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Hidden symmetries in the one-dimensional antiferromagnetic Heisenberg model

K. Fabricius and K.-H. Mütter
Fachbereich Physik, Bergische Universität Wuppertal, Postfach 10 0127, 5600 Wuppertal 1, West Germany

H. Grosse
Institut für Theoretische Physik, Universität Wien, Wien, Austria

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Nontrivial conservation laws of the antiferromagnetic Heisenberg (AFH) model for periodic rings are used for an efficient computation of the energy eigenstates in the *SP*0 sector, a sector with zero total spin and momentum. The appearance of twofold-degenerate eigenvalues is consequence of conserved operators, which are antisymmetric under reflection.

I. INTRODUCTION

Antiferromagnetic ordering in two-dimensional planes has been observed in La_2CuO_4,[1] which is one of the new high-T_c superconductors.[2] It has been suggested[3] that for undoped materials such a behavior can be described by the two-dimensional antiferromagnetic Heisenberg (AFH) model. This is one of the most simple but nontrivial quantum statistical models, whose thermodynamical properties are of principal interest in any case.

An exact computation of the ground state of the AFH model has been performed first for small lattices (up to 16 sites) by Oitma and Betts.[4] Using a modified Lanczos algorithm, the authors of Ref. 5 were able to enlarge the size of the lattice to 24 sites. These results confirm antiferromagnetic ordering in the ground state. Low-lying energy eigenvalues have been computed for larger systems (up to 64 sites) employing Monte Carlo techniques.[6] Of course, these results carry statistical errors.

For a systematic study of the AFH model, it would be desirable to know the distribution of the energy eigenvalues, their degeneracies, and their dependence on the number of lattice sites. In particular, degenerate energy eigenvalues might help to find the "hidden" symmetries of the model. So far, nothing is known about hidden conservation laws of the AFH model in two and more dimensions. A whole family of conservation laws was shown to exist for the one-dimensional *XYZ* model by Baxter.[7] This important result could be generalized to a wide class of one-dimensional quantum systems by means of the quantum inverse-scattering method.[8] A new type of conservation law that does not follow from this method has recently been discovered in the one-dimensional Hubbard model.[9] In this paper we want to demonstrate how to use nontrivial conservation laws for an efficient computation of the AFH energy eigenstates.

In Sec. II, we review the "obvious" conservation laws of the AFH model, which hold in any dimension, as there are conservation of the total spin **S**, the momentum *P*, and the reflection symmetry *R*. It is convenient to divide the Hilbert space of the energy eigenstates into sectors labeled by the quantum numbers s, s_3, and p for $\mathbf{S}^2$, S_3, and *P*, respectively. The ground state is supposed to be in the *SP*0 sector of states with zero total spin and zero respective π momentum. In this paper, we will be concerned only with this sector.

In Sec. III, we present the explicit form of the higher conservation laws,[10] as they can be found in the one-dimensional AFH model with periodic boundary conditions. Most important for the construction of the *SP*0 sector are operators that commute with the Hamiltonian but anticommute with the reflection operator *R*. Such operators allow for a further separation of the *SP*0 sector into two subsectors, which we denote by *SP*01 and *SP*02: *SP*01 contains only nondegenerate energy eigenstates, which are symmetric under the reflection operator *R*. *SP*02 contains twofold-degenerate energy eigenstates. One of these states is symmetric, the other one is antisymmetric under *R*. The *SP*02 sector appears first for periodic rings with $N=12$ sites.

In 1938, Hulthén[11] had already computed the eigenvalue spectrum of the *SP*0 sector for rings when $N=4,6,8,10$. If he had gone one step further to $N=12$, he would have found the degeneracy of reflection symmetric and antisymmetric states—a first hint of the existence of higher conservation laws.

In Sec. IV, we construct an appropriate basis for total-spin-zero states, following a procedure first introduced by Hulthén.[11] We describe how the AFH Hamiltonian and the conserved operators act on these states.

In Secs. V and VI we diagonalize the AFH Hamiltonian in the *SP*01 and *SP*02 sectors, respectively.

In Sec. VII, we present the expectation values of the spin-spin correlators for the lower eigenstates of the AFH Hamiltonian.

II. THE OBVIOUS SYMMETRIES IN THE AFH MODEL

The isotropic AFH Hamiltonian is defined by

$$H=\tfrac{1}{2}\sum_{\langle x,y\rangle}(x,y)\ . \tag{2.1}$$

The sum extends over the spin-$\frac{1}{2}$-coupling operators,

$$(x,y)=(\sigma(x)\sigma(y))\ , \tag{2.2}$$

between nearest-neighbor points $\langle x,y\rangle$. For periodic boundary conditions, H is invariant under translations, rotations, and reflections on the lattice. Moreover, the total spin,

$$\mathbf{S}=\sum_x \mathbf{S}(x),\quad [H,\mathbf{S}]=0\ , \tag{2.3}$$

is conserved as well. $H,\mathbf{S}^2,S_3$ and the momentum operators P as generators of the translations can be diagonalized simultanuously. The corresponding set of eigenstates is labeled by

$$|E,s,s_3,p,\lambda\rangle\ , \tag{2.4}$$

where λ stands for further possible degeneracies.

For a periodic ring with an even-number N of sites the reflection operator R,

$$R\,\mathbf{S}(N+1-x)R=\mathbf{S}(x),\quad x=1,2,\ldots,N\ , \tag{2.5}$$

commutes with H, $\mathbf{S}^2$, and S_3, but anticommutes with P:

$$[R,H]=[R,\mathbf{S}]=0,\quad \{R,P\}=0\ . \tag{2.6}$$

Therefore, the reflection operator transforms states with momentum p into states with momentum $-p$. The energy eigenvalues corresponding to these states are degenerate. In the subspace of states with momentum $p=0,\pi$, we can diagonalize the reflection operator R as well:

$$R\,|E,s,s_3,\ldots,p=0,\pi,r,\lambda\rangle=r\,|E,s,s_3,\ldots,p=0,\pi,r,\lambda\rangle\ . \tag{2.7}$$

III. HIDDEN SYMMETRIES IN THE ONE-DIMENSIONAL AFH MODEL

It was first observed by Baxter,[7] that the traces of the transfer matrix $T_N(\lambda)$ in the one-dimensional XYZ model commute for different values of the spectral parameter λ. Expanding the logarithm of $T_N(\lambda)$ around $\lambda=i/2$ yields "local" conservation laws,[10] where only neighboring spins are coupled. In case of the one-dimensional AFH model, the lowest terms have the following form:

$$F_3=\sum_x \epsilon(x,x+1,x+2)\ , \tag{3.1}$$

$$F_4=\sum_x[(x,x+3)(x+1,x+2)-(x,x+2)(x+1,x+3)-(x,x+2)]\ . \tag{3.2}$$

In the definition of F_4 and F_3, we introduced the abbreviations (2.2) and

$$\epsilon(x_1,x_2,x_3)=\epsilon_{abc}S_a(x_1)S_b(x_2)S_c(x_3)\ . \tag{3.3}$$

If we expand $T_N(\lambda)$ in a power series of λ we are led to a set of nonlocal translation-invariant operators G_k, which commute with the AFH Hamiltonian (2.1) and the total-spin operator $\mathbf{S}$ [Eq. (2.3)]:

$$[G_k,H]=[G_k,\mathbf{S}]=0\ . \tag{3.4}$$

G_k is constructed from the spin operators $S_a(x)$ on an ordered set of sites on the ring $x_1<x_2<x_3\cdots<x_k$:

$$G_k=\sum_{x_1<x_2\cdots<x_k} S_{a_1}(x_1)\cdots S_{a_k}(x_k)T_{a_1\cdots a_k}\ , \tag{3.5}$$

where

$$T_{a_1\cdots a_k}=\mathrm{tr}\,\sigma_{a_1}\cdots\sigma_{a_k}\ . \tag{3.6}$$

A derivation of these conservation laws is found in the Appendix.

The operator G_2 is related to the total spin $\mathbf{S}^2$. The operator G_3,

$$G_3=i\sum_{x_1<x_2<x_3}\epsilon(x_1,x_2,x_3)\ , \tag{3.7}$$

looks like a nonlocal version of F_3 [Eq. (3.1)].

F_3 and G_3 anticommute with the reflection operator R:

$$\{R,F_3\}=\{R,G_3\}=0\ . \tag{3.8}$$

This property of F_3 and G_3 leads to certain degeneracies of energy eigenvalues. To see this, let us consider the zero-momentum eigenstates of H and R:

$$|E,s,s_3,p=0,r\rangle\ . \tag{3.9}$$

The commutation and anticommutation rules of F_3 (G_3) allow for two possibilities.

(a) F_3 (G_3) transforms reflection-symmetric eigenstates $(r=1)$ of H into antisymmetric states $(r=-1)$ and vice versa. The corresponding eigenvalues of H are degenerate.

(b) F_3 (G_3) annihilates reflection-symmetric (antisymmetric) eigenstates of H. Of course in this case there is no degeneracy of eigenvalues due to reflection symmetry.

The ground state of the Heisenberg Hamiltonian is supposed to have zero total spin, zero (π) momentum, and to be symmetric (antisymmetric) under reflection for $N=4n$, $n=1,2,3,\ldots$ $(N=2n,\ n=1,3,5,\ldots)$. If a zero-momentum state, as the ground state, is nondegenerate, it will be annihilated by the operators F_3 and G_3.

This property allows us to separate the $SP0$ sector of the zero-spin and zero-momentum states into the subsectors $SP01$ and $SP02$ of nondegenerate and twofold-degenerate energy eigenstates. It will play an important role in our approach for constructing the ground state of the AFH Hamiltonian.

IV. SPIN-ZERO STATES

In this section, we will construct an appropriate basis in the sector with zero total spin $\mathbf{S}=0$.[11] This construction can be performed on hypercubical lattices in any dimension.

We first associate to each site x on the odd lattice L_-,

$$(-1)^x=-1\quad\text{for } x\in L_-\ , \tag{4.1}$$

in a unique way a site y on the even lattice L_+,

$$(-1)^y=+1\quad\text{for } y\in L_+\ . \tag{4.2}$$

Each of these one-to-one mappings,

$$M:\ x\in L_-\rightarrow y\in L_+\ , \tag{4.3}$$

defines a state of total spin zero,

$$\left|\begin{matrix} x_1 & x_2 & x_3 & \cdots \\ y_1 & y_2 & y_3 & \cdots \end{matrix}\right\rangle = [x_1,y_1][x_2,y_2]\cdots , \tag{4.4}$$

if we couple the spins $n(x),n(y)$ of all pairs $x,y(x)$ to zero,

$$[x,y]=1/\sqrt{2}[\,|n(x)=1,n(y)=-1\rangle - |n(x)=-1,n(y)=1\rangle\,] .$$

Anderson's resonating-valence-bond (RVB) state[12] is a superposition of such states, where the pairing is restricted to nearest-neighbor points $\langle x,y\rangle$. On a ring with an even number N of sites, there are just two states with nearest-neighbor pairings:

$$|a\rangle = \left|\begin{matrix} 1 & 3 & 5 & \cdots \\ 2 & 4 & 6 & \cdots \end{matrix}\right\rangle, \quad |b\rangle = \left|\begin{matrix} 1 & 3 & 5 & \cdots \\ N & 2 & 4 & \cdots \end{matrix}\right\rangle . \tag{4.5}$$

The basis of spin-zero states, defined in Eq. (4.4), is not orthogonal and linear independent.

Note, that two spin-zero couplings between the odd sites $x_1,x_2 \in L_-$ and the even sites $y_1,y_2 \in L_+$ can be expressed in terms of the basis (4.4):

$$[x_1,x_2][y_1,y_2]=[x_1,y_1][x_2,y_2]-[x_1,y_2][x_2,y_1] . \tag{4.6}$$

To see, how the Heisenberg Hamiltonian acts, we start with the operation of the spin-coupling operators (x,y) [cf. Eq. (2.2)] on states with all spins quantized in z direction:

$$(x,y)|\ldots,n(x),\ldots,n(y),\ldots\rangle = [-1+2P(x,y)]|,\ldots,n(x),\ldots,n(y),\ldots\rangle . \tag{4.7}$$

$P(x,y)$ interchanges spins $n(x)$ and $n(y)$ at sites x and y. The operator,

$$O(x,y)=1-P(x,y) , \tag{4.8}$$

acts on the total-spin-zero state as follows:

$$\begin{aligned} &O(x,y)[x,y]=2[x,y] , \\ &O(x_1,y_2)[x_1,y_1][x_2,y_2]=[x_1,y_2][x_2,y_1] . \end{aligned} \tag{4.9}$$

The conserved quantities F_3 and G_3 on periodic rings [Eqs. (3.1) and (3.6)] are built from operators of the type (3.2). Their action on the spin-zero states is found to be

$$\begin{aligned} &4\epsilon(x_1,x_2,x_3)[x_1,y_1][x_2,y_2][x_3,y_3]=[x_1,y_3][x_2,x_3][y_1,y_2]-[x_1,x_2][x_3,y_1][y_2,y_3] , \\ &4\epsilon(x_1,x_2,x_3)[x_1,x_2][x_3,y_3]=-[x_2,x_3][x_1,y_3]-[x_1,x_3][x_2,y_3] . \end{aligned} \tag{4.10}$$

V. THE *SP*01 SECTOR OF NONDEGENERATE SPIN AND MOMENTUM-ZERO STATES

In this section we will diagonalize the AFH Hamiltonian for periodic rings with $N=4n$ sites in the *SP*01 sector. The latter contains the ground state.

In terms of the nearest-neighbor operators $O(x,x+1)$, defined in Eq. (4.8), the Hamiltonian reads

$$H=\tfrac{1}{2}(N-2O), \quad O=\sum_{x=1}^{N} O(x,x+1) . \tag{5.1}$$

We start from an initial state $|1\rangle$ with zero total spin and zero momentum. This initial state is symmetric under reflection and is annihilated by the operators F_3 and G_3 [Eq. (3.1) and (3.6)]. The one-dimensional analog of the RVB state [the sum of the two states in Eq. (4.5)] is an example for such a state:

$$|1\rangle=|a\rangle+|b\rangle . \tag{5.2}$$

Application of the operator O on the initial state $|1\rangle$ creates a new state $|2\rangle$,

$$|2\rangle=\sum_{x=2,4,\ldots} P(x,x+2)|1\rangle , \tag{5.3}$$

which is obtained from $|1\rangle$ by the permutations $P(x,x+2)$ of even sites.

A graphic representation of these states, known in the literature as "valence-bond graphs,"[13] is quite instructive. We connect each pair of even-odd sites with a spin-zero coupling using a straight line. Then the initial state (5.2) only contains nearest-neighbor spin-zero couplings, as shown in Fig. 1(a). The second state (5.3) can be seen in Fig. 1(b). It contains one new "spin-coupling

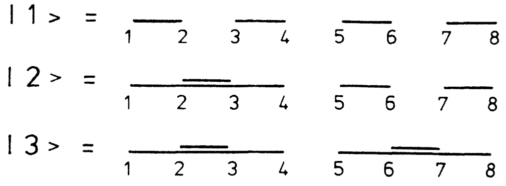

FIG. 1. Graphical representation of spin-zero couplings in the three states of Eq. (5.4) for $N=8$.

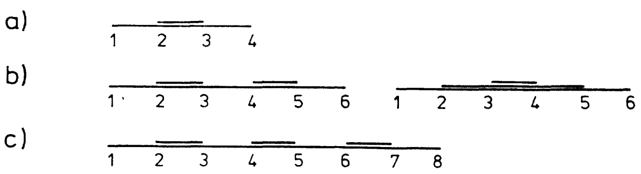

FIG. 2. Spin-zero couplings, as they appear in the new states generated by successive application of the operator O on the initial state (5.2).

TABLE I. Energy eigenvalues (divided by the number of sites N) in the $SP01$ sector for a periodic ring with $N=16$. The corresponding eigenstates are nondegenerate and symmetric under the reflection operator R.

−0.892 787 0	−0.427 478 9	−0.188 502 1	−0.032 405 0	0.124 910 2
−0.698 911 3	−0.351 035 7	−0.181 594 6	0.008 665 6	0.209 418 2
−0.601 780 5	−0.347 946 1	−0.132 980 8	0.009 193 4	0.216 838 3
−0.570 723 5	−0.292 474 5	−0.129 844 6	0.027 506 8	0.246 919 1
−0.496 243 7	−0.281 594 4	−0.108 335 0	0.092 624 6	0.327 402 7
−0.486 390 6	−0.233 353 3	−0.080 928 9	0.109 061 3	0.360 702 9
−0.439 402 8	−0.211 379 4	−0.061 323 3	0.119 779 8	0.394 393 0

configuration."

We now proceed and apply O successively to $|2\rangle$ and all new states. In this way more and more complicated coupling configurations are created. Some of them are shown in Fig. 2.

For illustration, let us consider a ring with eight sites, which has been treated already by Hulthén.[11] In this case, there are only three linear-independent states, which span the $SP01$ sector:

$$\begin{aligned}|1\rangle &= \left|\begin{matrix}1&3&5&7\\2&4&6&8\end{matrix}\right\rangle + \left|\begin{matrix}1&3&5&7\\8&2&4&6\end{matrix}\right\rangle ,\\ |2\rangle &= \left|\begin{matrix}1&3&5&7\\4&2&6&8\end{matrix}\right\rangle + \text{translations} ,\\ |3\rangle &= \left|\begin{matrix}1&3&5&7\\4&2&8&6\end{matrix}\right\rangle + \text{translations} .\end{aligned} \tag{5.4}$$

Following the rules of Eq. (4.9), the action of O on these three states is found to be

$$\begin{aligned}O|1\rangle &= 8|2\rangle + |2\rangle ,\\ O|2\rangle &= 8|1\rangle + 8|2\rangle + 2|3\rangle ,\\ O|3\rangle &= 4|1\rangle + 2|2\rangle + 4|3\rangle .\end{aligned} \tag{5.5}$$

Eigenvalues and right-handed eigenstates of O in the $SP01$ sector turn out to be

$$\begin{aligned}&O|\lambda\rangle = \lambda|\lambda\rangle ,\\ &\lambda_1 = 11.3022,\ |\lambda_1\rangle = |1\rangle + 0.363|2\rangle + 0.099|3\rangle ,\\ &\lambda_2 = 5.452\,22,\ |\lambda_2\rangle = |1\rangle - 0.189|2\rangle - 0.26|3\rangle ,\\ &\lambda_3 = 3.245\,59,\ |\lambda_3\rangle = |1\rangle + 1.826|2\rangle - 4.84|3\rangle .\end{aligned} \tag{5.6}$$

According to Eq. (5.1), the largest eigenvalue λ is related to the ground-state energy per site:

$$E_0/8 = 1/2(1 - 1/4\lambda_1) = -0.912\,775 . \tag{5.7}$$

Note, that the corresponding ground-state vector is not a pure RVB-like state with only nearest-neighbor spin-zero couplings. The more complicated spin-zero-coupling configurations, as they appear in the states $|2\rangle$ and $|3\rangle$ [cf. Figs. 1(a)–1(c)], contribute as well.

TABLE II. The dimensions of the sectors $SP01$ and $SP02$ and the number of states with zero momentum and zero total spin in the z direction for periodic rings with $N=8,\ldots,24$. The numbers marked with an asterix were computed by means of the approximate formula of Ref. 14.

N	8	12	16	20	24
$D(SP01)$	3	10	35	126	462
$D(SP02)$	0	4	60	728	8252
$D(s_3=0,\ p=0)$	7	44	415	4620*	56 366*

In Table I, we present the energy eigenvalues of the AFH Hamiltonian in the $SP01$ sector for a periodic ring with $N=16$ sites. They are obtained by diagonalizing a matrix of dimension $D(SP01)$, which is just the dimension of the $SP01$ sector. It is quite instructive to see how $D(SP01)$ increases with the number of sites N, which can be seen from Table II. This number tells us what effort is needed to compute the ground state for large rings. For comparison, we also quote the dimension $D(SP02)$ of the second sector of twofold-degenerate energy eigenstates with zero total spin and momentum, which will be investigated in the next section. As explained at the end of Sec. III, one of these states is symmetric, the other one antisymmetric under the reflection operator R. Therefore, $D(SP02)$ is just twice the number of states with zero spin and momentum, which are antisymmetric under R, whereas $D(SP01)$ is the difference of the numbers of symmetric and antisymmetric states. We get these numbers by simply counting the R symmetric and antisymmetric valence-bond graphs (cf. Fig. 1 for $N=8$ and Fig. 3 for $N=12$). Note, that for large N, $D(SP02)$ increases approximately as $O(2^N)$, whereas the increase of $D(SP01)$ seems to be much slower:

$$D(SP01) = O(2^{N/2}) .$$

Previous computations of the ground state[14] were per-

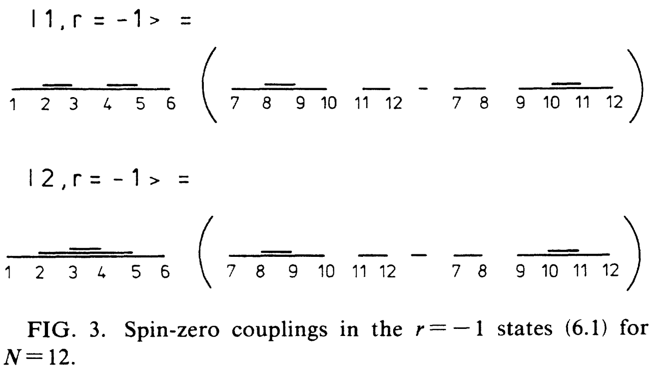

FIG. 3. Spin-zero couplings in the $r=-1$ states (6.1) for $N=12$.

formed in the space of states with zero total spin in the z direction and zero momentum. The dimension of this space is given in the fourth line of Table II.

Finally, it should be mentioned that the sectors with zero total spin, but nonvanishing momentum, can be treated in a completely analogous way. The only difference is that we start from an initial state, which is a momentum eigenstate with $p=2\pi n/N$, $n=0,1,2,\ldots,N-1$.

VI. THE *SP*02 SECTOR OF DEGENERATE SPIN AND MOMENTUM-ZERO STATES

For periodic rings with $N=4n$ sites, states with zero momentum and total spin, which are antisymmetric under the reflection operator R [Eq. (2.5)], appear first for $N=12$. In this case, we have two such states, which look in the graphic representation of the preceding section as shown in Fig. 3:

$$\begin{aligned} |1,r=-1\rangle &= \left|\begin{matrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 6 & 2 & 4 & 10 & 8 & 12 \end{matrix}\right\rangle - \left|\begin{matrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 6 & 2 & 4 & 8 & 12 & 10 \end{matrix}\right\rangle + \text{translations} , \\ |2,r=-1\rangle &= \left|\begin{matrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 6 & 4 & 2 & 10 & 8 & 12 \end{matrix}\right\rangle - \left|\begin{matrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 6 & 4 & 2 & 8 & 12 & 10 \end{matrix}\right\rangle + \text{translations} . \end{aligned} \tag{6.1}$$

The action of the operator O [Eq. (5.1)] on these states is found to be

$$\begin{aligned} O|1,r=-1\rangle &= 8|1,r=-1\rangle + |2,r=-1\rangle , \\ O|2,r=-1\rangle &= 2|1,r=-1\rangle + 5|2,r=-1\rangle . \end{aligned} \tag{6.2}$$

Eigenvalues and eigenvectors of O in this two-dimensional subspace turn out to be

$$\begin{aligned} &O|\lambda\rangle = \lambda|\lambda\rangle , \\ &\lambda_1 = 1/2(13+\sqrt{17}),\ |\lambda_1\rangle = |1,r=-1\rangle + \tfrac{1}{4}(\sqrt{17}-3)|2,r=-1\rangle , \\ &\lambda_2 = \tfrac{1}{2}(13-\sqrt{17}),\ |\lambda_2\rangle = |1,r=-1\rangle - \tfrac{1}{4}(\sqrt{17}+3)|2,r=-1\rangle . \end{aligned} \tag{6.3}$$

The operator F_3 transforms these eigenstates of the AFH Hamiltonian into states

$$F_3:\ |\lambda_j, r=-1\rangle \to |\lambda_j, r=1\rangle,\ j=1,2 , \tag{6.4}$$

which are symmetric under the reflection operator R. This means that the energy eigenvalues,

$$E_j = 6-\lambda_j , \tag{6.5}$$

are twofold degenerate, one eigenstate is symmetric and the other one antisymmetric. In Table III we present the energy eigenvalues in the *SP*02 sector for a periodic ring with 16 sites. The connection (6.4) between symmetric and antisymmetric eigenstates has been verified explicitly.

VII. SPIN-SPIN CORRELATIONS IN THE *SP*0 SECTOR

The approach outlined in Secs. V and VI provides us with the complete information on all the energy eigenstates in the sectors *SP*01 and *SP*02. The expectation values of the spin-spin correlators,

$$C(x) = \sum_{y=1}^{N} S_3(y) S_3(y+x) , \tag{7.1}$$

in the lower-energy eigenstates are of special interest.

In Fig. 4, we compare the correlators in the lowest- and highest-energy eigenstates of the *SP*01 and *SP*02 sectors. Note, that the correlators for degenerate energy eigenstates are identical for separations $x=2$. This feature is not yet understood. For states with zero total spin, there exists an obvious sum rule for spin-spin correlators, which has been checked explicitly.

VIII. CONCLUSIONS AND PERSPECTIVES

In this paper we have presented a new approach for computing the energy eigenstates of the one-dimensional

TABLE III. Energy eigenvalues (divided by the number of sites) in the *SP*02 sector for $N=16$. The corresponding eigenstates are twofold degenerate; one state is symmetric, and the other one antisymmetric under R.

−0.456 719 1	−0.239 107 9	−0.129 040 3	−0.035 953 6	0.077 254 2
−0.401 616 5	−0.202 254 2	−0.100 098 3	0.001 207 3	0.136 470 6
−0.395 658 1	−0.189 925 3	−0.077 596 6	0.039 863 5	0.164 919 8
−0.345 374 2	−0.151 483 6	−0.059 064 4	0.044 498 6	0.220 455 9
−0.277 012 1	−0.147 401 7	−0.050 855 1	0.053 952 2	0.220 589 5
−0.276 459 1	−0.135 091 6	−0.037 049 5	0.056 041 3	0.317 508 1

AFH model. So far, we have only exploited the sectors *SP*01 and *SP*02 of nondegenerate and twofold-degenerate energy eigenstates with zero total spin and momentum for periodic rings with $N=8$, 12, and 16 sites. Extensions to larger rings and to the total-spin-zero sectors with nonvanishing momentum are straightforward.

Our approach is based on three ingredients: (1) the "obvious" symmetries of Sec. II, (2) the hidden conservation laws (3.1) and (3.6) for the operators F_3 and G_3, and (3) the fact, that zero-momentum states with spin-zero couplings between nearest-neighbor sites ("RVB"-like states) are annihilated by the conserved quantities F_3 and G_3. It is just this property that reduces the dimension of the *SP*01 sector.

The obvious conservation laws hold in two and more dimensions as well. So far the approach is applicable in these cases too. However, we do not yet know, whether two-dimensional analogs of the conserved operators F_3

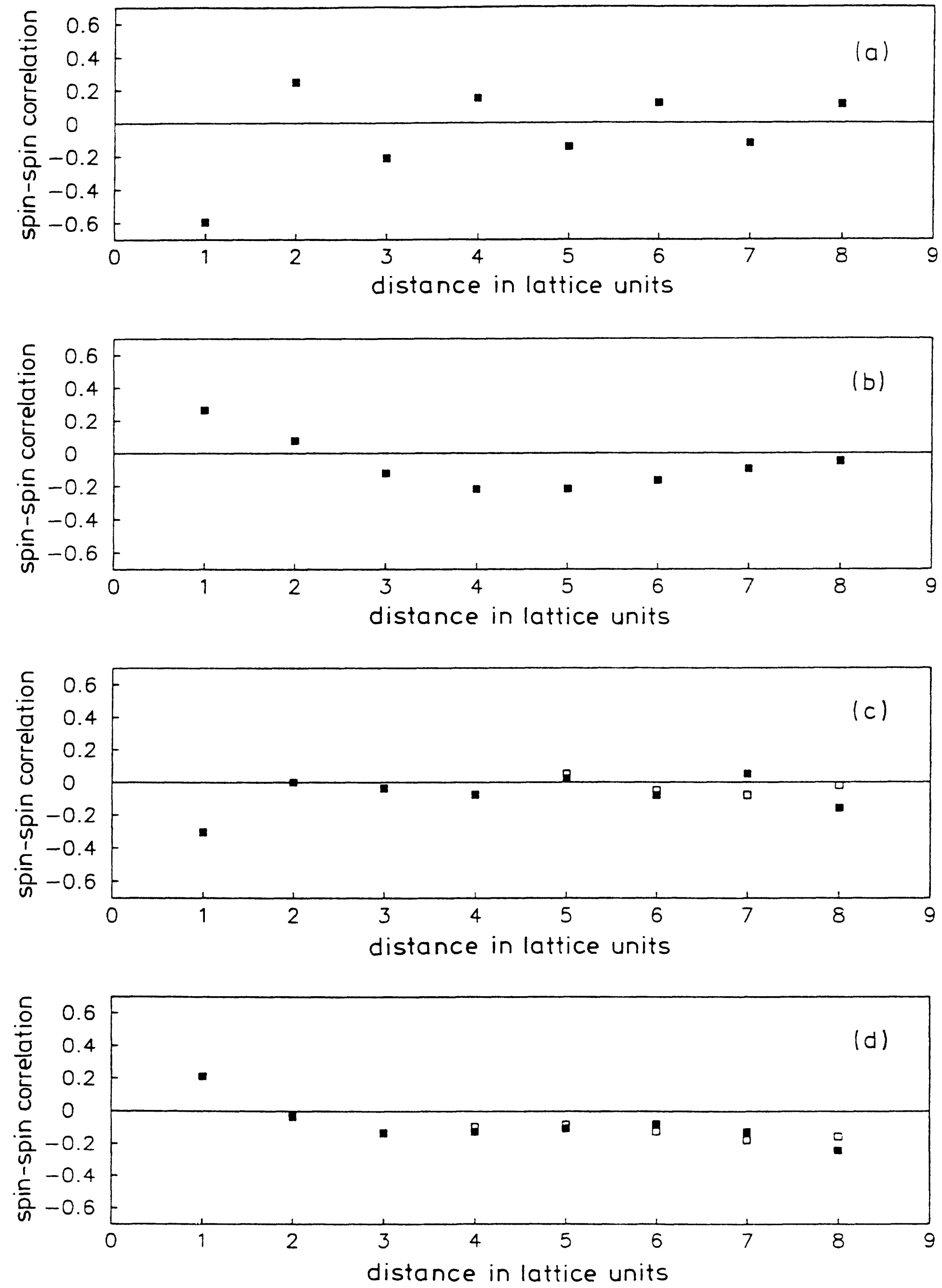

FIG. 4. Spin-spin correlators vs distance in lattice units in the lowest- and highest-energy eigenstates of the *SP*01 and *SP*02 sectors. Open and solid squares in parts (c) and (d) denote correlators in *R*-symmetric and -antisymmetric states, respectively. $E/N=-0.892\,787$, $0.394\,393$, $-0.456\,719$, and $0.317\,508$ for (a), (b), (c), and (d), respectively.

and G_3 do exist. One can attack this problem analytically (searching for conserved quantities by trial and error) and numerically by computing the energy-eigenvalue spectrum on small systems with high precision and looking for "nontrivial" degeneracies. The system should not be too small, since degeneracies might appear only beyond a certain size. Indeed, that is what happened for periodic rings with $N=4n$ sites: Twofold degenerate eigenstates in the $SP0$ sector appeared first for $N=12$.

APPENDIX: PROOF OF THE HIGHER CONSERVATION LAWS

We follow standard inverse-scattering transform methods[8] and start with the transfer matrix $T_N(\lambda)$, which is defined by the ordered product,

$$T_N(\lambda)=L_1(\lambda)L_2(\lambda)\cdots L_N(\lambda)\ , \tag{A1}$$

where the operators $L_N(\lambda)$ act in $(C^2)^N C^2$ and are given by

$$L_n(\lambda)=\lambda 1+i1(1)\otimes\cdots 1(n-1)\otimes S_a(n)\sigma_a\otimes 1(n+1)\cdots 1(N)\ . \tag{A2}$$

$1(x)$, $S_a(x)$, and $x=1,2,\ldots,n,\ldots,N$ are 2×2 unit and spin matrices at the sites x. The Pauli matrices σ_a act in a two-dimensional subsidiary space. The operators L_n and T_N obey the following commutation rules:

$$\begin{aligned}R(\lambda-\mu)L_n(\lambda)\otimes L_n(\mu)&=L_n(\mu)\otimes L_n(\lambda)R(\lambda-\mu)\ ,\\ R(\lambda-\mu)T_N(\lambda)\otimes T_N(\mu)&=T_N(\mu)\otimes T_N(\lambda)R(\lambda-\mu)\ ,\end{aligned} \tag{A3}$$

where R denotes a 4×4 matrix of the form

$$R(\lambda)=\begin{bmatrix}a&0&0&0\\0&b&c&0\\0&c&b&0\\0&0&0&a\end{bmatrix} \tag{A4}$$

with

$$a=1,\ b=\frac{i}{\lambda+i},\ c=\frac{\lambda}{\lambda+i}\ .$$

The direct products in Eq. (A3) connect the Pauli matrices in the two subsidiary spaces. The matrix R acts in this product space. Equations (A3) and (A4) imply that $R(\lambda)$ solves the Yang-Baxter equations.

More remarkable, for our purpose, is the commutativity of $\mathrm{tr}[T_N(\lambda)]$ for different values of the spectral parameter,

$$[\mathrm{tr}T_N(\lambda),\mathrm{tr}T_N(\mu)]=0\ , \tag{A5}$$

where tr means the trace in the two-dimensional subsidiary space. This scheme is directly applicable to the AFH Hamiltonian, which is related to $T_N(\lambda)$ via

$$H=\frac{i}{4}\frac{\partial}{\partial\lambda}\ln\mathrm{tr}T_N(\lambda)\Big|_{\lambda=i/2}-N/8\ . \tag{A6}$$

We may therefore expand $\mathrm{tr}T_N(\lambda)$ into a power series in λ:

$$\mathrm{tr}T_N(\lambda)=\sum_{k=0}^{N}\lambda^{N-k}c_kG_k\ . \tag{A7}$$

The operators G_k, defined in Eq. (3.4) appear as coefficients in this expansion. By construction, they commute with the AFH Hamiltonian, which completes the proof of the nonlocal conservation laws. The local conservation laws (3.1) and (3.2) follow from the higher-order derivatives of the logarithm of the transfer matrix [cf. Eq. (A6)]. We found a more direct access to these conservation laws by means of generalized Jacobi identities.

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[1] D. Vaknin *et al.*, Phys. Rev. Lett. **58**, 2802 (1987); G. Shirane *et al.*, *ibid.* **59**, 1613 (1987).

[2] J. G. Bednorz and K. A. Müller, Z. Phys. B **64**, 188 (1986); C. W. Chu *et al.*, Phys. Rev. Lett. **58**, 405 (1987).

[3] P. W. Anderson, Science **235**, 1196 (1987); S. Chakravarty, B. Halperin, and D. Nelson, Phys. Rev. Lett. **60**, 1057 (1988).

[4] J. Oitmaa and D. D. Betts, Can. J. Phys. **56**, 897 (1978).

[5] E. Dagotto and A. Moreo, Phys. Rev. B **38**, 5087 (1989).

[6] T. Barnes and E. Swanson, Phys. Rev. B **37**, 9405 (1988).

[7] R. J. Baxter, Ann. Phys. (N.Y.) **70**, 323 (1972).

[8] L. D. Faddeev, in *Structural Elements in Particle Physics and Statistical Mechanics,* edited by J. Honerkamp, K. Pohlmeyer, and H. Römer (Plenum, New York, 1983); P. P. Kulish and E. K. Sklyanin, in *Proceedings of the International Symposium on Integrable Quantum Fields,* Vol. 151 of *Lecture Notes in Physics*, edited by C. Montonen and J. Hietarinta (Springer, New York, 1982); M. Wadati, T. Deguchi, and Y. Akutsu, Phys. Rep. **180**, 247 (1989).

[9] B. Sriram Shastry, Phys. Rev. Lett. **56**, 1529 (1986); H. Grosse, Lett. Math. Phys. **18**, 151 (1989).

[10] M. Lüscher, Nucl. Phys. **B117**, 475 (1976).

[11] L. Hulthén, Ark. Mat. Astron. Fys. **26A**, N11 (1938).

[12] P. W. Anderson, Mater. Res. Bull. **8**, 153 (1973); S. Kivelson, D. Rokhsar, and J. Sethna, Phys. Rev. B **35**, 8865 (1987).

[13] S. Ramasesha and Z. Soos, Solid State Commun. **46**, 509 (1983); Phys. Rev. B **29**, 5410 (1984).

[14] E. Gagliano, E. Dagotto, A. Moreo, and F. Alcaraz, Phys. Rev. B **35**, 1677 (1986).