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Phase transitions in Josephson-junction ladders in a magnetic field

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A model of a Josephson-junction ladder in a perpendicular magnetic field with $f_0=p/q$ flux quanta per plaquette is considered. It is shown that the topological features of the zero-temperature phase diagram, as a function of charging energy and small deviations $f-f_0$ from commensurability of the vortex lattice, are strongly dependent on q. In addition to a superconductor-insulator transition, a commensurate-incommensurate transition is also possible within the superconducting phase for $q \gtrsim 2$. For $q>3$, an intermediate incommensurate phase occurs for $f \rightarrow f_0$.

Arrays of Josephson junctions can currently be fabricated in any desired geometry in one and two dimensions with well-controlled parameters. A Josephson-junction ladder provides the simplest one-dimension version of an array in a magnetic field. The latter system has attracted much attention in the recent years due to the possibility of different transitions as a function of the magnetic field, temperature, disorder, quantum fluctuations, and dissipation.[1-3] The behavior is strongly dependent on a dimensionless parameter $f=\Phi/\Phi_o$, the magnetic flux through an elementary cell in units of the flux quantum $\Phi_o=hc/2e$. Finite-temperature effects destroy phase coherence in a ladder since the system is one dimensional but one expects the zero-temperature phase diagram to show similarly interesting possible phase transitions. In this work we concentrate on the effects of quantum fluctuations and the magnetic field. We show that as a result of the competition between the periodicity of the vortex lattice introduced by the external field and the underlying pinning potential provided by the ladder, different phase transitions are possible as a function of the charging energy of the grains and the fields. For $f=p/q$ (p and q are relative primes) the resulting phase diagrams are strongly dependent on q.

We consider a periodic Josephson-junction ladder as indicated in Fig. 1. With each site **r** we associate a phase θ_r and charge $2en_r$ representing a superconducting grain which is coupled to its neighbors by Josephson couplings. n_r and θ_r are conjugate variables satisfying the commutation relation $[\theta,n]=i$. The interaction Hamiltonian is

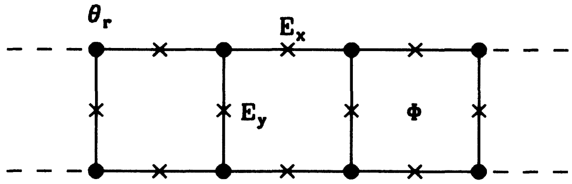

FIG. 1. Periodic Josephson-junction ladder. Josephson-junctions are indicated by crosses and superconducting grains by solid circles. Φ is the magnetic flux through an elementary cell.

given by a self-charging model[4,5]

$$H=-\frac{E_c}{2}\sum_r\left(\frac{\partial}{\partial\theta_r}\right)^2-\sum_{\langle rr'\rangle}E_{rr'}\cos(\theta_r-\theta_{r'}-A_{rr'}), \quad (1)$$

where the first term is the electrostatic energy of a grain $(2e)^2n^2/2C$, C is the capacitance, and $E_c=(2e)^2/C$, while the second term is the Josephson coupling energy. $A_{rr'}=(2\pi/\Phi_0)\int_r^{r'}\mathbf{A}dr$ and $\mathbf{A}$ is the vector potential due to the external field $\mathbf{B}$ and the gauge-invariant sum around a plaquette $\sum A_{rr'}=2\pi f$ with $f=\Phi/\Phi_o$. Since the Hamiltonian is periodic under $f\rightarrow f+n$ (n integer), $f=0$ is equivalent to $f=p/q$ with $q=1$.

Kardar[5,6] has shown the connection of this system with a discrete quantum sine-Gordon chain. As the magnetic field is increased from zero, a transition into a vortex state can occur where the magnetic flux first penetrates the ladder. This transition can be viewed as a commensurate-incommensurate transition described by the sine-Gordon Hamiltonian.[7,8] In the commensurate state the phases in different branches of the ladder are locked to each other while in the vortex state exponentially interacting kinks (vortices) appear that unlock the phases. Inclusion of charging effects leads to a normal phase in the vicinity of this transition and a direct commensurate-incommensurate transition is not possible. In the vortex state, however, the vortex lattice can become commensurate with the ladder at rational values of the flux quanta per plaquette $f_0=p/q$. The behavior of this commensurate phases as a function of small deviations from commensurability $f-f_0$ and charging energy has not been investigated so far and is studied in this work. We show that the topological features of the phase diagram for rational values are strongly dependent on q and are sketched in Fig. 2. In addition to a superconductor-insulator transition, direct commensurate-incommensurate transitions are also possible within the superconducting phase for $q\gtrsim 2$ and for $q \gtrsim 3$ an intermediate incommensurate phase appears when $f\rightarrow f_0$. The case $q=1$ corresponds to the result first obtained by Kardar[6] where vortices are absent in the commensurate phase.

The analysis which leads to the above conclusions is based on an effective free energy describing fluctuations from the commensurate state which can be derived from the model of Eq. (1). We start from a path-integral representation and introduce an auxiliary field $\xi(r,\tau)$ coupling linearly to $e^{i\theta(r,\tau)}$ via a Hubbard-Stratonovich transformation.[9] This yields a partition function $Z=Z_0\int D\xi \times\exp[-S(\xi)]$ with an effective Euclidean action

$$S=\frac{1}{2}\int d\tau\sum_r \xi^*(r,\tau)J_{rr'}\xi(r,\tau) -\left\langle \exp\left[\frac{1}{2}\int d\tau\sum_r\left(\sum_{r'}J_{rr'}\xi_{r'}\right)^* e^{i\theta_r}+\text{c.c.}\right]\right\rangle_0 , \qquad (2)$$

where the expectation value is taken with respect to

$$S_0=\frac{1}{2E_c}\sum_r\int d\tau\left[\frac{\partial}{\partial\tau}\theta(r,\tau)\right]^2 \qquad (3)$$

and $J_{rr'}=E_{rr'}\exp(-iA_{rr'})$. For convenience we have set $\hbar=1$ and $c=1$.

Now we choose a particular gauge in which **A** is parallel to the ladder taking opposite values in the upper and lower branches such that $A_{rr'}=-\pi f(x_r-x_{r'})$ and $A_{rr'}=\pi f(x_r-x_{r'})$, respectively, along these directions and $A_{rr'}=0$ in the vertical direction. Denoting by ξ_1 and ξ_2 the values of $\xi(r,\tau)$ in the lower and upper branches and performing a cumulant expansion in the second term of Eq. (2) we obtain

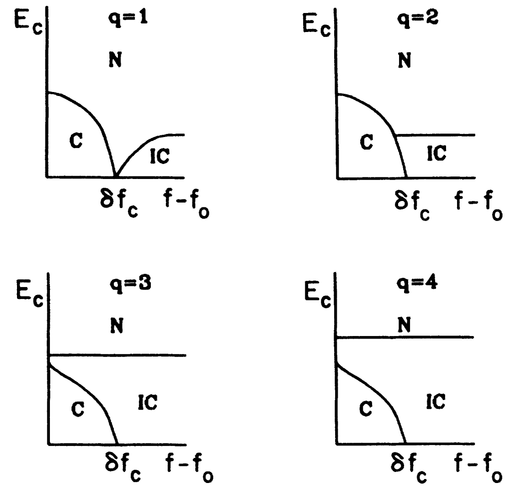

FIG. 2. Qualitative zero-temperature phase diagrams as a function of charging energy E_c and small deviations $f-f_0$ from a commensurate field $f_0=p/q$. *C*, IC, and *N* denote commensurate, incommensurate, and normal phases. For $q=3$, the intermediate phase at $f\to f_0$ may vanish when finite fugacities are considered. For $q=1$ vortices are absent in the *C* phase (Ref. 6).

$$S=\sum_q\int dw\left[\frac{1}{2}\sum_{\alpha,\beta}\xi_\alpha^*(k,w)\left(J(q)-\frac{2}{E_c}J^2(k)\right)_{\alpha\beta}\xi_\beta(k,w)+\frac{4}{E_c^3}\sum_{\alpha,\beta}w^2\xi_\alpha^*(k,w)J_{\alpha\beta}^2(k)\xi_\beta(k,w)\right] +\sum_r\sum_\alpha\int d\tau\sum_{n=2}^{\infty}u_{2n}|\xi_\alpha(r,\tau)|^{2n} , \qquad (4)$$

where $J_{11}(k)=2E_x\cos(k+\pi f)$; $J_{22}(k)=2E_x\cos(k-\pi f)$ and $J_{12}=J_{21}=E_y$. To obtain (4) we have neglected the space and time dependence of the higher-order cumulants leading to u_{2n}.

Following the standard procedure,[9] an effective free energy describing fluctuations from the commensurate state can now be obtained by expanding about the most fluctuating modes for each field. Assuming a commensurate state at $f_0=p/q$, these occur at $k_1=-\pi f_0$ and $k_2=\pi f_0$, respectively, for ξ_1 and ξ_2, where $J_{11}(k)$ and $J_{22}(k)$ reach a maximum. Note, however, that ξ_2 acts as an external field for ξ_1 via the off-diagonal coupling J_{12} and vice versa so we must also consider in addition $k_1'=\pi f_0$ and $k_2'=-\pi f_0$. When $q>1$ these wave vectors are not equivalent and must be retained in the expansion. Denoting by Ψ_1,Ψ_1' and Ψ_2,Ψ_2' the corresponding modes and neglecting for the momentum space and time fluctuations we get $F=\int dx\int d\tau f[\Psi_1,\Psi_2,\Psi_1',\Psi_2']$ where

$$f=\frac{1}{2}r(|\Psi_1|^2+|\Psi_2|^2)+\frac{r'}{2}(|\Psi_1'|^2+|\Psi_2'|^2)-2h[\text{Re}(\Psi_1'^*\Psi_2)+\text{Re}(\Psi_1^*\Psi_2')]+u_4(|\Psi_1|^4+|\Psi_2|^4+|\Psi_1'|^4+|\Psi_2'|^4) +4u_4(|\Psi_1|^2|\Psi_1'|^2+|\Psi_1|^2|\Psi_2'|^2)+2u_{2q}\text{Re}(\Psi_1^*\Psi_1')^q+2u_{2q}\text{Re}(\Psi_2^*\Psi_2')^q , \qquad (5)$$

where we have considered the terms coupling the amplitudes only up to fourth order and retained the lowest-order term coupling the phases. This coupling arises from the u_{2n} term in (4) since the Fourier components $\Psi^*(k_i),\Psi(k_i')$ are restricted as usual by $\sum_{i=1}^n k_i-\sum_{i=1}^n k_i'=0\bmod(2\pi)$. If $k_i=\pi f_0$, $k_i'=-\pi f_0$, the lowest n satisfying this restriction is $n=q$. Minimizing with respect to Ψ_1',Ψ_2' and substituting back into Eq. (5) yields an effective free energy only in terms of Ψ_1,Ψ_2 with renormalized coupling constants and a coupling between the phases of the form $\text{Re}(\Psi_1^*,\Psi_2)^q$. Since in two dimensions amplitude fluctuations are irrelevant to the critical behavior, after allowing a small deviation of the external field $\delta f=f-f_0$, we finally obtain an effective free energy in the form

$$F=\int dx\int d\tau\left\{\frac{1}{2}K_x\left[\left(\frac{\partial}{\partial x}\theta_1+\pi\delta f\right)^2+\left(\frac{\partial}{\partial x}\theta_2-\pi\delta f\right)^2\right]+\frac{1}{2}K_\tau\left[\left(\frac{\partial}{\partial\tau}\theta_1\right)^2+\left(\frac{\partial}{\partial\tau}\theta_2\right)^2\right]-w\cos q(\theta_1-\theta_2)\right\} , \qquad (6)$$

where θ_1, θ_2 are the phases of Ψ_1, Ψ_2 and $K = K_x K_\tau \approx E_x/E_c$ and w are effective couplings. As the coupling constant w arises from higher-order terms it is a rapidly decreasing function of q. The precise relation of these couplings to the original ones is of no concern here since, as will be shown, the topology of the phase diagram is independent of them but depends strongly on q. Note that θ_1, θ_2 now measure phase deviations of each order parameter from the commensurate phase and should not be identified with the phases of the superconducting grains in Eq. (1).

When $\delta f = 0$, Eq. (6) is in the form of a Gaussian approximation of coupled classical XY models which has been studied previously with E_c playing the role of an effective temperature.[10,11] When the vortices in θ_1, θ_2 are included different behavior occurs as a function of q. If $q = 1$ a single XY-model-like transition occurs as E_c increases separating a commensurate (superconducting) phase with long-range order in $\theta_1 - \theta_2$ and algebraic order in θ_1, θ_2, from a disordered (insulating) phase where correlations decay exponentially. For $q = 2$, the nature of the transition is still unknown but a recent study indicates it could be either nonuniversal or first order if it is a single transition.[12] Interestingly enough, the same effective free energy with $q = 2$ is also believed to describe the finite-temperature transition in a two-dimensional array of junctions precisely at the same external field.[10–14] For $q > \sqrt{8}$, an intermediate (incommensurate) phase is possible with algebraic order in all correlations and, therefore, superconducting.

To investigate the effect of δf, we follow Kardar[5] and perform a change of variables in (6) $\phi = \theta_1 - \theta_2$; $\psi = \theta_1 + \theta_2$ which leads to

$$F = \int dx \int d\tau \left[\tfrac{1}{4} K_\tau \left(\frac{\partial}{\partial \tau} \psi \right)^2 + \tfrac{1}{4} K_x \left(\frac{\partial}{\partial x} \psi \right)^2 + \tfrac{1}{4} K_\tau \left(\frac{\partial}{\partial \tau} \phi \right)^2 + \tfrac{1}{4} K_x \left(\frac{\partial}{\partial x} \phi - 2\pi \delta f \right)^2 - w \cos q\phi \right], \qquad (7)$$

which now have decoupled into a Gaussian in ψ and a sine-Gordon in ϕ which describes the commensurate-incommensurate transition. For small δf the phase difference ϕ is zero and the vortex lattice is commensurate with the ladder. Above a critical value $\delta f_c \approx \sqrt{w}$ kinks appear separating commensurate domains and the lattice is incommensurate. The stable region decreases with increasing charging energy E_c and vanishes at a critical value. Expressing the correlation functions of the original variables in terms of ψ and ϕ gives $\langle \exp[-i(\theta_r - \theta_0)] \rangle = r^{-\eta}$ where $\eta = (\eta_G + \eta_{SG})/4$ and $\eta_G = 1/\pi\sqrt{K_x K_\tau}$ is the correlation function exponent of the Gaussian part and η_{SG} of the sine-Gordon part. For η_{SG}, we can use the known results of the sine-Gordon model:[5,6] $\eta_{SG} = 0$ for $\delta f \ll \delta f_c(K_x, K_\tau)$; $\eta_{SG} = 1/\pi\sqrt{K_x K_\tau}$ in the incommensurate phase and right at the commensurate-incommensurate transition $\eta_{SG} = 2/q^2$; when $\delta f \to 0$, $\eta_{SG} \to 4/q^2$ at $\pi\sqrt{K_x K_\tau} = q^2/4$. Guided by the $\delta f = 0$ case discussed above, we require $\eta < \frac{1}{4}$ in order for the ordered phase to be stable against vortex-pair unbinding in the phases θ_1 and θ_2 which will result in exponentially decaying correlation functions. For different values of q this leads to the phase diagrams indicated in Fig. 2. Note that we have used the limit of very weak vortex fugacity which amounts to take the values of η_G and η_{SG} in the absence of these excitations. Allowing finite fugacity can shrink to zero the intermediate phase at $f \to f_0$ for $q = 3$, as a more careful renormalization-group analysis indicates.[10] However, this should persist for $q \geq 4$.

In conclusion, we have studied the phase diagram of a periodic Josephson-junction ladder in a perpendicular magnetic field. The topology of the phase diagram is shown to be strongly dependent on q for $f_0 = p/q$, displaying direct vortex commensurate-incommensurate transitions in addition to superconductor-insulator transitions. For $q > 3$, an intermediate incommensurate phase is also possible when $f \to f_0$. In this work we have been mainly concerned with the global features of the phase diagram and have not studied the critical behavior in detail. Although the analysis carried out here is only strictly valid at zero temperature, we expect similar algebraic decrease of resistance in temperature $R \approx T^\lambda$ as found in chains of Josephson junctions.[15] the exponent λ may attain universal values at the transitions. Also anomalous behavior is expected in the amplitude of the δ-function singularity in the frequency-dependent conductivity. Understanding these effects is important for experiments in the system as they may provide a signature of the transitions. Certainly, more work along these lines is necessary to provide more quantitative information on these effects.

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