(x_i-L)][1-\delta^p(x_j-L)] \right\} \\ &\quad\times\exp\left\{ i\theta\sum_{i\neq j}\delta^p(y_i-L)\{2\delta^p(x_j-L)[1-\delta^p(x_i-L)]+\delta^p(x_i-L)\delta^p(x_j-L)\} \right\} \\ &= U_y\exp\left\{ 2i\theta\sum_{i\neq j}\delta^p(y_i-L)\delta^p(x_j-L) \right\}\exp\left\{ -2i\theta\sum_{i\neq j}\delta^p(y_i-L)\delta^p(x_i-L) \right\} \ ,\end{aligned} \tag{5.21}$$

where $\mathcal{S}_2 = \sum_i \delta^p(y_i-L)$. Since $e^{i2\theta N}=1$, the second exponential in the last line of (5.21) can be simplified:

$$\begin{aligned}\exp\left\{ -2i\theta\sum_{i\neq j}\delta^p(y_i-L)\delta^p(x_i-L) \right\} &= \exp\left\{ -2i\theta(N-1)\sum_{i}\delta^p(y_i-L)\delta^p(x_i-L) \right\} \\ &= \exp\left\{ 2i\theta\sum_{i}\delta^p(y_i-L)\delta^p(x_i-L) \right\} \ .\end{aligned} \tag{5.22}$$

Using (5.22) we may write (5.21) as

$$t_x^{-1} U_y t_x = U_y \exp\left[2i\theta \sum_{i,j} \delta^p(y_i - L)\delta^p(x_j - L)\right] . \tag{5.23}$$

Since $\tilde{T}_1$ and $\tilde{T}_2$ satisfy the algebra (2.2), we can show that

$$U_y U_x = \exp\left[-2i\theta \sum_i \delta^p(y_i - L) \sum_j \delta^p(x_j - L)\right] U_x U_y . \tag{5.24}$$

Equations (5.23) and (5.24) imply (5.20). One can also easily check that (5.20) implies

$$t_x U_x t_y U_y = t_y U_y t_x U_x , \tag{5.25}$$

using the fact that $t_y^{-1} U_x t_y = U_x$. Hence the two translations T_x and T_y commute with each other.

Summarizing, we found that the Hamiltonian (4.1) is translational invariant in both $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions. We have shown that the translation in the $\hat{\mathbf{x}}$ direction T_x commutes with the translation in the $\hat{\mathbf{y}}$ direction T_y, and the energy eigenstates of (4.1) can be labeled by the crystal momenta. We stress that this is a highly nontrivial result for the Hamiltonian (4.1), which at first sight does not look translationally invariant. H in (4.1) also respects the 90° rotation symmetry. If our lattice has different sizes in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions the Hamiltonian (4.1) will only respect the 180° rotation.

VI. NUMERICAL TEST OF THE ANYON HOPPING HAMILTONIAN

In order to demonstrate the consistency of our Hamiltonian more completely, we also perform some numerical tests. We implemented numerically the lattice Hamiltonian presented in Sec. IV corresponding to "free" spinless anyons on 2×2 and 4×4 lattices for the special case of two semions ($\theta=\pi/2$). To diagonalize it we used a modified Lanczos method, although for these special cases the matrices are very small. The size of the Hilbert space grows quickly with the lattice size and the number of anyons so for a larger system a Lanczos technique would be the only way to obtain the ground-state energy and wave function.

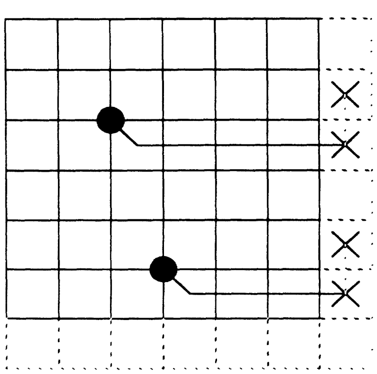

FIG. 12. The gauge transformation (5.11) transforms the vertical strings into horizontal strings.

For the 2×2 lattice with periodic boundary conditions the Hilbert space of two semions contains twelve states (while for a system of fermions or bosons it has only six states). In this case we constructed the 12×12 matrix explicitly in addition to solving numerically (Lanczos) the Hamiltonian. To monitor the translational invariance properties of the system we studied the density of semions at each site. Our result clearly showed that it is uniform in spite of the fact that the particular gauge we used makes the Hamiltonian look nonuniform. We also studied the density-density correlation functions at a distance one defined as

$$G(x) = \langle n(x) n(x+\hat{\mathbf{e}}_\mu)\rangle ,$$

where $n(x)$ is the anyon number operator and $\hat{\mathbf{e}}_\mu$ is a unit vector in the direction of the lattice axis. We found that these correlations are also uniform and, in addition, equal in both directions. This result implies that the Hamiltonian is also rotational invariant.

The ground state of the 2×2 system is not degenerate if to satisfy the algebra Eq. (2.2) we choose as matrices $\tilde{T}_1$ and $\tilde{T}_2$ the following:

$$\tilde{T}_1 = \tau_3, \quad \tilde{T}_2 = i\tau_1 ,$$

where τ_1 and τ_3 are the standard Pauli matrices. We repeated the calculation for a 4×4 lattice with two semions. There are 240 states in this case. We again found that the ground state is not degenerate if the matrices $\tilde{T}_1$ and $\tilde{T}_2$ are selected as above. The density of anyons is uniform,

$$\langle n(x)\rangle = 0.125 ,$$

and the correlation functions at distance of one lattice spacing are also uniform,

$$G(x) = 0.005\,016 ,$$

and rotational invariant as for the 2×2 lattice.

Then we conclude that numerically we have shown that our method to construct a Hamiltonian for anyons on a lattice with periodic boundary conditions is consistent. Work is in progress to extend this numerical study to larger lattices and more anyons.

VII. EFFECTIVE HAMILTONIAN OF HOLONS IN THE CHIRAL SPIN STATE AND ANYONS IN A MAGNETIC FIELD

For $\theta=\pi/2$, although (4.1) describes a translationally and rotationally invariant semion system, (4.1) is not the correct Hamiltonian for the holons in the (level-2) chiral spin state. This is because for the chiral spin state the consistent number of holons is odd when L is odd and even when L is even. While the consistent number of semions in (4.1) is always even, independent on whether L is even or odd. From the mean-field theory[2] and the variational approach[13] we see that as a holon hops around a plaquette, it sees a π flux. Therefore the correct Hamiltonian of the holons has a π flux per plaquette:

$$H_{\text{holon}}=-t\sum_{(x,y),\tau}\chi_{(x,y)(x+\tau_x,y+\tau_x)}H(x,y;\tau)\ , \tag{7.1}$$

where $H(x,y;\tau)$ is given by (4.2)–(4.6) with $\theta=\pi/2$. χ_{ij} in (7.1) satisfies $|\chi_{ij}|=1$ and

$$\chi_{i,i+\hat{x}}\chi_{i+\hat{x},i+\hat{x}+\hat{y}}\chi_{i+\hat{x}+\hat{y},i+\hat{y}}\chi_{i+\hat{y},i}=-1 \tag{7.2}$$

except for the last plaquette P_{LL}: $[(L,L),(1,L),(1,1),(L,1)]$ [i.e., the plaquette (7890) in Fig. 11]. The flux going through P_{LL} is fixed by (7.2),

$$\prod_{P_{LL}}\chi_{ij}=(-1)^{L^2-1}\ , \tag{7.3}$$

so that the total flux is a multiple of 2π. Now the condition (a) becomes (a′), a semion (or holon) hopping around a plaquette induces a phase -1. The consistency condition (4.11) changes to

$$\prod_{P_{LL}}\chi_{ij}e^{-i\pi N}=-1\ , \tag{7.4}$$

which implies the consistent number of the semions is odd for odd L and is even for even L. We can also show that H_{holon} is translationally invariant.

VIII. DISCUSSION

Before ending the paper we would like to make some remarks on the flux quantization. Notice that the gauge transformation

$$|\{(x_i,y_i)\};\alpha\rangle\to(\tilde{T}_2)^n_{\alpha\beta}|\{(x_i,y_i)\};\beta\rangle \tag{8.1}$$

introduces a phase $e^{i(2\pi/q)}$ to $\tilde{T}_1$:

$$\tilde{T}_1\to e^{i(2\pi/q)}\tilde{T}_1\ , \tag{8.2}$$

where n is an integer such that $e^{i2\pi(p/q)=n}=e^{i2\pi/q}$. From Eqs. (4.3) and (4.4) we see that the gauge transformation (8.1) effectively adds a $2\pi/q$ flux to the hole of the torus. Or more precisely, an anyon hopping all the way around the torus in the $\hat{x}$ direction obtains an additional phase $e^{i2\pi/q}$. This result implies that the ground-state energy $E(\Phi_{\text{EM}})$ is a periodic function with a period $2\pi/q$:

$$E(\Phi_{\text{EM}})=E\left[\Phi_{\text{EM}}+\frac{2\pi}{q}\right]\ , \tag{8.3}$$

where Φ_{EM} is the *electromagnetic* flux going through the hole of the torus. Because the flux Φ_{EM} changes sign under a 180° rotation, the 180° rotation symmetry implies that $E(\Phi_{\text{EM}})$ is symmetric:

$$E(\Phi_{\text{EM}})=E(-\Phi_{\text{EM}})\ . \tag{8.4}$$

In particular, for a semion system $E(\Phi_{\text{EM}})$ is a symmetric periodic function with period π. These are actually the characteristic properties of a charge $2e$ system.

Here we would like to discuss an important subtle point. The anyon hopping amplitudes in general have nontrivial phases due to the fractional statistics. It is difficult to distinguish which part of the phases is due to the fractional statistics and which part is due to the external electromagnetic flux. Therefore it is nontrivial to determine the value of Φ_{EM} from the anyon hopping Hamiltonian. The correct definition should be derived from the microscopic theory from which the anyons are generated. Based on the physical picture provided by the chiral spin state, we would like to propose a way to define the external electromagnetic flux. Consider, for example, the semion system. We know from the chiral spin state that two semions are equivalent to two electrons. As we move two semions around the torus once, the state $|\{(x_i,y_i)\};\alpha\rangle$ comes back to itself (i.e., α remains unchanged). The phase $e^{i\phi}$ induced by this process can be used to define the external flux Φ_{EM}. Because two semions are equivalent to two electrons, we will set $e^{i2\Phi_{\text{EM}}}=e^{i\phi}$. This only defines the Φ_{EM} up to an $n\pi$ ambiguity. However, this is the best we can do, because the semion Hamiltonian contains an intrinsic $n\pi$ ambiguity in its flux. According to this definition, the numerical results in Sec. VI indicate that the total energy $E(\Phi_{\text{EM}})$ is minimized at $\Phi_{\text{EM}}=0,\pi$ and that the minimum of the total energy is not shifted away from $\Phi_{\text{EM}}=0$. The shift of the minimum observed in Ref. 4 could be due to a different definition of the magnetic flux or the choice of boundary conditions. We would like to remark that moving one semion around the torus twice also leaves the state $|\{(x_i,y_i)\};\alpha\rangle$ invariant. However, moving two semions around the torus once and moving one semion around the torus twice induce two *different* phases. The latter induces a phase $e^{i2\Phi_{\text{EM}}+\pi}$. If we use the latter to define the flux, the minimum of the total energy will appear at $\pi/2$ flux, which corresponds to the result in Ref. 4. However, we believe that the first definition (i.e., moving two semions around the torus once) gives rise to a correct value of the electromagnetic flux.

The ground states of anyon systems closed geometries have many interesting topological properties. These properties can be studied systematically using the explicit lattice anyon Hamiltonian as the one derived in this paper. Observing those topological properties in numerical calculations may help to reveal the nontrivial topological structure hidden in the anyon ground states. Work is in progress.

We have received a related paper by Einarsson, who also discuss the multiple components of the anyon wave function on a torus. We would like to thank him for making a copy of his work available to us prior to publication.

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