Spin-Peierls, valence-bond solid, and Néel ground states of low-dimensional quantum antiferromagnets

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We examine large-N limits of the nearest-neighbor SU(N) quantum antiferromagnets on bipartite lattices in $d=1,2$. In $d=2$ the model displays a transition from a Néel to a disordered phase. The properties of the disordered phase close to the phase boundary are crucially dependent upon the nature of "hedgehog"-like instanton tunneling events. We calculate the Berry phases of the instantons and show that, at scales larger than the spin-correlation length, the system can be described by a Coulomb plasma of instantons with complex fugacities. The properties of the Coulomb plasma vary periodically with the "spin" n_c of the states at each site, with periodicity given by the coordination number Z of the lattice [$n_c=2S$ for SU(2)]. For $n_c\neq 0$ (mod Z) the disordered phase has a broken lattice symmetry with spin-Peierls order, while for $n_c=0$ (mod Z), the ground state is a valence-bond solid state with no broken symmetry. Related topological effects for the $d=1$ chain lead to spin-Peierls order for odd n_c. These results are for a class of models which have, at sites of the A sublattice, representations of SU(N) described by a Young tableau with a single row, and the conjugate on the B sublattice. Similar results are also obtained for representations with m rows, using U(m) gauge theory.

I. INTRODUCTION

A long-standing problem in the theory of quantum spin systems has been the classification of the different types of possible ground states of quantum antiferromagnets. Its solution has been given renewed importance by the recent discovery of high-temperature superconductivity.[1] Following a suggestion by Anderson,[2] the occurrence of superconductivity in $La_{1-x}Sr_xCuO_4$ and $YBa_2Cu_3O_{6+x}$ may be related to novel properties of the Cu $3d$ electron spins in the CuO_2 layers of these materials. The low-lying spin fluctuations in the insulating material La_2CuO_4 are known to be well described by an effective spin-$\frac{1}{2}$ Heisenberg antiferromagnet on a square lattice.[3]

In this paper we shall present new results on the properties of SU(N) antiferromagnets on bipartite lattices in $d=1,2$, with nearest-neighbor exchange interactions. Some earlier results[4] and a shortened version of the results of this paper[5] have already appeared. Our results are obtained by a combination of semiclassical and large-N expansions on the following SU(N) antiferromagnet:

$$\mathcal{H}=\frac{J}{N}\sum_{\langle i,j\rangle}\hat{S}^{\beta}_{\alpha}(i)\hat{S}^{\alpha}_{\beta}(j)\ , \tag{1.1}$$

where $\hat{S}^{\beta}_{\alpha}(i)$ are the generators of SU(N), $\langle ij\rangle$ denotes pairs of nearest-neighbor links on a d-dimensional bipartite lattice and repeated indices $\alpha,\beta=1,\ldots,N$ are summed over. We will study the ground state of $\mathcal{H}$ as a function of N and the integer n_c which labels the SU(N) representation under which the states at each site transform [$n_c=2S$ for the group SU(2), where S is the spin]. An important issue which we will not resolve is the extent to which our results are applicable to SU(2) antiferromagnets with frustrating, non-nearest-neighbor interactions. A restriction on antiferromagnets to which our results could be extended is that their semiclassical limit ($n_c\to\infty$) be described at long wavelengths and low frequencies by a nonlinear σ model. For the case of SU(2), as was first shown by Haldane,[6] the partition function of $\mathcal{H}$ is equivalent in the naive continuum limit and for sufficiently large S to that of an O(3) nonlinear σ (NLσ) model:

$$Z=\int\mathcal{D}\mathbf{n}\exp^{-S_n}\ ,$$
$$S_n=\frac{1}{2ga^{d-1}}\int_0^\beta cd\tau\int d^d\mathbf{r}\left[(\nabla_{\mathbf{r}}\mathbf{n})^2+\frac{1}{c^2}(\partial_\tau\mathbf{n})^2\right]+S_B\,, \tag{1.2}$$

where $\mathbf{n}$ is the Néel order parameter satisfying $\mathbf{n}^2=1$, d is the spatial dimensionality, a is the spacing, and c is the spin-wave velocity; g is a dimensionless coupling constant which depends upon S. For models with additional non-nearest-neighbor interactions, g will also depend upon the ratios of the exchange constants. In all cases we have $g\to 0$ as $S\to\infty$. The term S_B denotes additional Berry phases which depend upon the value of S and are crucial in determining the structure of the non-Néel phase. Similar nonlinear σ models can also be written down for more general SU(N) antiferromagnets[4,7] with the integer-valued parameter n_c playing the role of $2S$.

We now discuss the structure of the different ground states of $\mathcal{H}$ on the square lattice. For simplicity we will use the language of SU(2) although the results were obtained in a large-N calculation. The two types of ground states we find are the following.

A. Néel state

This state has long-range order in the $\mathbf{n}$ field with

$$\langle \mathbf{n} \rangle \neq 0 \tag{1.3}$$

and is the ground state for $g < g_c$. The critical value g_c can be calculated either in a $d = 1+\epsilon$ expansion[3] ($g_c \sim \epsilon$) or by a large-N method.[4,8] The low-lying excitations are spin waves with two polarizations and an energy-momentum relation $\omega = ck$.

B. Disordered state

This is the ground state for $g > g_c$ and has $\langle \mathbf{n} \rangle = 0$ with exponentially decaying two-spin-correlation functions

$$\langle \mathbf{S}(i)\cdot\mathbf{S}(j) \rangle \sim \varepsilon_i \varepsilon_j \exp\left[-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{\xi} \right] , \tag{1.4}$$

where $\varepsilon_i = 1$ for $i \in A$ and $\varepsilon_i = -1$ for $i \in B$, and ξ is a finite spin-correlation length. For the case of SU(2) it now appears that $g < g_c$ even for $S = \frac{1}{2}$,[9] so it is necessary to include frustrating interactions to have a chance of stabilizing any such phase.

The surprising new feature of the disordered state is the presence of spin-Peierls or valence-bond solid order. To describe the nature of this order we introduce the field Q on every link of the square lattice

$$Q_{i,i+\hat{\eta}} = -\mathbf{S}(i)\cdot\mathbf{S}(i+\hat{\eta}) , \tag{1.5}$$

where $\hat{\eta}$ takes the values $\hat{x}$, $-\hat{x}$, $\hat{y}$, and $-\hat{y}$ and all careted vectors are a lattice spacing in length. Except for the case $2S = 0$ (mod 4), the symmetry group of rotations about lattice points is spontaneously broken and the values of $\langle Q_{i,i+\hat{\eta}} \rangle$ depend upon the orientation and location of the link as shown in Figs. 1(a)–1(c) (with $n_c = 2S$). Thus, for $2S = 1,3$ (mod 4), the Z_4 lattice rotation symmetry is completely broken and there is a fourfold degeneracy in the ground state. For $2S = 2$ (mod 4), the Z_4 symmetry is broken down to Z_2 and there is a twofold ground-state degeneracy. The symmetry breaking is conveniently described by the complex spin-Peierls order parameters Ψ_p

$$\Psi_p(i) = \sum_{\hat{\eta}} (\vartheta_{i,i+\hat{\eta}})^p Q_{i,i+\hat{\eta}} , \tag{1.6}$$

where the sum is over the four links ending at the site i, and the $\vartheta_{i,i+\hat{\eta}}$ take the fixed values 1, i, -1, and $-i$ on the links as shown in Fig. 2. The values have been chosen such that $\Psi_p \to e^{in\pi p/2}\Psi_p$ under a rotation by $n\pi/2$ about a point on the A sublattice and $\Psi_p \to e^{-in\pi p/2}\Psi_p$ under a rotation by $n\pi/2$ about a point on the B sublattice. In the Néel phase we have $\langle \Psi_p \rangle = 0$ except for $p = 0$ (mod 4). The broken lattice rotation symmetry in the disordered phase shown in Fig. 1 implies that

$$|\langle \Psi_p \rangle| \neq 0 \quad \text{for } p = 2S \text{ (mod 4)} . \tag{1.7}$$

In the regime $\xi \gg a$, N large, and $2S \neq 0$ (mod 4), we find

$$|\langle \Psi_{2S} \rangle| \sim \exp(-NE_c) , \tag{1.8}$$

where E_c is the action of a charge-1 "hedgehog" instanton[10] in the disordered phase. The quantity E_c has been calculated recently[11] in the limit $N \to \infty$, with ξ large but fixed:

$$E_c = (0.124\,592\,18\ldots)\ln\left[\frac{\xi}{a}\right] . \tag{1.9}$$

The elementary excitations in this phase for all S are confined (i.e., permanently bound) pairs of spin-$\frac{1}{2}$ bose particles with total spin 1 or, possibly, zero. There is also a spinless collective mode with a gap at all wave vectors,

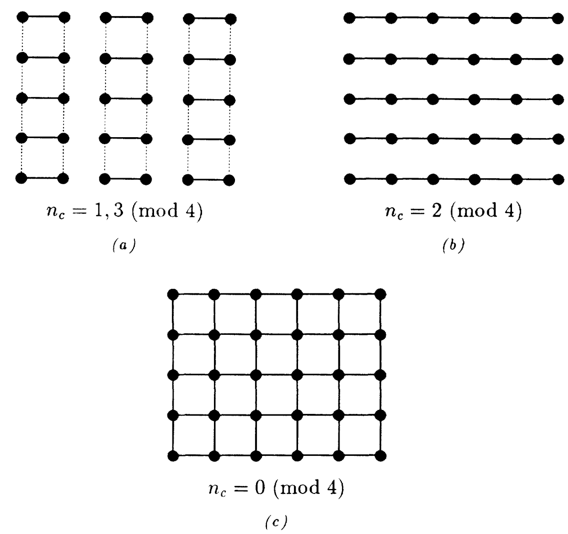

FIG. 1. Symmetry of non-Néel ground states of H as a function of n_c (mod 4) with the minimum possible degeneracies 4,2,1, respectively [$n_c = 2S$ for SU(2)]: solid lines denote larger values of $\langle \hat{S}(i)\cdot\hat{S}(i+1) \rangle$ for a link, no line, smaller values, and dashed line, intermediate values.

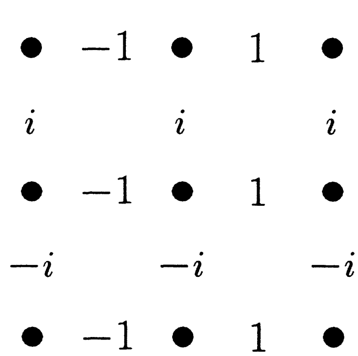

FIG. 2. Values of ϑ on the links of the square lattice [Eq. (1.6)].

at least in the analogous models for large N. The case $2S (\mathrm{mod}\ 4)=0$ gives ground states with properties the same as the valence-bond solid states of Affleck *et al.*[12]

The critical properties of the transition at $g=g_c$ require an understanding of the limit $\xi \rightarrow \infty$, with N possibly large but fixed. In this paper we have examined the limit $N \rightarrow \infty$ with ξ large but fixed and are thus not able to make any definite predictions on the nature of the transition.

Our results can be generalized to other bipartite lattices in $d=2$. The properties of the disordered phase are now sensitive to the value of $2S$ (mod Z), where Z is the coordination number of the lattice. Spin-Peierls order is now present for all $2S \neq 0$ (mod Z).

Several investigators[13-18] have recently considered frustrated, SU(2), spin-$\frac{1}{2}$, antiferromagnets on the square lattice. Many of the models considered are so strongly frustrated that their classical ground state is not the usual two-sublattice Néel state with ordering wave vector (π,π): the results of this paper cannot be extended to such models. Gelfand *et al.*[13] have performed a systematic series expansion on a model with nearest-neighbor exchange J_1 and second-neighbor exchange J_2; in the classical $S \rightarrow \infty$ limit this model is ordered for $J_2/J_1 < 0.5$. They found a Néel ground state for $J_2/J_1 < 0.33$, and a disordered ground state which appears to have the columnar spin-Peierls order shown in Fig. 1(a) for $0.33 < J_2/J_1 < 0.6$. A mean-field analysis[14] of the models of Gelfand *et al.* yields similar results. Additional evidence for this scenario has emerged from exact diagonalization studies of Dagotto and Moreo[15] on finite systems (≤ 20 sites) of the J_2-J_1 model. They found an enhanced susceptibility towards the spin-Peierls ordering of Fig. 1(a) near $J_2/J_1 \approx 0.5$. We note, however, that Dagotto and Moreo[15] also observed signals of "spin-nematic"[17] ordering in the quantum disordered state; this is one among the several alternative structures that have been suggested as ground state for frustrated quantum antiferromagnets.[17,18]

We now turn to a discussion of the methods used to establish the results of this paper. We study the properties of the Hamiltonian $\mathcal{H}$ [Eq. (1.1)]. The SU(N) generators satisfy the commutation relation

$$[\hat{S}^{\beta}_{\alpha}(i),\hat{S}^{\delta}_{\gamma}(j)]=\delta_{i,j}[\delta^{\beta}_{\gamma}\hat{S}^{\delta}_{\alpha}(i)-\delta^{\delta}_{\alpha}\hat{S}^{\beta}_{\gamma}(i)] \ . \qquad (1.10)$$

At each site on sublattice A we place a "spin" transforming under the representation of SU(N) given by the Young tableau in Fig. 3, with $0 < m < N$ rows and n_c columns. On sites on sublattice B we place the conjugate representation which has N-m rows and n_c columns. For the case $N=2$ all representations have $m=1$ and $n_c=2S$. For general N we find that n_c continues to play the role of $2S$.

The ground states of $\mathcal{H}$ are presented in a phase diagram in Fig. 4 as a function of N and n_c; the properties of the system are relatively insensitive to the value of m. We first present a catalog of previously established results on this phase diagram.

(1) In $d=2$, there is a finite region in this plane where the model displays long-range Néel order. The low-

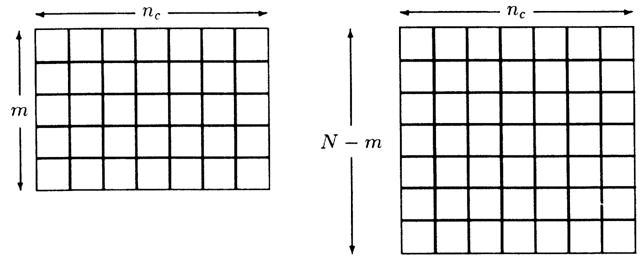

FIG. 3. Young tableau of the SU(N) representations of the "spins" on sublattices A and B, respectively.

energy, long-wavelength, semiclassical ($n_c \rightarrow \infty$, N,m fixed) dynamics in or near the Néel phase are described by a $(d+1)$-dimensional

$$\mathrm{U}(N)/[\mathrm{U}(m)\times \mathrm{U}(N-m)]$$

NLσ model with additional Berry phases.[4,7] This model possesses a critical coupling $g=g_c$ beyond which the Néel order vanishes; for sufficiently large N, or in a $d=1+\epsilon$ expansion, this determines a line $n_c=\kappa_c N$ marking the limit of stability of the Néel phase[4,8] (Fig. 4).

(2) Haldane[10] noted the importance of "hedgehog" space-time point singularities in the O(3) NLσ model. Such singularities, in fact, occur for all values of N,m (because

$$\pi_2[\mathrm{U}(N)/\mathrm{U}(m)\times \mathrm{U}(N-m)]=Z$$

and their accompanying Berry phases led Haldane to suggest that all low-lying states in the disordered phase have a minimum degeneracy of 1,4,2,4 for $n_c=0,1,2,3$ (mod 4).[4]

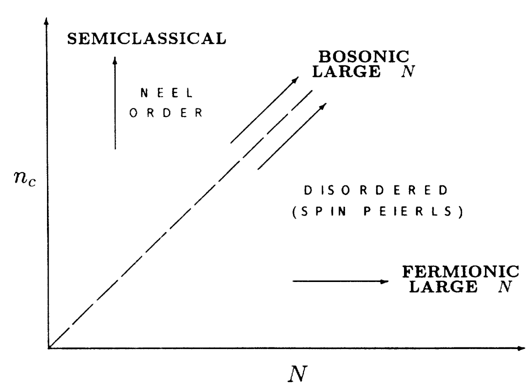

FIG. 4. Phase diagram of the square lattice SU(N) antiferromagnet as a function of the "spin" n_c [$=2S$ for SU(2)]. The phase boundary between Néel order and its absence behaves as $N_c/N \rightarrow 0.19$ as $N \rightarrow \infty$ (Ref. 8). Earlier work examined the semiclassical (Refs. 4 and 10) and the fermionic large-N limits (Refs. 4, 8, and 19); the latter has spin-Peierls order with the symmetry of Fig. 1(a) for all n_c. This paper examines the bosonic large-N region in the disordered phase close to the transition line. In $d=1$, the Néel region is absent, while for $d>2$, a similar phase boundary is found (Ref. 4).

(3) The properties of the disordered phase can be examined directly in the extreme quantum limit ($N \to \infty$, n_c fixed). Two cases have been considered.

(a) $m = N/2$. This case can be solved exactly in the large-N limit[4] using the functional integral method first applied to this problem by Affleck and Marston.[19] The ground state has spin-Peierls order with the symmetry of Fig. 1(a) for all values of n_c. However, there are low-lying metastable states with the symmetry of Figs. 1(b) and 1(c) for the respective values of n_c. The lower bound on the degeneracy noted in (2) is always satisfied. The low-lying excitations consists of two unpaired spins connected by a string of singlet bonds. The string is out of alignment with the columns of singlet in the remaining lattice (Fig. 5) and therefore carries a finite string tension. Consequently, unpaired spins are, in fact, confined in pairs by the unbreakable string, the ends of which must be on opposite sublattices. A similar picture works for all the spin-Peierls and valence-bond solid states considered in this paper (Figs. 1 and 8) and provides a simple physical picture for the origin of the "confinement."

(b) $m = 1$. The $1/N$ fluctuations now map onto a generalized quantum dimer model of the type first considered by Rokhsar and Kivelson.[20] Finite-size exact diagonalizations[21] have been carried out for $n_c = 1$ and show, quite convincingly for the parameters obtained in this large-N limit ($V = 0$ in the language of Ref. 20), that the ground state has spin-Peierls order of the type shown in Fig. 1(a).

In this paper we will analyze the properties of the disordered phase in the region close to the transition to the Néel phase. As is clear from Fig. 4, this can be done by fixing m and the ratio n_c/N and then taking the large-N limit. We will, for simplicity, consider the case $m = 1$; the generalization of these results to all m is discussed in Appendix C. We will also review in Sec. II some important features of the semiclassical limit ($n_c \to \infty$, N, m fixed). In addition, we show that the limit $N \to \infty$ with n_c fixed, $m = 1$ can also be settled using the duality transforms of Fradkin and Kivelson[22] (Appendix A).

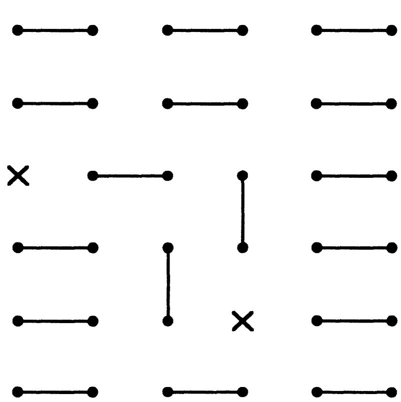

FIG. 5 Excitation of the spin-Peierls ground state for $n_c = 1$. The crosses represent unpaired spinons. Notice the "string" of bonds connecting them which are out of alignment with the spin-Peierls ordering. A similar picture holds for all the cases in Figs. 1 and 8.

The $m = 1$, $n_c/N = \text{const}$, $N \to \infty$ limit is most conveniently obtained by a Schwinger boson representation of the spin states in which

$$\hat{S}^{\beta}_{\alpha}(i) = b^{\dagger}_{\alpha}(i) b^{\beta}(i), \quad i \in A \text{ sublattice},$$
$$\hat{S}^{\beta}_{\alpha}(j) = -\bar{b}^{\beta\dagger}(j) \bar{b}_{\alpha}(j), \quad j \in B \text{ sublattice}. \tag{1.11}$$

The $\bar{b}$ bosons are implied by the placement of indices to transform as the conjugate representation to b, which are in the fundamental representation of SU(N). The states on sublattice A (B) will transform under the SU(N) representation with a Young tableau of $m = 1$ ($m = N - 1$) rows and n_c columns after imposition of the constraint

$$b^{\dagger}_{\alpha}(i) b^{\alpha}(i) = n_c ,$$
$$\bar{b}^{\alpha\dagger}(j) \bar{b}_{\alpha}(j) = n_c , \tag{1.12}$$

on every site i (j) on the A (B) sublattice.

The mean-field solution in the $N = \infty$ limit has been examined earlier by Arovas and Auerbach.[8] In $d = 2$ there is a critical value of $n_c/N = \kappa_c$ above which the Schwinger bosons condense[23] ($\langle b \rangle \neq 0$, $\langle \bar{b} \rangle \neq 0$) leading to the appearance of long-range Néel order. For $n_c/N < \kappa_c$ we obtain a spin-disordered ground state with a finite energy gap Δ for excitations. In $d = 1$, the constant $\kappa_c = \infty$ and the system is always in the disordered state. Two-spin-correlation functions decay exponentially in the disordered phase with a spin-correlation length ξ given by

$$\xi = \frac{c}{2\Delta} , \tag{1.13}$$

where the velocity c goes continuously into the spin-wave velocity on the Néel ordered side. The energy gap Δ vanishes at $n_c/N = \kappa_c$ leading to a divergence in the spin-correlation length. The ground-state wave function in the disordered state has the form

$$|\Omega\rangle = C \exp\left[\sum_{\mathbf{k}} f_{\mathbf{k}} b^{\dagger}_{\mathbf{k}\alpha} \bar{b}^{\alpha\dagger}_{-\mathbf{k}} \right] |0\rangle , \tag{1.14}$$

where C is a normalization constant and $|0\rangle$ is the state with no bosons. The wave function $|\Omega\rangle$ represents a condensate of singlet pairs of bosons ("valence bonds"); the bonds have ends on opposite sublattices and their characteristic size is ξ. When projected onto n_c bosons per site, $|\Omega\rangle$ is an SU(N) generalization of the short-range resonating-valence-bond states of Sutherland and Liang *et al.*[24] which are thus exact in the present mean-field limit provided the distribution of bond lengths is chosen correctly. The mean-field excitation spectrum consists of two free bosons ("spinons") at each point in the reduced Brillouin zone, transforming under the fundamental representation of SU(N) and with an energy-momentum relation which reduces at long wavelengths to the relativistic form

$$\omega_{\mathbf{k}} = (\Delta^2 + c^2 \mathbf{k}^2)^{1/2} . \tag{1.15}$$

The bulk of this paper is devoted to showing how topological effects in the fluctuation corrections dramatically alter the simple mean-field picture of the disordered state outlined in the previous paragraph. In particular, we will show how fluctuations lead to the appearance of spin-Peierls order whose symmetry is controlled by the value n_c (mod Z) and, for the square lattice, has the form shown in Figs. 1(a)–1(c). Moreover, we will find that the spinon excitations are confined in pairs. The final physical picture is therefore similar to that obtained in the large-N, fixed n_c limit discussed in Ref. 4 and outlined in point (3) above.

We examine in Sec. III the long-wavelength effective action for the fluctuations about the mean-field state (1.14). It contains a complex relativistic charged boson ("spinon") minimally coupled to a compact U(1) gauge field which acquires the standard electrodynamic action in the disordered phase. The charged boson transforms under the fundamental representation of SU(N) and the velocity c plays the role of the velocity of light. The effect of additional topological Berry phase terms in the action will be crucial. We discuss the results in $d=1,2$ in turn.

C. $d=1$

The Berry phase terms in the action give rise to a topological Θ term[25] in the action for the U(1) gauge field with $\Theta=\pi p$ where the integer p is even (odd) if n_c is even (odd). Each choice of Θ corresponds to a different metastable state of the spin chain with a spin-Peierls order parameter proportional to p. The ground state for n_c even is obtained with the choice $p=0$ and is nondegenerate; the linear Coulomb force confines the spinons in pairs. For n_c odd, the ground state corresponds to $p=\pm 1$ and is twofold degenerate with a nonzero spin-Peierls order parameter; the spinons are domain walls interpolating between the two ground states. A schematic of the two ground states is shown in Figs. 6(a) and 6(b). The spin-Peierls order for n_c odd was anticipated by Affleck[26] through not shown directly for $n_c \sim N$. This picture is now expected to be correct for all $N>2$.[4,26]

D. $d=2$

The physics of a compact U(1) gauge theory in 2+1 dimensions has been considered in detail by Polyakov[27] and we shall make extensive use of his methods and results. Polyakov identified pointlike instanton configurations of the (2+1)-dimensional compact U(1) gauge theory[27,28] which have

$$\int_\Sigma F_{\mu\nu} dS_{\mu\nu} = 2\pi m_s \ , \tag{1.16}$$

where $F_{\mu\nu}$ is field tensor associated with the U(1) gauge field, the integral is over the surface, Σ, of a sphere surrounding the singular point and m_s is the integer-valued "charge" of the instanton. It can be shown (Ref. 28 and Sec. II B) that these objects are the remnants of the hedgehogs of the Néel phase. Moreover, as we shall see in Sec. III A, the Berry phase associated with these instantons is identical to the hedgehog Berry phase calculated by Haldane[10] and its extension to SU(N).[4]

For a gas of sufficiently dilute instantons, we can evaluate the action for each instanton configuration and obtain the following effective partition function valid for N large:

$$Z = \sum_{K\{m_s\}} \frac{1}{K!} \prod_{s=1}^{K} \left[\sum_{\mathbf{R}_a} \int_0^\beta \frac{c\,d\tau_s}{\rho a} \right] \exp[-S_m(\{m_s\})] \ ,$$

$$S_m(\{m_s\}) = \frac{N\pi}{2e^2} \sum_{s\neq t} \frac{m_s m_t}{[(\mathbf{R}_s-\mathbf{R}_t)^2 + c^2(\tau_s-\tau_t)^2]^{1/2}} + \sum_s \left[N E_c(|m_s|) + i\frac{n_c \pi}{2}\zeta_s m_s \right] . \tag{1.17}$$

The instantons are represented by integer charges m_s located at space-time coordinates $(\mathbf{R}_s, \tau_s)$, where the $\mathbf{R}_s$ are the centers of the plaquettes of the two-dimensional lattice and the τ_2 are the imaginary time coordinates. We have also introduced the coupling constant e, the instanton core action NE_c which is a function of $|m_s|$, and ρ a dimensionless constant of order unity. The last term in S_m is the all-important Berry phase and can be obtained from the results of Haldane or from the calculations in Sec. III A. For the case of the square lattice, the integer $\zeta_s = 0,1,2,3$ for $\mathbf{R}_s$ on four dual sublattices W,X,Y,Z (Fig. 7). There is a gauge choice involved in specifying the values of ζ_s: gauge transformations will rotate the values of ζ_s among the four dual sublattices. We note that Fradkin and Kivelson[22] have recently introduced duality transformations on the quantum dimer models [which can be obtained rigorously from SU(N) antiferromagnets in the $m=1$, large-N, fixed n_c limit[4]] which yield a Coulomb gas partition function which we show in Appendix A is essentially the same as (1.17). This confirms the nature of the Berry phases and suggests a solution in this limit also.

An important feature of S_m is the Coulombic $1/r$ interaction between the instantons. This is in contrast to the linear r interaction between the hedgehogs in the Néel state. In Sec. III B we use a standard duality argument to map the instanton plasma of the disordered phase into a frustrated sine-Gordon model. This model is used to show that the plasma is in a Debye-screening phase[29] and that the presence of the Berry phases leads to

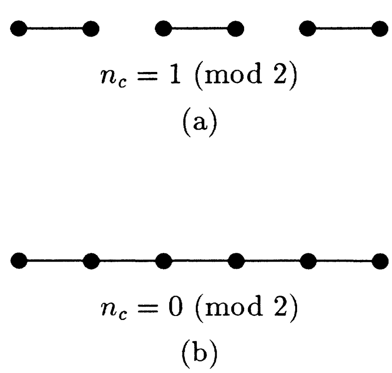

FIG. 6. Symmetry of the ground states for the $d=1$ chain.

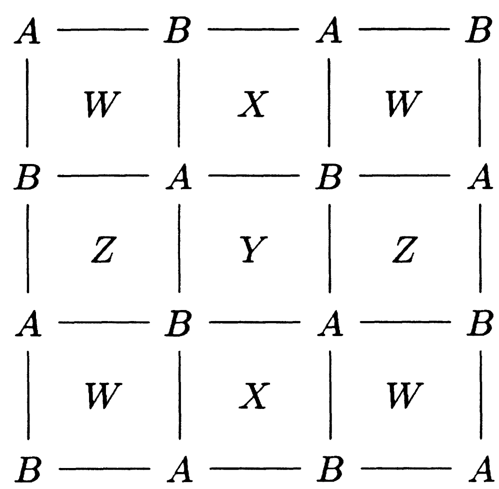

FIG. 7. The A,B sublattices of the lattice of spins and the sublattices W,X,Y,Z of the dual lattice.

a spontaneous broken symmetry with a nonzero mean instanton charge on the dual plaquette sublattices (the total mean charge obtained by summing over the sublattices is, of course, always zero). For the case of $n_c=0$ (mod Z), the Berry phase is always an integer multiple of 2π and the mean instanton charge is zero everywhere. For $n_c=1,3$ (mod 4), the mean instanton charge configuration breaks the Z_4 lattice rotational symmetry completely while for $n_c=2$ (mod 4) it is broken down to Z_2. A detailed calculation in Sec. III B which retains the coupling between the instanton charges and the spin-Peierls order shows that the condensation of instanton charges implies spin-Peierls order with the symmetry of Figs. 1(a)–1(c): Moreover, results of Polyakov[27] imply that, in all cases, the spinons are confined into pairs of size of the order of a combination of the spin-correlation and Debye-screening lengths. These are the central results of this paper.

The outline of the rest of the paper is as follows. Section II reviews the semiclassical nonlinear σ model description of $\mathcal{H}$ ($n_c \to \infty$). Section II A reviews topological properties of the order-parameter manifold of the nonlinear σ models and displays the Berry phases associated with the topological excitations. Section II B reviews the connection between the CP^{N-1} model (the special case $m=1$ of our models) and a theory of complex scalars interacting with a U(1) gauge field. Section III contains the central new results of this paper; its contents have been described above. Section IV gives more physical discussion and presents some speculations on the critical properties of the transitions between the Néel and disordered phases.

Some additional results are contained in the appendices. Appendix A review the connection between our results and those of Ref. 22. Appendix B presents the extension to the case of honeycomb lattice. Appendix C discusses the generalization to $m>1$.

II. SEMICLASSICAL THEORY

This section reviews aspects of the semiclassical theory of SU(N) antiferromagnets.[7,4] The semiclassical limit is obtained by choosing SU(N) representations with n_c large while keeping the values of N and m fixed. The system behaves classically at $n_c=\infty$ and has a Néel ground state. The quantum fluctuations about the classical Néel ground state have been shown to be described by an effective $(d+1)$-dimensional nonlinear σ (NLσ) model. The NLσ model is a nontrivial interacting field theory but many of its properties can be understood in the framework of a $d=1+\epsilon$ expansion.[3] We shall concentrate in this section upon results obtained via a large-N expansion of the NLσ model partition function. The results will turn out to be closely related to the n_c/N fixed, large-N, theory of the original spin Hamiltonian $\mathcal{H}$ to be discussed in the subsequent sections. As noted earlier, we will restrict our attention in this section to representations with $m=1$. The results can be generalized to arbitrary values of m by the the methods of MacFarlane[30] (Appendix C).

As shown in detail in Ref. 4, the large-n_c limit of the partition function of the SU(N) antiferromagnet $\mathcal{H}$ yields the following functional integral over a NLσ model action with an additional Berry phase term:

$$Z=\int \mathcal{D}\Omega \exp^{-S_n} ,$$
$$S_n=\frac{1}{4ga^{d-1}}\int_0^\beta c\,d\tau\int d^d\mathbf{r}\,\mathrm{Tr}\left[(\nabla_\mathbf{r}\Omega)^2+\frac{1}{c^2}(\partial_\tau\Omega)^2\right]+S_B , \tag{2.1}$$

where $g=c/\rho_s$, the spin-wave stiffness $\rho_s=Jn_c^2/2N$, and the spin-wave velocity $c=2\sqrt{d}\,Jn_c a/N$. The form of the Berry phase term will be considered in greater detail below. The order parameter Ω is a $N\times N$ matrix which satisfies $\Omega^2=1$ and belongs to the

$$\mathrm{U}(N)/[\mathrm{U}(m)\times\mathrm{U}(N-m)]$$

manifold. Equation (1.2) of the Introduction can be obtained by using the SU(2) parametrization $\Omega=\mathbf{n}\cdot\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the triplet of Pauli matrices. For arbitrary N and $m=1$ we parametrize Ω as follows:

$$\Omega_\alpha^\beta=-\delta_\alpha^\beta+2z_\alpha^* z^\beta , \tag{2.2}$$

where z^α are N complex fields ($\alpha=1,\ldots,N$) satisfying the constraint $\sum_\alpha|z^\alpha|^2=1$ everywhere in space-time. Notice, however, the presence of a residual gauge invariance: the transformation $z^\alpha(\mathbf{r},\tau)\to z^\alpha(\mathbf{r},\tau)e^{i\phi(\mathbf{r},\tau)}$ leaves the value of $\Omega(\mathbf{r},\tau)$ unchanged. In terms of the z's the action, S_n takes the form

$$S_z=\frac{2}{ga^{d-1}}\int d^{d+1}x\,(|\partial_\mu z^\alpha|^2-|z_\alpha^*\partial_\mu z^\alpha|^2)+S_B , \tag{2.3}$$

where x is the space-time coordinate $(\mathbf{r},\tilde{\tau})$, $\tilde{\tau}=c\tau$, μ extends over the $d+1$ coordinates $x,y,\ldots,\tilde{\tau}$ and we have taken the zero-temperature limit. Without the Berry phase term, S_z is the action for the well-known CP^{N-1}

model.[31,32] The term S_B has a simple form in terms of the z^α fields:[4]

$$S_B = n_c \sum_i \varepsilon_i \int_0^\beta d\tau\, z_\alpha^*(\mathbf{r}_i,\tau) \frac{dz^\alpha(\mathbf{r}_i,\tau)}{d\tau} , \tag{2.4}$$

where we have now placed the z^α fields on the underlying lattice. As before, $\varepsilon_i = 1(-1)$ for $i \in A(B)$. For the subsequent analysis it is useful to make the gauge invariance of the CP^{N-1} model manifest. We introduce the field A_μ as a Hubbard-Stratonovich field to decouple the quartic term in Eq. (2.3)

$$\begin{aligned} Z &= \int \mathcal{D}z^\alpha \mathcal{D}A_\mu\, \delta(|z^\alpha|^2 - 1)\exp(-S_z) , \\ S_z &= \frac{2}{ga^{d-1}} \int d^{d+1}x\, |(\partial_\mu - iA_\mu)z^\alpha|^2 + S_B , \end{aligned} \tag{2.5}$$

with an implicit gauge-fixing term in the functional integral. We note that the quadratic term in A_μ in Eq. (2.5) is simply A_μ^2: the equations of motion therefore constrain[31,32]

$$A_\mu = \frac{i}{2}(z^\alpha \partial_\mu z_\alpha^* - z_\alpha^* \partial_\mu z^\alpha) . \tag{2.6}$$

Inserting this into Eq. (2.5) we regain the action as written in Eq. (2.3). Under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \phi$; A_μ has therefore all the characteristics of a U(1) gauge field.

The remaining semiclassical analysis is divided into two subsections. In Sec. II A we recall results on the evaluation of the spatial summation of the Berry phases of the individual spins in Eq. (2.4). In Sec. II B we review previous results on the U(1) gauge-field description of the disordered phase of the CP^{N-1} model.

A. Berry phases

We discuss the form of S_B in spatial dimensionality $d=1$ and 2 in turn.

1. d=1

With the assumption that $z^\alpha(\mathbf{r},\tau)$ varies slowly on the scale of the lattice spacing, we can evaluate the summation in Eq. (2.4) and obtain[31,32]

$$S_B = \frac{\Theta}{2\pi} \int d^2x\, \epsilon_{\mu\nu} \partial_\mu z_\alpha^* \partial_\nu z^\alpha \tag{2.7}$$

with $\Theta = \pi n_c$. The integrand is the topological invariant associated with mappings from a plane with periodic boundary conditions to the order-parameter space

$$\mathrm{U}(N)/[\mathrm{U}(1)\times\mathrm{U}(N-1)] ;$$

such mappings are classified by the second homotopy group and we have

$$\pi_2\{\mathrm{U}(N)/[\mathrm{U}(1)\times\mathrm{U}(N-1)]\} = Z ,$$

the group of integers. With periodic boundary conditions, S_B can only take the values $i\Theta p$, where p is an arbitrary integer, so the physics should depend only on Θ modulo 2π.

Using Eq. (2.6) we can also show

$$S_B = \frac{i\Theta}{4\pi} \int d^2x\, \epsilon_{\mu\nu} F_{\mu\nu} , \tag{2.8}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field. The topological invariant is thus also related to the net electromagnetic flux. In this form, S_B is the well-known Θ term of (1+1)-dimensional quantum electrodynamics.[25]

2. d=2

The spatial summation in S_B in Eq. (2.4) has been shown to vanish[33] for any order-parameter configuration which is smooth on the scale of the lattice spacing. However,[10] there are also topologically stable configurations with point singularities in space-time. Such singularities can be classified by considering smooth maps from a two-sphere S^2 surrounding the singular point to the order-parameter manifold

$$\mathrm{U}(N)/[\mathrm{U}(1)\times\mathrm{U}(N-1)] .$$

This again introduces

$$\pi_2\{\mathrm{U}(N)/[\mathrm{U}(1)\times\mathrm{U}(N-1)]\} = Z ;$$

the singularities therefore have integer charges. These are the "hedgehogs" and are characterized by the integers m_s

$$\begin{aligned} m_s &= \frac{i}{2\pi} \int_\Sigma dS_{\mu\nu} (\partial_\mu z_\alpha^* \partial_\nu z^\alpha - \partial_\nu z_\alpha^* \partial_\mu z^\alpha) \\ &= \frac{1}{2\pi} \int_\Sigma dS_{\mu\nu} F_{\mu\nu} , \end{aligned} \tag{2.9}$$

where Σ is a sphere surrounding the point $(\mathbf{R},\tau)$ at which the hedgehog is centered: $\mathbf{R}$ is located at the center of a plaquette of the lattice of spins. The second equation above shows that the hedgehogs have a net flux $2\pi m_s$ emanating from the center and identifies them as Dirac monopoles in the U(1) gauge field. The hedgehogs can also be interpreted as instantons in tunneling events involving a change in the total "Skyrmion" number Q of the instantaneous spin configuration:[34]

$$\begin{aligned} Q &= \frac{i}{2\pi} \int dx\, dy (\partial_x z_\alpha^* \partial_y z^\alpha - \partial_y z_\alpha^* \partial_x z^\alpha) \\ &= \frac{1}{2\pi} dx\, dy F_{xy} , \end{aligned} \tag{2.10}$$

which is the same integer-valued invariant, this time for a time slice. In the language of the U(1) gauge field, the Skyrmion number is linked to the total magnetic flux, F_{xy}, piercing the lattice and an instanton of charge m changes this flux by $2\pi m$. The Berry phase for such tunneling events has been evaluated in the ordered phase of the CP^{N-1} model using Eq. (2.4);[4,10] for the case of the square lattice we find

$$S_B = \sum_s i \frac{n_c \pi}{2} \zeta_s m_s , \tag{2.11}$$

where the integer $\zeta_s = 0,1,2,3$ for $\mathbf{R}_s$ on four dual sublattices W,X,Y,Z (Fig. 7). We note there is a gauge choice involved in specifying the values of ζ_s: a gauge transformation can rotate the values of ζ_s among the four dual sublattices.

B. Gauge theory of the disordered phase of the CP^{N-1} model

D'Adda *et al.*[31] and Witten[32] have presented an analysis of the disordered phase of the CP^{N-1} model in $d=1$ in which they emphasized the importance of the fluctuations of the U(1) gauge field A_μ. For completeness, we recall features of their results which will be useful in the subsequent sections. We will also present the straightforward generalization of their results to $d=2$.[35] All the results in this subsection can be interpreted as the lowest nontrivial order in a $1/N$ expansion, provided n_c is of order N (using the expressions for ρ_s and c at the beginning of this section, this implies $g=c/\rho_s$ is of order $1/N$).

We begin by expressing the constrain in Eq. (2.5) by a Lagrange multiplier field λ:

$$Z=\int \mathcal{D}z^\alpha \mathcal{D}A_\mu \mathcal{D}\lambda \times\exp\left[-\frac{2}{ga^{d-1}}\int d^{d+1}x[|(\partial_\mu-iA_\mu)z^\alpha|^2 +i\lambda(|z^\alpha|^2-1]+S_B(A_\mu)\right] . \quad (2.12)$$

We have also indicated that the S_B term is to be evaluated using the A_μ-dependent expressions in Sec. II A. Integrating out the z^α we find

$$Z=\int \mathcal{D}A_\mu \mathcal{D}\lambda \exp[-NS_{\text{eff}}(A_\mu,\lambda)+S_B(A_\mu)] , \quad (2.13)$$

where the effective action S_{eff} will be preceded by a factor N provided $g\sim 1/n_c \sim 1/N$. In the large-N limit we may evaluate the partition function by expanding about a minimum of S_{eff}. We search for a minimum with $A_\mu=0$ (thus eliminating S_B) and $i\lambda=\Delta^2/c^2$, where Δ is a positive constant. Such a minimum always exists in $d=1$ and for $g>g_c$ in $d=2$ (g_c is a number of order $1/N$); for $g<g_c$ in $d=2$, the CP^{N-1} model is in the Néel phase.

In the disordered phase, and at length scales larger than c/Δ, we may perform a gradient expansion of S_{eff} and obtain

$$Z=\int \mathcal{D}z^\alpha \mathcal{D}A_\mu \exp\left\{-\int d^{d+1}x\left[\frac{2}{ga^{d-1}}\left[|(\partial_\mu-iA_\mu)z^\alpha|^2+\frac{\Delta^2}{c^2}|z^\alpha|^2\right]+\frac{N}{4e^2}F_{\mu\nu}^2\right]+S_B(A_\mu)\right\} . \quad (2.14)$$

We have reintroduced the z^α bosons to display their coupling to A_μ, neglected the massive λ fluctuations, and introduced the coupling constant $e^2\sim(\Delta/c)^{3-d}$. Notice the dynamical generation of the A_μ kinetic energy and the absence of an explicit constraint on the magnitude of z^α. We postpone further analysis of this partition function to Sec. III where we will obtain a very similar result by performing a large-N calculation (with n_c of order N) directly on the spin Hamiltonian $\mathcal{H}$. The advantage of the latter procedure is that it retains the coupling to lattice spin-Peierls order parameter which has been lost in the continuum limit of this section.

III. BOSONIC LARGE N

In this section we shall develop a large-N theory which investigates the transition between the Néel and disordered phases (Fig. 4) in greater detail. We shall examine the properties of $\mathcal{H}$ [Eq. (1.1)] by fixing n_c proportional to N and then taking the large-N limit. This is most conveniently done by using the bosonic representation of the SU(N) operators discussed in Sec. I [Eqs. (1.11) and (1.12)] and Ref. 8. For simplicity, we shall restrict our discussion in this section to $m=1$; the generalization to arbitrary m is discussed in Appendix C. We may represent the partition function of $\mathcal{H}$ by the functional integral

$$Z=\int \mathcal{D}Q\,\mathcal{D}b\,\mathcal{D}\bar{b}\mathcal{D}\lambda \exp\left[-\int_0^\beta \mathcal{L}d\tau\right] , \quad (3.1)$$

where

$$\mathcal{L}=\sum_{i\in A}\left[b_\alpha^\dagger(i)\left[\frac{d}{d\tau}+i\lambda(i)\right]b^\alpha(i)-i\lambda(i)n_c\right] +\sum_{j\in B}\left[\bar{b}^{\alpha\dagger}(j)\left[\frac{d}{d\tau}+i\lambda(j)\right]\bar{b}_\alpha(j)-i\lambda(j)n_c\right] +\sum_{i\in A,\hat{\eta}}\left[\frac{N}{J}|Q_{i,i+\hat{\eta}}|^2 -[Q^*_{i,i+\hat{\eta}}b^\alpha(i)\bar{b}_\alpha(i+\hat{\eta})+\text{H.c.}]\right] . \quad (3.2)$$

Here the $\lambda(i)$ fix the boson number of n_c at each site, the τ dependence of all fields is implicit, Q was introduced by a Hubbard-Stratonovich decoupling of H, and $\hat{\eta}$ runs over nearest-neighbor vectors and has length a. An important feature of the Lagrangian $\mathcal{L}$ is its U(1) gauge invariance under which

$$b_\alpha^\dagger(i)\to b_\alpha^\dagger(i)\exp[i\phi(i,\tau)] ,$$
$$\bar{b}^{\alpha\dagger}(j)\to b^{\alpha\dagger}(j)\exp[i\phi(j,\tau)] ,$$
$$Q_{i,i+\hat{\eta}}\to Q_{i,i+\hat{\eta}}\exp[-i\phi(i,\tau)-i\phi(i+\hat{\eta},\tau)] ,$$
$$\lambda(i)\to\lambda(i)+\frac{\partial\phi}{\partial\tau}(i,\tau) .$$

The functional integral over $\mathcal{L}$ faithfully represents the partition function as long as we fix a gauge, e.g., by the condition $d\lambda/d\tau=0$ at all sites.

The $1/N$ expansion of the free energy can be obtained by integrating out of $\mathcal{L}$ the N-component $b,\bar{b}$ fields to leave an effective action for Q,λ having coefficient N (since $n_c \propto N$). Thus, the $N\to\infty$ limit is given by minimizing the effective action with respect to "mean-field" values of Q,λ. This is, in turn, equivalent to solving the mean-field (MF) Hamiltonian

$$H_{\rm MF}=\sum_{i\in A,\hat{\eta}}\left[\frac{N|\bar{Q}|^2}{J}-[\bar{Q}b^{\alpha}(i)\bar{b}_{\alpha}(i+\hat{\eta})+\text{H.c.}]\right]+\bar{\lambda}\sum_{i\in A}[b_{\alpha}^{\dagger}(i)b^{\alpha}(i)-n_c]+\bar{\lambda}\sum_{j\in B}[\bar{b}^{\alpha\dagger}(i)\bar{b}_{\alpha}(i)-n_c] . \quad (3.3)$$

In writing $H_{\rm MF}$ we used the fact that $i\lambda(i)=\bar{\lambda}$ and $Q_{i,i+\hat{\eta}}$ are found to be uniform and independent of $\hat{\eta}$ at the saddle point. (We have numerically verified that this saddle point is locally stable and has the global minimum action for all configurations with a period of two lattice spacings.) The constant $\bar{\lambda}$ is found to be real, and $\bar{Q}$ can be taken real, positive by a gauge transformation. The Hamiltonian $H_{\rm MF}$ can be diagonalized by Bogoliubov's method and we find two modes for each wave vector in the (reduced) Brillouin zone of energy

$$\omega_{\mathbf{k}}=(\bar{\lambda}^2-4d^2\bar{Q}^2\gamma_{\mathbf{k}}^2)^{1/2} , \quad (3.4)$$

where

$$\gamma_{\mathbf{k}}=(1/2d)\sum_{\hat{\eta}}e^{i\mathbf{k}\cdot\hat{\eta}}=\tfrac{1}{2}[\cos(k_x a)+\cos(k_y a)] \text{ in } d=2 , \quad (3.5)$$

and $\bar{\lambda}\sim\bar{Q}\sim J$. At $\mathbf{k}=0$,

$$\omega_{\mathbf{k}}=\Delta=(\bar{\lambda}^2-4d^2\bar{Q}^2)^{1/2}\geq 0$$

is the energy gap; a nonzero Δ implies the absence of long-range Néel order. In $d=1$, $\Delta\to 0$ as $n_c/N\to\infty$; thus, in agreement with the general arguments (see, e.g., Ref. 4), we find that there is no Néel ordered phase for finite n_c. In $d=2$, $\Delta\to 0$ as temperature $T\to 0$ for all $n_c/N\geq 0.19$, while for $n_c/N<0.19$, the gap Δ remains nonzero at $T=0$.[8] Thus, this mean-field analysis determines a line in the n_c-N plane (shown in Fig. 4), with slope 0.19 for large N, above which there is long-range Néel order. This conclusion is again in agreement with the nonlinear σ model analysis of Ref. 4. For $d>2$, Δ vanishes above some critical value of n_c/N for all $T<T_{\text{Néel}}(n_c/N)$, the Néel ordering temperature. The vanishing of Δ may be physically identified with the presence of long-range Néel order by noting that $\Delta=0$ requires $\langle b\rangle,\langle\bar{b}\rangle$ to be nonzero due to condensation into the zero-energy states.[23] In the remainder of this section, we shall focus exclusively on the properties of the disordered state at $T=0$ in $d=1,2$ ($n_c/N<0.19$ for $d=2$), where SU(N) symmetry is unbroken.

The subsequent analysis is simplest close to the transition line in Fig. 4 where $\Delta\ll J$; the bosonic spectrum has the relativistic form

$$\omega_{\mathbf{k}}=(\Delta^2+c^2\mathbf{k}^2)^{1/2} , \quad (3.6)$$

where the speed of "light" (spin-wave velocity) $c=\bar{\lambda}a/d^{1/2}$. We can also define a spin-correlation length ξ

$$\xi=\frac{c}{2\Delta}\gg a , \quad (3.7)$$

which is much greater than the lattice spacing. This length diverges as one approaches the transition to the Néel phase (Fig. 4): we therefore expect to obtain a simple continuum description of the system in this limit. The ground state of $H_{\rm MF}$ has the form for $\Delta>0$

$$|\Omega\rangle\propto\exp\left[\sum_{\mathbf{k}}f_{\mathbf{k}}b_{\mathbf{k}\alpha}^{\dagger}\bar{b}_{-\mathbf{k}}^{\alpha\dagger}\right]|0\rangle , \quad (3.8)$$

which represents a condensate of singlet pairs of bosons ("valence bonds"); the bonds have ends on opposite sublattice and their characteristic size is ξ. The mean-field results for the disordered phase are thus close in spirit to the resonating-valence-bond scenario of Anderson[2] and Kovelson *et al.*,[36] and consist of a featureless fluid of singlet pairs of bosons. However, the form of (3.8) exhibits the relative phases of Ref. 24 and the excitations are clearly not fermions[2,36] but bosons as found in Ref. 37. We will show that topological effects in the fluctuations about the mean field dramatically alter the nature of the disordered phase. We will begin in this section by examining the structure of a straightforward $1/N$ expansion. The consequences of topological fluctuations will be examined in the subsequent subsections.

We begin by parametrizing the fluctuations of Q and λ in a manner which makes the gauge invariance of $\mathcal{L}$ manifest:

$$Q_{i,i+\hat{\eta}}=[\bar{Q}+q_{\hat{\eta}}(i)]\exp[i\hat{\eta}\cdot\mathbf{A}(i)] ,$$
$$i\lambda(i)=\bar{\lambda}+iA_{\tau}(i)+i\hat{\lambda}(i) \text{ for } i\in A , \quad (3.9)$$
$$i\lambda(j)=\bar{\lambda}-iA_{\tau}(j)+i\hat{\lambda}(j) \text{ for } j\in B .$$

The parametrization of λ appears redundant; the ambiguity can, however, be resolved by demanding that the fields A_τ and $\hat{\lambda}$ vary smoothly on the scale of the lattice spacing. The magnitude fluctuation $q_{\hat{\eta}}$ is a real field residing on the lattice links. It is related to physically measurable correlation functions by

$$\langle\hat{S}(i)\cdot\hat{S}(i+\hat{\eta})\rangle=\frac{N^2}{J^2}\langle[\bar{Q}+q_{\hat{\eta}}(i+\tfrac{1}{2}\hat{\eta})]^2\rangle . \quad (3.10)$$

The lattice gauge field $\mathbf{A}=(A_{\hat{x}},A_{\hat{y}})$ resides on the links of the lattice. As is usual in the lattice gauge theory literature,[38] $A_{-\hat{\eta}}(i+\hat{\eta})=-A_{\hat{\eta}}(i)$ and $q_{-\hat{\eta}}(i+\hat{\eta})=q_{\hat{\eta}}(i)$. After the bosons have been integrated out, the gauge invariance of $\mathcal{L}$ under the transformation (3.3) implies that the effective action for the A fields can only depend upon certain gauge-invariant fields. In $d=2$, these are the "magnetic" field B through each plaquette

$$B(i)=\Delta_{\hat{x}}A_{\hat{y}}(i)-\Delta_{\hat{y}}A_{\hat{x}}(i) , \quad (3.11)$$

where $\Delta_{\hat\eta}$ is the lattice derivative [i.e., $\Delta_{\hat\eta}f(i)=f(i+\hat\eta)-f(i)$] and the electric field $E_{\hat\eta}(i)$ on each link of the lattice

$$E_{\hat\eta}(i)=\frac{1}{c}\left[\Delta_{\hat\eta}A_\tau(i)-\frac{\partial A_{\hat\eta}}{\partial\tau}(i)\right] . \quad (3.12)$$

Note that $E_{-\hat\eta}(i+\hat\eta)=-E_{\hat\eta}(i)$.

We discuss first the nature of the long-wavelength and low-energy fluctuations. Upon inserting the parametrizations of Eq. (3.9) into $\mathcal{L}$, and assuming that the fields b, $\bar b$, A, and q vary slowly in space and time, we obtain

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1 .$$

The term $\mathcal{L}_0$ contains terms involving the U(1) gauge field which dominate the long-time and long-wavelength fluctuations

$$\mathcal{L}_0=\int\frac{d^dr}{2a^d}\left[b^\dagger_\alpha\left[\frac{d}{d\tau}+iA_\tau\right]b^\alpha+\bar b^{\alpha\dagger}\left[\frac{d}{d\tau}-iA_\tau\right]\bar b_\alpha+(\bar\lambda+i\hat\lambda)(|b^\alpha|^2+|\bar b_\alpha|^2)-4\bar Q(b^\alpha\bar b_\alpha+b^\dagger_\alpha\bar b^{\alpha\dagger})\right]$$
$$+\int\frac{d^d\mathbf{r}}{2a^{d-2}}\bar Q[(\nabla_\mathbf{r}+i\mathbf{A})b^\alpha(\nabla_\mathbf{r}-i\mathbf{A})\bar b_\alpha+(\nabla_\mathbf{r}-i\mathbf{A})b^\dagger_\alpha(\nabla_\mathbf{r}+i\mathbf{A})\bar b^{\alpha\dagger}] . \quad (3.13)$$

It is clear from this equation that b and $\bar b$ have charges $+1$ and -1 under the U(1) gauge transformation, i.e., $\mathcal{L}_0$ (and also $\mathcal{L}_1$ below) is invariant under the transformations $b\to be^{i\phi}$, $\bar b\to\bar b e^{-i\phi}$, $\mathbf{A}\to\mathbf{A}-\nabla\phi$, and $A_\tau\to A_\tau-\partial_\tau\phi$. All remaining terms from $\mathcal{L}$ have been absorbed into $\mathcal{L}_1$

$$\mathcal{L}_1=\int\frac{d^dr}{2a^d}\left[\sum_{\hat\eta}\frac{N(\bar Q+q_{\hat\eta})^2}{J}-\left[\sum_{\hat\eta}q_{\hat\eta}\right](b^\alpha\bar b_\alpha+b^\dagger_\alpha\bar b^{\alpha\dagger})+a\sum_{\hat\eta>0}(q_{\hat\eta}-q_{-\hat\eta})\left[\frac{\partial b^\alpha}{\partial r_{\hat\eta}}\bar b_\alpha-\frac{\partial\bar b_\alpha}{\partial r_{\hat\eta}}b^\alpha+\frac{\partial b^\dagger_\alpha}{\partial r_{\hat\eta}}\bar b^{\alpha\dagger}-\frac{\partial\bar b^{\alpha\dagger}}{\partial r_{\hat\eta}}\right]\right]\cdots . \quad (3.14)$$

We have explicitly displayed the couplings between the $b,\bar b$ fields and the amplitude fluctuation mode q; the ellipses denote additional terms involving higher spatial derivatives and couplings of the $b,\bar b$ to fluctuations of the gauge field which are not near the Brillouin-zone center.

At distances much larger than the lattice spacing, $\mathcal{L}_1$ can be neglected and the fluctuations are controlled by $\mathcal{L}_0$. We find it convenient to introduce the boson fields

$$z^\alpha=(b^\alpha+\bar b^{\alpha\dagger})/2 ,$$

$$\pi^\alpha=(b^\alpha-\bar b^{\alpha\dagger})/2 .$$

From Eq. (3.13), it is clear that the π fields turn out to have mass $\bar\lambda+4\bar Q$, while the z fields have a mass $\bar\lambda-4\bar Q$ which vanishes at the transition to the Néel phase. The π fields can therefore be safely integrated out and $\mathcal{L}_0$ yields the following effective action, valid at distances much larger than the lattice spacing:

$$S'_{\rm eff}=\infty d^dr\int_0^{c\beta}d\tilde\tau\frac{a^{1-d}}{2\sqrt{d}}\left[|(\partial_\mu-iA_\mu)z^\alpha|^2+\frac{\Delta^2}{c^2}|z^\alpha|^2\right] . \quad (3.15)$$

Here $\tilde\tau=c\tau$, $A_{\tilde\tau}=A_\tau/c$, and μ runs over the $d+1$ coordinates $\hat x,\hat y,\ldots,\tilde\tau$. Remarkably, this action is identical in form to the CP^{N-1} action (2.14) obtained in the semiclassical limit in Sec. II B. The constraint $|z^\alpha|^2=1$ was imposed in the CP^{N-1} model by the fluctuations of the Lagrange multiplier [see Eq. (2.12) and Refs. 31 and 32]; a similar role will be played here by the field $\hat\lambda$ which couples to $i|z^\alpha|^2$. In the large-N limit these fluctuations are unimportant at distances larger than ξ (Refs. 31 and 32) and have therefore been omitted here.

However, there are several advantages to the present derivation of the action $S'_{\rm eff}$ over that in Sec. II.

(1) We have a microscopic interpretation of the spatial components of the gauge field $\mathbf{A}$ as the phase of a bond variable.

(2) The compact nature of the gauge-field fluctuations are apparent from the definition in Eq. (3.9)—this fact will be crucial in understanding topological effects below.

(3) As shown in Eq. (3.17) below, the present large-N limit is useful in describing the lattice-scale coupling to the spin-Peierls order parameter.

To complete the large-N calculation we must now integrate out the N-component b^α and $\bar b_\alpha$ bosons to obtain an effective action $S_{\rm eff}$ for the A, M, and q fields. We obtain

$$S_{\rm eff}=S^0_{\rm eff}+S^1_{\rm eff} .$$

The first term, $S^0_{\rm eff}$ contains the contributions of $\mathcal{L}_0$; at distances much larger than ξ, this is most easily determined by integrating out the z^α fields from Eq. (3.15) yielding

$$S^0_{\rm eff}=N\int d^dr\int_0^{c\beta}d\tilde\tau\frac{1}{4e^2}F^2_{\mu\nu} , \quad (3.16)$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromagnetic field; in

$d=2$, $F_{\hat{x}\hat{y}}$, $F_{\hat{x}\hat{\tau}}$, and $F_{\hat{x}\hat{\tau}}$ are the continuum limits of the lattice fields B, $E_{\hat{x}}$, and $E_{\hat{y}}$ [Eqs. (3.11) and (3.12)]. The coupling constant $e^2 \sim (\Delta/c)^{3-d}$.

The terms in S^1_{eff} will be irrelevant in the discussion of the long-wavelength fluctuations. They will, however, be necessary later in understanding the consequence of global symmetry breaking on lattice scale physics. Anticipating some of our subsequent results, we write S^1_{eff} in a form in which the continuum limit is taken with two sites per unit cell. In $d=2$ there will therefore be four electric fields $E_{\hat{x}}(i)$, $E_{\hat{y}}(i)$, $E_{-\hat{x}}(i)$, and $E_{-\hat{y}}(i)$, the two magnetic fields $B(i)$ and $B(i+\hat{x})$, and the four amplitude fields $q_{\hat{x}}(i)$, $q_{\hat{y}}(i)$, $q_{-\hat{x}}(i)$, and $q_{-\hat{y}}(i)$, all of which are assumed to be smooth functions of $i \in A$. We obtain, in $d=2$,

$$S^1_{\text{eff}} = N \int d^2r \int_0^\beta d\tau \sum_{p=0}^{3} \left(\tfrac{1}{2} c_p \Psi_p \Psi_{4-p} + i\gamma_p \Psi_p E_{4-p}\right), \tag{3.17}$$

where c_p and γ_p are coupling constants and

$$\Psi_p(i) = \sum_{\hat{\eta}} (\vartheta_{i,i+\hat{\eta}})^p q_{\hat{\eta}}(i), \qquad E_p(i) = \sum_{\hat{\eta}} (\vartheta_{i,i+\hat{\eta}})^p E_{\hat{\eta}}(i) \tag{3.18}$$

for $i \in A$. The complex numbers $\theta_{i,i+\hat{\eta}}$ take the fixed values 1, i, -1, and $-i$ as shown in Fig. 2; thus, $E_p = E^*_{4-p}$, $\Psi_p = \Psi^*_{4-p}$, $c_p = c_{4-p}$, $\gamma_p = \gamma_{4-p}$, $E_0 = E_{\hat{x}} + E_{\hat{y}} + E_{-\hat{x}} + E_{-\hat{y}}$, $E_1 = E_{\hat{x}} + iE_{\hat{y}} - E_{-\hat{x}} - iE_{-\hat{y}}$, etc. From the relationship (3.10) it is clear that the Ψ_p are proportional to the spin-Peierls order parameters Ψ_p discussed in the Introduction (Sec. I). The coupling constants c_p have the orders of magnitude $c_0 \sim 1/(\Delta a^2)$ and $c_p \sim 1/(\bar{\lambda} a^2)$ for $p \neq 0$. Similarly we find $\gamma_0 \sim \bar{\lambda}/\Delta$ and $\gamma_p \sim 1$ for $p \neq 0$. Note that to obtain an accurate value of the couplings c_p, γ_p for $p \neq 0$, it is necessary to use the full lattice Lagrangian $\mathcal{L}$ in Eq. (3.2) and not its approximate continuum limit forms in Eqs. (3.13) and (3.14). The form of S^1_{eff} in $d=1$ is very similar to the one above, with the summation over p extending over the two values 0 and 1 and ϑ taking the values ± 1 on alternate links of the chain.

It is clear that the terms of S_{eff} so far do not change any essential features of the mean field-analysis. One expects that, in all orders in a $1/N$ expansion, the fluctuation corrections will merely renormalize the mean-field parameters and be insensitive to the value of n_c (mod 4). Such a sensitivity, however, arises when we consider the effects of topologically nontrivial gauge-field configurations. Such configurations are expected to give rise to Berry phase factors in the functional integral. These phases should be obtainable by integrating out $b, \bar{b}$ in the presence of a nontrivial background gauge field. However, it is much simpler to perform the equivalent procedure of calculating the phase due to adiabatic evolution of the ground state (3.8) in such a gauge-field background. We discuss the results of such a calculation in the next two subsections.

A. Calculation of Berry phases

In this section we will present the calculation of the Berry phase acquired by the ground state $|\Omega\rangle$ under a topologically nontrivial adiabatic evolution in the spatial lattice gauge field $\mathbf{A}$, the phase of the Q_{ij} field [Eq. (3.9)]. We begin by developing the framework of the calculation; applications to the case $d=1$ and 2 will be presented in the subsequent subsections.

We will be interested in the ground state of Hamiltonians of the following form:

$$H_B = -\sum_{i\in A, j\in B} [Q_{ij}(\tau) b^\alpha(i) \bar{b}_\alpha(j) + \text{H.c.}] + \sum_{i\in A} \lambda_i(\tau) b^\dagger_\alpha(i) b^\alpha(i) + \sum_{j\in B} \lambda_j(\tau) \bar{b}^{\alpha\dagger}(j) \bar{b}_\alpha(j), \tag{3.19}$$

which is the generalization of H_{MF} [Eq. (3.3)] to the case of a space-dependent link field Q_{ij} (with $Q_{ji} = Q_{ij}$) which evolves adiabatically as a function of τ. The Lagrange multiplier field λ also has a τ dependence: this is required to maintain the average boson occupation number constraint

$$\langle b^\dagger_\alpha b^\alpha \rangle = \langle \bar{b}^{\alpha\dagger} \bar{b}_\alpha \rangle = n_c$$

for all values of τ.[11] We will determine the ground state of H_B for arbitrary values of Q_{ij} and λ_i, λ_j sufficiently large and positive so that there is a gap towards excitations above the ground state. The consequences of any special symmetry properties of H_B will be explored at the end of this section. The τ dependence of all quantities will be implicitly assumed.

A compact vector notation is particularly useful for the subsequent manipulations. In a system with N_s sites, we introduce the N_s component vector of operators Γ^α:

$$\Gamma^\alpha = \begin{bmatrix} b^\alpha(i) \\ \bar{b}^{\alpha\dagger}(j) \end{bmatrix}. \tag{3.20}$$

This operator clearly satisfies the commutation relations $[\Gamma^\alpha, \Gamma^\dagger_\beta] = \delta^\alpha_\beta \tau_3$, where

$$\tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.21}$$

with the entries being $(N_s/2)\times(N_s/2)$ matrices. The Hamiltonian has the form $H_B = \Gamma^\dagger_\alpha M \Gamma^\alpha$ with

$$M = \begin{bmatrix} \lambda_i & -Q_{ij} \\ -Q^*_{ji} & \lambda_j \end{bmatrix}. \tag{3.22}$$

We introduce a second N_s component vector of operators which will ultimately label the eigenmodes of H_B

$$\Lambda^\alpha = \begin{bmatrix} \gamma^\alpha_1(\sigma) \\ \gamma^{\alpha\dagger}_2(\rho) \end{bmatrix}. \tag{3.23}$$

Here $\sigma = 1, \ldots, N_s/2$ and $\rho = 1, \ldots, N_s/2$ label the components of Λ and $\gamma^\alpha_1, \gamma_{2\alpha}$ are operators satisfying the canonical bosonic commutation relations; thus,

$[\Lambda^\alpha,\Lambda^\dagger_\beta]=\delta^\alpha_\beta\tau_3$. We express Γ in terms of Λ by $\Gamma^\alpha=T\Lambda^\alpha$, where T is a $N_s\times N_s$ matrix. Consistency of the commutation relations requires

$$T\tau_3T^\dagger=\tau_3,\quad T^\dagger\tau_3T=\tau_3\ , \tag{3.24}$$

which shows that T is an element of the group $\mathrm{U}(N_s/2,N_s/2)$. In terms of the Λ operators, the Hamiltonian becomes

$$H_B=\Gamma^\dagger_\alpha M\Gamma^\alpha=\Lambda^\dagger_\alpha T^\dagger MT\Lambda^\alpha\equiv\Lambda^\dagger_\alpha\epsilon\Lambda^\alpha\ . \tag{3.25}$$

The transformation T will be chosen so that the matrix ϵ is diagonal. Thus, up to an additive constant,

$$H_B=\sum_\sigma\epsilon_{1\sigma}\gamma^\dagger_{1\alpha}(\sigma)\gamma^\alpha_1(\sigma)+\sum_\rho\epsilon_{2\rho}\gamma^{\alpha\dagger}_2(\rho)\gamma_{2\alpha}(\rho)\ . \tag{3.26}$$

Stability requires that all the diagonal elements $\epsilon_{1\sigma},\epsilon_{2\rho}$ be positive. Combining Eqs. (3.24) and (3.25) we observe

$$\tau_3MT=T\tau_3\epsilon\ . \tag{3.27}$$

In other words, the columns of T are the right eigenvectors of the matrix τ_3M with the positive eigenvalues labeled by the index σ and the negative eigenvalues by ρ. In terms of the parametrization

$$T=\begin{bmatrix}U_1 & V_2\\ V_1 & U_2\end{bmatrix}, \tag{3.28}$$

where the entries are $(N_s/2)\times(N_s/2)$ matrices, the eigenvalue equations are

$$\begin{aligned}\begin{bmatrix}\lambda_i & -Q_{ij}\\ Q^*_{ji} & -\lambda_j\end{bmatrix}\begin{bmatrix}U_{1i\sigma}\\ V_{1j\sigma}\end{bmatrix}&=\epsilon_{1\sigma}\begin{bmatrix}U_{1i\sigma}\\ V_{ij\sigma}\end{bmatrix},\\ \begin{bmatrix}\lambda_i & -Q_{ij}\\ Q^*_{ji} & -\lambda_j\end{bmatrix}\begin{bmatrix}V_{2i\rho}\\ U_{2j\rho}\end{bmatrix}&=-\epsilon_{2\rho}\begin{bmatrix}V_{2i\rho}\\ U_{2j\rho}\end{bmatrix}.\end{aligned} \tag{3.29}$$

The separation above into the sets of eigenvalues ϵ_1 and ϵ_2 is made by requiring that they all be positive. For sufficiently large λ_i,λ_j (which we assume is the case), there will be equal numbers in both sets. In the absence of any special symmetries in H_B, the eigenvalues in the two sets will not be equal in pairs. Solution of the eigenvalue equations on (3.29) thus yields the complete set of excitation energies of H_B via (3.26). We note the eigenvectors are determined up to the following overall global phase changes:

$$(U_{1i\sigma},V_{1j\sigma})\mapsto\exp(i\varphi_{1\sigma})(U_{1i\sigma},V_{1j\sigma})\ , \tag{3.30}$$

$$(U_{2j\rho},V_{2i\rho})\mapsto\exp(i\varphi_{2\rho})(U_{2j\rho},V_{2i\rho})\ , \tag{3.31}$$

where $\varphi_{1\sigma},\varphi_{2\sigma}$ can be chosen arbitrarily.

Finally, the ground-state wave function $|\Omega\rangle$ of H_B can be determined by requiring $\gamma^\alpha_i(\sigma)|\Omega\rangle=0$ and $\gamma_{2\alpha}(\rho)|\Omega\rangle=0$. This yields

$$|\Omega\rangle=\frac{|\tilde\Omega\rangle}{\langle\tilde\Omega|\tilde\Omega\rangle^{1/2}}\ ,$$

where

$$|\tilde\Omega\rangle=\exp\left[\sum_{i,j}f_{ij}b^\dagger_\alpha(i)\bar b^{\,\alpha\dagger}(j)\right]|0\rangle\ . \tag{3.32}$$

The pair wave function f_{ij} is given by

$$f_{ij}=\sum_\sigma(U_1^{-1\dagger})_{i\sigma}(V_1^\dagger)_{\sigma j}=\sum_\rho V_{2i\rho}(U_2^{-1})_{\rho j}\ , \tag{3.33}$$

where the second equality follows from one of the equations implied by (3.24). We make the important observation that the phase changes in Eq. (3.31) do not affect the value of f_{ij}. Thus, the specification (3.32) uniquely fixes the phase of $|\Omega\rangle$ in a natural manner.

One portion of the gauge-invariant Berry phase associated with the evolution of H_B as τ evolves from $\tau=0$ to β is the integral of

$$\left\langle\Omega\left|\frac{d}{d\tau}\right|\Omega\right\rangle=\frac{i}{\langle\tilde\Omega|\tilde\Omega\rangle}\mathrm{Im}\left\langle\tilde\Omega\left|\frac{d}{d\tau}\right|\tilde\Omega\right\rangle\ . \tag{3.34}$$

Using the wave function in (3.32) and the two equivalent definitions of f_{ij} in (3.33), we obtain two equivalent expressions for $\langle\Omega|(d/d\tau)|\Omega\rangle$:

$$\begin{aligned}\left\langle\Omega\left|\frac{d}{d\tau}\right|\Omega\right\rangle&=i\,\mathrm{Im\,Tr}\left[V_2^\dagger V_2\left[V_2^{-1}\frac{dV_2}{d\tau}-U_2^{-1}\frac{dU_2}{d\tau}\right]\right]\\ &=-i\,\mathrm{Im\,Tr}\left[V_1^\dagger V_1\left[V_1^{-1}\frac{dV_1}{d\tau}-U_1^{-1}\frac{dU_1}{d\tau}\right]\right]\ .\end{aligned} \tag{3.35}$$

Note that both expressions are invariant under the global phase rotation in Eq. (3.31) and are nonzero in the absence of any special symmetry. However, for the configurations we shall consider, H_B will often have a sublattice symmetry. This symmetry may be realized by any mapping $\mathcal{S}$ which interchanges the two sublattices:

$$\mathcal{S}\begin{cases}i\in A\mapsto\bar j\in B\\ j\in B\mapsto\bar i\in A\end{cases}\text{ such that }Q_{\bar i\bar j}=Q^*_{ji},\ \lambda_{\bar i}=\lambda_j,\text{ and }\lambda_{\bar j}=\lambda_i\ . \tag{3.36}$$

Under these circumstances we may easily show from from the eigenvalue equations (3.29) that we can choose the set (U_2,V_2) and the dummy index ρ such that

$$\epsilon_{2\rho}=\epsilon_{1\sigma},\quad U_{2j\rho}=U_{1\bar i\sigma},\quad V_{2i\rho}=V_{1\bar j\sigma}\ . \tag{3.37}$$

For this case the two expressions in Eq. (3.35) are equal but have opposite signs and hence must vanish. We thus have the important result that

$$\left\langle\Omega\left|\frac{d}{d\tau}\right|\Omega\right\rangle=0 \tag{3.38}$$

for any system which has a sublattice symmetry $\mathcal{S}$.

The vanishing of $\langle\Omega|(d/d\tau)|\Omega\rangle$, however, does not imply that the Berry phase is zero. The second portion of the gauge-invariant Berry phase is simply the difference in phase between the ground state at $\tau=\beta$ and 0. Thus, if

$$|\Omega(\tau=\beta)\rangle=\exp(i\Upsilon)|\Omega(\tau=0)\rangle ,$$

then the total Berry phase iS_B is

$$S_B=\Upsilon+i\int_0^\beta d\tau\left\langle\Omega\left|\frac{d}{d\tau}\right|\Omega\right\rangle . \tag{3.39}$$

We will examine the details of its evaluation in $d=1,2$ in turn.

1. d=1

The calculations in Sec. II A showed that the topological term in the CP^{N-1} model could be written in the form

$$\frac{i\Theta}{2\pi}\int dx\, d\tilde{\tau}F_{x\tilde{\tau}} \tag{3.40}$$

with $\Theta=\pi n_c$. It is clear, therefore, that the phase of the Q_{ij} [which is gauge field $\mathbf{A}$ by Eq. (3.9)] must be chosen such that $\int dx\, d\tilde{\tau}F_{x\tilde{\tau}}$ is nonzero. This can be achieved in a spin chain with N_s sites (N_s even) and periodic boundary by choosing $q_{\hat{\eta}}(i)=0$ and

$$A_{\hat{x}}=\phi(\tau)/a , \tag{3.41}$$

where $\phi(\tau)$ increases slowly from 0 at $\tau=0$ to the gauge equivalent value $2\pi/N_s$ at $\tau=\beta$. This clearly yields

$$\int dx\, d\tilde{\tau}F_{x\tilde{\tau}}=2\pi . \tag{3.42}$$

Moreover, it is easy to show that the boson occupation number constraints are maintained by the choice $\lambda_i=\lambda_j=\bar{\lambda}$ for all τ. Under these conditions the eigenvalue equations can be solved exactly for all τ. The eigenvectors are labeled by the $N_s/2$ momenta k_l where $l=0,1,\ldots,(N_s/2-1)$ and $k_l=2\pi l/aN_s-\pi/(2a)$. We have

$$\epsilon_{1l}=\epsilon_{2l}=\{\bar{\lambda}^2-4\bar{Q}^2\cos^2[k_la+\phi(\tau)]\}^{1/2} \tag{3.43}$$

and

$$\begin{aligned} U_{1il}&=e^{ik_lr_i}\cosh\theta_l ,\\ V_{1jl}&=e^{ik_lr_j}\sinh\theta_l ,\\ U_{2jl}&=e^{ik_lr_j}\cosh\theta_l ,\\ V_{2il}&=e^{ik_lr_i}\sinh\theta_l ,\end{aligned} \tag{3.44}$$

where

$$\tanh 2\theta_l=2\bar{Q}\cos[k_la+\phi(\tau)]/\bar{\lambda} .$$

Notice that the indices σ and ρ in Eq. (3.9) have been replaced by a single index l.

It can now be checked by explicit evaluation of the expression (3.35) that $\langle\Omega|(d/d\tau)|\Omega\rangle=0$. Alternatively, we note that translation of the chain by one lattice spacing constitutes a valid sublattice symmetry operation $\mathcal{S}$ (3.36), the existence of which was shown above to lead to the vanishing of $\langle\Omega|(d/d\tau)|\Omega\rangle$.

To compute the phase difference Υ of the ground state between $\tau=0$ and β we now examine the τ dependence of the wave function more closely. It is not difficult to see that the eigenvalues and eigenfunctions (3.44) at $\tau=0$ and β are related by the gauge transformation that transforms $A_{\hat{x}}(0)$ to $A_{\hat{x}}(\beta)$:

$$\begin{aligned} \epsilon_{1l}(\tau=0)&=\epsilon_{1l'}(\tau=\beta) ,\\ \epsilon_{2l}(\tau=0)&=\epsilon_{2l'}(\tau=\beta) ,\\ U_{1il}(\tau=\beta)&=\exp(i\varphi_{1l})\exp\left[-i\frac{2\pi}{N_sa}r_i\right]U_{1il'}(\tau=0) ,\\ V_{1jl}(\tau=\beta)&=\exp(i\varphi_{1l})\exp\left[-i\frac{2\pi}{N_sa}r_j\right]V_{1jl'}(\tau=0) ,\\ U_{2jl}(\tau=\beta)&=\exp(i\varphi_{2l})\exp\left[-i\frac{2\pi}{N_sa}r_j\right]U_{2jl'}(\tau=0) ,\\ V_{2il}(\tau=\beta)&=\exp(i\varphi_{2l})\exp\left[-i\frac{2\pi}{N_sa}r_i\right]V_{2il'}(\tau=0) ,\end{aligned} \tag{3.45}$$

where $l'=l+1$ [mod $(N_s/2)$] and the position-independent phases $\varphi_{1l},\varphi_{2l}$ are

$$\begin{aligned} &\varphi_{1l}=0 \text{ for all } l ,\\ &\varphi_{2l}=\begin{cases}\pi & \text{for } l=N_s/2-1,\\ 0 & \text{otherwise} .\end{cases}\end{aligned} \tag{3.46}$$

Inserting the above into the pair wave function f_{ij} in Eq. (3.33) we find

$$f_{ij}(\tau=\beta)=\exp\left[-i\frac{2\pi}{N_sa}(r_i-r_j)\right]f_{ij}(\tau=0) . \tag{3.47}$$

As expected, the phases $\varphi_{1l},\varphi_{2l}$ have dropped out of the above expression. The evolution in the phase of f_{ij} will naturally lead to changes in $|\Omega\rangle$. It can easily be shown from the expression (3.32) that

$$\begin{aligned} P_{n_c}|\Omega(\tau=\beta)\rangle&=\exp\left[-in_c\frac{2\pi}{N_sa}\left[\sum_i r_i-\sum_j r_j\right]\right]\\ &\quad\times P_{n_c}|\Omega(\tau=0)\rangle ,\\ &=\exp(-i\pi n_c)P_{n_c}|\Omega(\tau=0)\rangle ,\end{aligned} \tag{3.48}$$

where P_{n_c} projects onto n_c bosons per site. The gauge-invariant Berry phase, $S_B=\Upsilon$, therefore equals πn_c (mod 2π). For the functional integral over the action S^0_{eff} in Eq. (3.16) to reproduce this phase, it is clear that we have to add a topological term as in Eq. (3.40) with $\Theta=p\pi$, where the integer p is restricted to be even (odd) if n_c is even (odd). We have thus shown, directly in the disordered phase, the existence of the topological Θ term obtained earlier (Sec. II A) by a semiclassical method.

2. $d=2$

As in $d=1$, we use the insight offered by the semiclassical analysis. The nontrivial Berry phases are expected to arise from Skyrmion number-changing "instanton"[27] tunneling events. We choose $q_{\hat{\eta}}(i)=0$, and the phase of Q_{ij} [Eq. (3.9)] given by the following gauge field for a configuration of instantons with integer charges m_s located at times $\tau=\tau_s$ and spatial coordinates $\mathbf{R}_s$ (at the centers of plaquettes of the lattice of spins):

$$A_{\hat{x}}(i)=-\sum_s m_s A_{\theta_s}(i)\sin[\theta_s(i)] ,\quad A_{\hat{y}}(i)=\sum_s m_s A_{\theta_s}(i)\cos[\theta_s(i)] , \tag{3.49}$$

where

$$\theta_s(i)=\tan^{-1}\left[\frac{y_i-Y_s}{x_i-X_s}\right] \tag{3.50}$$

is the azimuthal angle from $\mathbf{R}_s$ to the point i, The vector potential

$$A_{\theta_s}(i)=\frac{1}{2|\mathbf{r}_i-\mathbf{R}_s|}\left[1+\frac{c(\tau-\tau_s)}{[c^2(\tau-\tau_s)^2+(\mathbf{r}_i-\mathbf{R}_s)^2]^{1/2}}\right] \tag{3.51}$$

is identical to that associated with a Dirac monopole at $(\mathbf{R}_s,\tau_s)$ in three-dimensional space-time. Note that we have taken the zero-temperature limit and the time τ varies from $-\infty$ to $+\infty$. The Dirac "string" associated with the monopole extends "vertically" from $(\mathbf{R}_s,\tau_s)$ to $\tau=+\infty$. It follows from Eq. (3.51) that as $\tau\to-\infty$, the magnetic flux through a spatial circle of fixed radius r tends to zero. The charge neutrality condition $\sum_s m_s=0$ ensures that this holds even if we send $r\to\infty$ before taking $\tau\to-\infty$. Thus, at $\tau=-\infty$, we may safely take $A_{\hat{\eta}}(i)=0$ everywhere. However, in the limit $\tau\to+\infty$ it is easy to show that $A_{\hat{\eta}}(i)$ is very closely approximated by

$$A_{\hat{\eta}}(i)=\sum_s m_s[\theta_s(i+\hat{\eta})-\theta_s(i)] , \tag{3.52}$$

where there are no branch cuts in the θ_s functions across the link $(i,i+\hat{\eta})$. The magnetic flux associated with this vector potential is $2\pi m_s$ on the plaquettes $\mathbf{R}_s$ and zero on all other plaquettes. The local Skyrmion number Q has therefore been changed by m_s in the neighborhood of $\mathbf{R}_s$. The Lagrange multipliers λ_i,λ_j will now have to be position and τ dependent to properly enforce the boson occupation number constraint on the average. A closed-form determination of the λ_i,λ_j is not possible; certain asymptotic limits were determined in Ref. 11. Of course, in the limits $\tau\to\infty$ and $\tau\to-\infty$ we will have $\lambda_i=\lambda_j=\bar{\lambda}$.

Analytic evaluation of the wave function is also not possible at all τ. Considerable progress can, however, be made for the case of a single instanton which we now consider. The eigenvalue equations (3.29) are invariant under operations of the group Z_4 of 90° rotations about the center of the plaquette containing the instanton. A single 90° rotation constitutes a valid sublattice symmetry $\mathcal{S}$ (3.36), which implies that

$$\left\langle\Omega\left|\frac{d}{d\tau}\right|\Omega\right\rangle=0 . \tag{3.53}$$

Moreover, the wave functions $(U_1,V_1),(V_2,U_2)$ can be classified by an integer $p=0,1,2,3$ labeling the representations of Z_4 under which they transform. We will thus replace the indices σ and ρ in Eq. (3.29) by the pairs $(p,\bar{\sigma})$ and $(p,\bar{\rho})$. These indices remain fixed as a particular state evolves with τ. Although the wave functions are not known at intermediate times, the generalization of the relations (3.45) to $d=2$ is straightforward. By examining the structure of (3.29) and the vector potential at $\tau=-\infty$ and $\tau=+\infty$ [Eq. (3.52)], it can be shown that there is a one-to-one map from the eigenfunctions labeled by the pairs $(p,\bar{\sigma})$ and $(p,\bar{\rho})$ to those labeled by $(p',\bar{\sigma}')$ and $(p',\bar{\rho}')$, where $p'=p+1$ (mod 4) and

$$\begin{aligned}
&\epsilon_{1p\bar{\sigma}}(\tau=+\infty)=\epsilon_{1p'\bar{\sigma}'}(\tau=-\infty) ,\\
&\epsilon_{2p\bar{\rho}}(\tau=+\infty)=\epsilon_{2p'\bar{\rho}'}(\tau=-\infty) ,\\
&U_{1ip\bar{\sigma}}(\tau=+\infty)=\exp(i\varphi_{1p\bar{\sigma}})\exp[-im_s\theta_s(i)]\\
&\qquad\times U_{1ip'\bar{\sigma}'}(\tau=-\infty) ,\\
&V_{1jp\bar{\sigma}}(\tau=+\infty)=\exp(i\varphi_{1p\bar{\sigma}})\exp[-im_s\theta_s(j)]\\
&\qquad\times V_{1jp'\bar{\sigma}'}(\tau=-\infty) ,\\
&U_{2jp\bar{\rho}}(\tau=+\infty)=\exp(i\varphi_{2p\bar{\rho}})\exp[-im_s\theta_s(j)]\\
&\qquad\times U_{2jp'\bar{\rho}'}(\tau=-\infty) ,\\
&V_{2ip\bar{\rho}}(\tau=+\infty)=\exp(i\varphi_{2p\bar{\rho}})\exp[-im_s\theta_s(i)]\\
&\qquad\times V_{2ip'\bar{\rho}'}(\tau=-\infty) .
\end{aligned} \tag{3.54}$$

The position-independent phases $\varphi_{1p\bar{\sigma}},\phi_{2p\bar{\rho}}$ must be appropriately chosen and clearly depend upon the particular global phase choices made for the wave functions at intermediate times. Notice that the map between eigenstates is nontrivial as there is a change in the Z_4 quantum number from p to $p'=p+1$ (mod 4). Again, the relationship (3.54) is just the singular gauge transformation needed to remove the unit flux at $\tau=\infty$. While the expressions for U_1, V_1, U_2, and V_2 are quite involved, a considerable simplification occurs in the pair wave function f_{ij} in which the indices $(p,\bar{\sigma})$ and $(p,\bar{\rho})$ are summed over. Inserting the above results into (3.33) we find

$$f_{ij}(\tau=+\infty)=\exp\{-im_s[\theta_s(i)-\theta_s(j)]\}f_{ij}(\tau=-\infty) . \tag{3.55}$$

Notice that, as in $d=1$, all dependence on φ_1,φ_2 has dropped out.

The discussion now returns to the multi-instanton case.

There is now no general argument to show that $\langle \Omega | (d/d\tau) | \Omega \rangle$ vanishes. For instantons which are well separated we expect that this quantity will be quite small and, due to the finite gap in the spectrum, vanish exponentially with the separation between the instantons. The transformation (3.55) in the pair wave function f_{ij}, however, has a straightforward generalization. From Eqs. (3.29), (3.33), and (3.52) we can show

$$f_{ij}(\tau=+\infty)=\exp\left[-i\sum_s m_s[\theta_s(i) - \theta_s(j)]\right] f_{ij}(\tau=-\infty) . \tag{3.56}$$

The evolution in the phase of f_{ij} again leads to changes in $|\Omega\rangle$. As in $d=1$, it can be shown from (3.32) that

$$P_{n_c}|\Omega(\tau=+\infty)\rangle = e^{i\Upsilon} P_{n_c}|\Omega(\tau=-\infty)\rangle , \tag{3.57}$$

where

$$\Upsilon = S_B(n_c) = -in_c\sum_s m_s\left[\sum_{i\in A}\theta_s(i) - \sum_{j\in B}\theta_s(j)\right] . \tag{3.58}$$

In equating Υ to $S_B(n_c)$ we have used (3.39) and neglected any small correction from a nonzero integral of $\langle \Omega | (d/d\tau) | \Omega \rangle$. The spatial summation above is exactly the same as that considered by Haldane[10] in his semiclassical evaluation of the Berry phase of a hedgehog. We may therefore conclude from his analysis that

$$S_B(n_c) = \sum_s i\frac{n_c\pi}{2}\zeta_s m_s , \tag{3.59}$$

where the integer $\zeta_s = 0,1,2,3$ for $\mathbf{R}_s$ on four dual sublattices W,X,Y,Z (Fig. 7).

The Berry phase above has been calculated for a special gauge choice in $\mathbf{A}$, the phase of the Q_{ij}. However, as argued by Haldane,[10] it is clear that the final result in Eq. (3.59) is a topological invariant and is independent of the specific choice of the θ_s field. The minimum conditions necessary to obtain Eq. (3.59) from Eq. (3.58) are (*i*) the instantons are dilute, (ii) the field θ_s is smooth everywhere except near the cores of the instantons and at branch cuts of strength $2\pi m_s$ originating from each instanton, and (iii) the gradients of θ_s have the full rotational symmetry of the square lattice at the cores of the instantons.

B. Consequences of Berry phases

This section combines the results of Secs. III and III A to deduce physical consequences for the disordered ground state of $\mathcal{H}$. The discussion will begin by considering the simpler $d=1$ case, where our results agree with previous calculations on spin chains.[7,26,39] We then derive new results in $d=2$.

1. d=1

The phases associated with topologically nontrivial gauge-field configurations (Sec. III A) imply that S^0_{eff} [Eq. (3.16)] has to be modified to

$$S^0_{\text{eff}} = \int dx \int_0^{c\beta} d\tilde{\tau} \left[\frac{N}{2e^2}F^2_{x\tilde{\tau}} + \frac{i\Theta}{2\pi}F_{x\tilde{\tau}}\right] \tag{3.60}$$

with $\Theta = p\pi$ and $(-1)^p = (-1)^{n_c}$. The ground state of this action for fixed Θ has a mean static electric field[25]

$$\langle iF_{x\tilde{\tau}}\rangle = e^2 p/N \tag{3.61}$$

and energy per site $ce^2p^2a/(8N)$. For a given n_c, the ground state of the spin chain is therefore associated with $p=0$ for n_c even and with $p=\pm 1$ for n_c odd. Using the coupling between the electric field and the $q_{\hat{\eta}}$ fields in Eq. (3.17), we deduce the presence of a spin-Peierls order parameter

$$\langle \hat{S}(i)\cdot\hat{S}(i+1) - \hat{S}(i)\cdot\hat{S}(i-1)\rangle = \frac{2N^2\overline{Q}\langle q_{\hat{x}} - q_{-\hat{x}}\rangle}{J^2} = \frac{N\overline{Q}\gamma_1 e^2 p}{J^2 c_1} . \tag{3.62}$$

The ground state for n_c even is therefore obtained with the choice $p=0$ and is nondegenerate; the linear Coulomb force confines the z^α bosons (spinons) in pairs. For n_c odd the ground state corresponds to $p=\pm 1$ and is twofold degenerate with a nonzero spin-Peierls order parameter; the spinons are domain walls interpolating between the two ground states. A schematic of the two ground states is shown in Figs. 6(a) and 6(b). The spin-Peierls order for n_c odd was anticipated by Affleck[26] though not shown directly for $n_c \sim N$. This picture is now expected to be correct for all $N>2$.[4,26] Only for $N=2$ and odd values of n_c does a massless system described by the $k=1$ Wess-Zumino-Witten model appear.[39]

2. d=2

In the continuum limit, the field configurations with nonzero Berry phases considered in Sec. III A 2 are solutions of the equations[27]

$$H_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\lambda}F_{\nu\lambda} ,$$
$$\nabla\times\mathbf{H} = 0, \quad \nabla\cdot\mathbf{H} = 2\pi m\,\delta(r-r') ,$$

for an instanton at r'. These equations are identical in form to those of ordinary three-dimensional electrostatics with $\mathbf{H}$ playing the role of the electric field. The analogy implies a Coulombic $1/r$ interaction between the instanton charges. Note that the $1/r$ form for the interaction is only valid at length scales larger than ξ, the spin-correlation length. In contrast, in the ordered Néel phase (Fig. 4), the instantons are hedgehogs interacting with each other by an asymptotically linear r potential.

The subsequent analysis closely follows Polyakov's solution[27] of (2+1)-dimensional compact quantum electrodynamics (QED). Neglecting all fields except the gauge field A_μ at distances $>\xi=c/2\Delta$, the action $S^0_{\rm eff}$ [Eq. (3.16)] is evaluated for each instanton configuration; this yields the following effective partition function for a gas of instantons with integer charges m_s located at space-time coordinates $(\mathbf{R}_s,\bar{\tau}_s)$:

$$Z=\sum_{K,\{m_s\}}\frac{1}{K!}\prod_{s=1}^{K}\left[\sum_{\mathbf{R}_a}\int_0^{c\beta}\frac{d\bar{\tau}_s}{\rho a}\right]\exp[-S_m(\{m_s\})]\ ,$$

$$S_m(\{m_s\})=\frac{N\pi}{2e^2}\sum_{s\neq t}\frac{m_s m_t}{[(\mathbf{R}_s-\mathbf{R}_t)^2+(\bar{\tau}_s-\bar{\tau}_t)^2]^{1/2}}+\sum_s\left[NE_c(|m_s|)+i\frac{n_c\pi}{2}\zeta_s m_s\right]. \quad (3.63)$$

The term proportional to ζ_s is the crucial Berry phase term obtained from Eq. (3.59) where, as before, the spatial coordinates $\mathbf{R}_s$ are located at the centers of plaquettes of the lattice of spins. The dimensionless constant ρ is of order unity. The core action E_c cannot be determined from the long-wavelength effective free energy $S^0_{\rm eff}$. It is instead necessary to return to the boson action $S'_{\rm eff}$ [Eq. (3.15)] and integrate out the z^α quanta in the presence of a topologically nontrivial gauge field A_μ; particular attention has to be paid to the constraint on the average boson occupation number in this procedure by allowing the field $\hat{\lambda}$ to have space-time-dependent mean value. Such a calculation was carried out in Ref. 11 where it was found that

$$E_c(|m|)=2\rho_{|m|}\ln\left[\frac{\xi}{a}\right] \quad (3.64)$$

for large ξ/a. The $\rho_{|m|}$ are a set of universal constants with $\rho_1=0.062\,296\,09\ldots$ and $\rho_2=0.155\,547\,62\ldots$.

We now use the well-known equivalence between the d-dimensional Coulomb gas and the sine-Gordon model.[27] Introduce the sine-Gordon field $\chi(\mathbf{R},\bar{\tau})$ by the transformation

$$e^{-S_m}=\int\mathcal{D}\chi\exp\left\{-\frac{g}{2}\int_0^{c\beta}d\bar{\tau}\left[\sum_{\langle s,t\rangle}(\chi_s-\chi_t)^2+\sum_s a^2\left[\frac{\partial\chi_s}{\partial\bar{\tau}}\right]^2\right]-\sum_{\{m_s\}}\left[NE_c(|m_s|)+i\left[\frac{n_c\pi}{2}\zeta_s+\chi_s\right]m_s\right]\right\}, \quad (3.65)$$

where we have used the notation $\chi_s\equiv\chi(\mathbf{R}_s,\bar{\tau}_s)$ and $g=e^2/(4N\pi^2)$. The χ_s field consists of four sublattice fields χ_W, χ_X, χ_Y, and χ_Z which, separately, vary smoothly on the scale ξ, the spin-correlation length: thus, there is an upper cutoff in momentum space of $\sim\xi^{-1}$ for the fluctuations of the sublattice fields. The instantons have a fugacity $\sim\exp[-NE_c(|m_s|)]$; in the large-N limit the concentration of the instantons is exponentially small and the mean spacing of the instantons is much greater than ξ. We also neglect instantons with $|m_s|>1$: summing over instantons with charges ±1 we finally show that the properties of Z are equivalent to those of $\int\mathcal{D}\chi e^{-S_{sg}}$ with

$$S_{sg}=\frac{g}{2}\int_0^{c\beta}d\bar{\tau}\left\{\sum_{\langle s,t\rangle}(\chi_s-\chi_t)^2+\sum_s\left[a^2\left[\frac{\partial\chi_s}{\partial\bar{\tau}}\right]^2-M^2\cos[\chi_s-(n_c\pi/2)\zeta_s]\right]\right\}. \quad (3.66)$$

Here

$$\frac{gM^2}{2}=\frac{1}{\rho a}\exp[-NE_c(1)] \quad (3.67)$$

is the instanton fugacity which is exponentially small in N.

Before turning to an analysis of the properties of S_{sg}, we note that a Coulomb gas partition function closely related to S_m in Eq. (3.63) was also obtained by Fradkin and Kivelson[22] by duality transformations on a quantum dimer model.[40] The connection with their results is briefly reviewed in Appendix A. A renormalization-group analysis of S_{sg} based upon a $2+\epsilon$ expansion was carried out in Ref. 41. In this paper we shall focus on a careful examination of the large-N limit, when the parameter M is small. Our results are consistent with these other analyses.

We consider first the case $n_c=0$ (mod 4), when S_{sg} is the usual unfrustrated sine-Gordon model. In three dimensions, this model is expected to display only a massive phase in which the instanton gas forms a plasma with ordinary Debye screening.[29] For small M, S_{sg} is solved by expanding perturbatively around the uniform state $\chi_s=$const.[27] This gives a "screening length" in the instanton plasma $\sim aM^{-1}$ and, at length scales larger than this, a linear potential of strength $\sim e^2M/Na$ (Ref. 27) appears between the z spinons. Using a z mass of order $\Delta\sim e^2$ we can estimate, using a nonrelativistic approximation,[32] a spinon pair size of $\sim(e^4M/Na)^{-1/3}$. The fluctuations in χ give a collective mode of gap $\sim cM/a$. These properties closely resemble those of the valence-bond solid states recently introduced for $n_c=2S=4$ in an SU(2) model,[12] and give the full symmetry [Fig. 1(c)].

Qualitatively new phenomenon, however, occur for the cases $n_c\neq0$ (mod 4). It is easy to show that the uniform state χ_s = const is an unstable point of S_{sg}. All the stable minima spontaneously break the rotation symmetry between the four sublattices W,X,Y,Z. The analysis is distinct for the two cases $n_c=2$ (mod 4) and $n_c=1,3$ (mod 4) and we will consider them in turn.

(a) $n_c=2$ *(mod 4)*. In this case the cosine term in S_{sg} has only two different phase shifts: we therefore need only two dual sublattices X and Y in Fig. 7. Sublattice W is merged with sublattice Y and sublattice Z with sublattice X. Rewriting S_{sg} with sublattice labels on χ we have

$$S_{sg}=\frac{g}{2}\int_0^{c\beta}d\tilde{\tau}\left\{\sum_{\langle s,t\rangle}(\chi_{X_s}-\chi_{Y_t})^2+\sum_s\left[a^2\left[\frac{\partial\chi_{X_s}}{\partial\tilde{\tau}}\right]^2+M^2\cos\chi_{X_s}\right]+\sum_t\left[a^2\left[\frac{\partial\chi_{Y_t}}{\partial\tilde{\tau}}\right]^2-M^2\cos\chi_{Y_t}\right]\right\}, \tag{3.68}$$

where the sum on s extends over all sites on sublattices X and Z and that on t over all sites on sublattices W and Y. We take the continuum limit of S_{sg} by introducing the sum and difference variables

$$\chi_X=\chi_1+\chi_2,\quad \chi_Y=\chi_1-\chi_2\ , \tag{3.69}$$

and expanding the action in gradients of χ_1 and χ_2. Retaining only the lowest nontrivial gradients this procedure yields

$$S_{sg}=\frac{g}{2}\int\frac{d^2\mathbf{r}}{a^2}d\tilde{\tau}\left[a^2(\nabla_{\mathbf{r}}\chi_1)^2+a^2\left[\frac{\partial\chi_1}{\partial\tilde{\tau}}\right]^2+8\chi_2^2-M^2\sin\chi_1\sin\chi_2\right]. \tag{3.70}$$

An important property of this action is that the fluctuations of χ_2 are massive because of the presence of the $8\chi_2^2$ term—gradients in χ_2 are therefore irrelevant and were omitted for this reason. The minimum of the action, however, occurs at a nonzero value of χ_2:

$$\chi_2=\frac{M^2}{16}\sin\chi_1+O(M^6)\ . \tag{3.71}$$

We may thus safely replace χ_2 by this optimum value and neglect its fluctuations. This produces the following effective action for the field χ_1:

$$S_{sg}=\frac{g}{2}\int\frac{d^2\mathbf{r}}{a^2}d\tilde{\tau}\left[a^2(\nabla_{\mathbf{r}}\chi_1)^2+a^2+\frac{M^4}{256}\cos2\chi_1\right]+O(M^8)\ . \tag{3.72}$$

This is of the usual sine-Gordon form and can be analyzed in a manner parallel to the $n_c=0$ (mod 4) case discussed above. The field χ_1 will fluctuate in a small neighborhood of either $\pi/2$ or $-\pi/2$. These two minima are physically distinct and the choice of one or the other leads to a broken lattice symmetry. The analysis of Polyakov[27] can be used to show that perturbation theory in powers of M^4 is well defined around either of the two minima. The χ_1 fluctuations are massive and form a spinless collective mode with a gap $\sim cM^2/a$. The fluctuations decay over a length scale $\sim a/M^2$, which is also the scale beyond which a linear confining force between the z^α quanta (spinons)[27] appears. For the minimum at $\chi_1=\pi/2$ we have

$$\langle\chi_W\rangle=\langle\chi_Y\rangle=\frac{\pi}{2}-\frac{M^2}{16},\quad \langle\chi_X\rangle=\langle\chi_Z\rangle=\frac{\pi}{2}+\frac{M^2}{16} \tag{3.73}$$

and a second minimum near $-\pi/2$. The mean instanton charge density per unit time, $i\rho_s=g\Delta^2\chi_s/c$ (Ref. 27) (Δ^2 is the lattice Laplacian), is therefore nonzero and has the values

$$\langle i\rho_W\rangle=\langle i\rho_Y\rangle=-\frac{gM^2}{4c},\quad \langle i\rho_X\rangle=\langle i\rho_Z\rangle=+\frac{gM^2}{4c}\ . \tag{3.74}$$

This condensation of the instantons leads to static electric fields $iE_\alpha=g\sum_\beta\epsilon_{\alpha\beta}\Delta_\beta\chi$ (Ref. 27) ($\alpha,\beta=\hat{x},\hat{y}$) which reside on the links of the lattice of spins. We find, in terms of the electric fields E_p introduced in Sec. III [Eq. (3.18)], that

$$\langle iE_2\rangle=\frac{gM^2}{2a},\quad \langle iE_0\rangle=\langle iE_1\rangle=\langle iE_3\rangle=0\ . \tag{3.75}$$

Finally, the couplings between the spin-Peierls order parameters Ψ_p [Eq. (3.18)] and the gauge fields S^1_{eff} [Eq. (3.17)] imply the expectation values

$$\langle\Psi_2\rangle=\frac{\gamma_2 gM^2}{2ac_2},\quad \langle\Psi_1\rangle=\langle\Psi_3\rangle=0\ . \tag{3.76}$$

The structure of the spin-Peierls order implied by these expectation values is exactly that shown in Fig. 1(b). The minimum $\chi_1=-\pi/2$ will lead to spin-Peierls order which is rotated from that in Fig. 1(b) by 90°. We have now obtained one of the central results of this paper: we have shown how the dynamical properties of the plasma S_m lead to a spontaneous breaking of lattice symmetry.

(b) $n_c=1,3$ *(mod 4)*. The analysis for these cases is very similar to that displayed above for $n_c=2$ (mod 4). However, the presence of four distinct phase shift terms on the four sublattices W,X,Y,Z leads to a greater degree of algebraic complexity. We will discuss the case $n_c=1$ (mod 4); the results for $n_c=3$ (mod 4) are very similar. As in (*a*), we rewrite S_{sg} in terms of the four sublattice fields (Fig. 7). χ_W, χ_X, χ_Y, and χ_Z. To perform a continuum expansion we introduce the parametrization

$$\begin{aligned}\chi_W&=\chi_1+\chi_2+\chi_3\ ,\\ \chi_X&=\chi_1-\chi_2+\chi_4\ ,\\ \chi_Y&=\chi_1+\chi_2-\chi_3\ ,\\ \chi_Z&=\chi_1-\chi_2-\chi_4\ ,\end{aligned} \tag{3.77}$$

and expand in gradients of χ_1, χ_2, χ_3, and χ_4. This yields

$$S_{sg}=\frac{g}{2}\int\frac{d^2\mathbf{r}}{a^2}d\tilde{\tau}\left[a^2(\nabla_{\mathbf{r}}\chi_1)^2+a^2\left[\frac{\partial\chi_1}{\partial\tilde{\tau}}\right]^2+8\chi_2^2+2\chi_3^2+2\chi_4^2-\frac{M^2}{2}\sin(\chi_1+\chi_2)\sin\chi_3-\frac{M^2}{2}\cos(\chi_1-\chi_2)\sin\chi_4\right] . \quad (3.78)$$

The fields χ_3 and χ_4 are massive and, as in (*a*), may be replaced by the values which minimize S_{sg}:

$$\chi_3=-\frac{M^2}{8}\sin(\chi_1+\chi_2)+O(M^6) ,$$
$$\chi_4=\frac{M^2}{8}\cos(\chi_1-\chi_2)+O(M^6) . \quad (3.79)$$

The effective action for χ_1 and χ_2 then becomes

$$S_{sg}=\frac{g}{2}\frac{d^2\mathbf{r}}{a^2}d\tilde{\tau}\left[a^2(\nabla_{\mathbf{r}}\chi_1)^2+a^2\left[\frac{\partial\chi_1}{\partial\tilde{\tau}}\right]^2+8\chi_2^2-\frac{M^4}{32}\sin2\chi_1\sin2\chi_2\right] . \quad (3.80)$$

We now replace the massive χ_2 field by the value which minimizes S_{sg}

$$\chi_2=\frac{M^4}{256}\sin2\chi_1 \quad (3.81)$$

and finally obtain the effective action for the field χ_1

$$S_{sg}=\frac{g}{2}\int\frac{d^2\mathbf{r}}{a^2}d\tilde{\tau}\left[a^2(\nabla_{\mathbf{r}}\chi_1)^2+a^2\left[\frac{\partial\chi_1}{\partial\tilde{\tau}}\right]^2+\frac{M^8}{16\,384}\cos4\chi_1\right] . \quad (3.82)$$

As in (*a*), the final action for χ_1 is a pure sine-Gordon theory. The field χ_1 will fluctuate in a small neighbor of $\pi/4$, $3\pi/4$, $-3\pi/4$, or $3\pi/4$ with each minimum leading to a physically distinct state with a broken lattice symmetry. The χ_1 fluctuations lead to a spinless collective mode with a gap $\sim cM^4/a$ and decay over a length scale $\sim a/M^4$; this is also the scale beyond which a linear confining force between the z^α quanta (spinons)[27] appears. For the minimum at $\chi_1=-\pi/4$ we have

$$\langle\chi_W\rangle=-\frac{\pi}{4}+\frac{M^2}{8\sqrt{2}}-\frac{M^4}{256},$$
$$\langle\chi_X\rangle=-\frac{\pi}{4}+\frac{M^2}{8\sqrt{2}}+\frac{M^4}{256} ,$$
$$\langle\chi_Y\rangle=-\frac{\pi}{4}-\frac{M^2}{8\sqrt{2}}-\frac{M^4}{256},$$
$$\langle\chi_Z\rangle=-\frac{\pi}{4}-\frac{M^2}{8\sqrt{t}}+\frac{M^4}{256} . \quad (3.83)$$

The mean instanton charge per unit time has the values

$$\langle i\rho_W\rangle=\frac{gM^2}{8\sqrt{2}c}-\frac{gM^4}{64c},\quad \langle i\rho_X\rangle=\frac{gM^2}{8\sqrt{2}c}+\frac{gM^4}{64c} ,$$
$$\langle i\rho_Y\rangle=-\frac{M^2}{8\sqrt{2}c}-\frac{gM^4}{64c}, \quad (3.84)$$
$$\langle i\rho_Z\rangle=-\frac{gM^2}{8\sqrt{2}c}+\frac{gM^4}{64c} .$$

The electric fields [defined in Eq. (3.18)] are

$$\langle iE_0\rangle=0,\quad \langle iE_1\rangle=\langle iE_3\rangle=\frac{gM^2}{2\sqrt{2}a},\quad \langle iE_2\rangle=\frac{gM^4}{32a} . \quad (3.85)$$

Finally, the couplings between the spin-Peierls order parameters Ψ_p [Eq. (3.18)] and the gauge fields in S_{eff}^1 [Eq. (3.17)] imply the expectation values

$$\langle\Psi_1\rangle=\langle\Psi_3\rangle=\frac{\gamma_1 gM^2}{2\sqrt{2}ac_1},\quad \langle\Psi_2\rangle=\frac{\gamma_2 gM^4}{32ac_2} . \quad (3.86)$$

These order parameters imply a state with the broken lattice symmetry of Fig. 1(a). Choosing $\chi_1=\pi/4$, $3\pi/4$, $-3\pi/4$ will yield the three states related to Fig. 1(a) by 90° and 180° rotations. Thus, as in (*a*), the Berry phases in S_{sg} have led to a broken lattice symmetry and the appearance of spin-Peierls order.

IV. CONCLUSIONS

This paper has addressed the zero-temperature properties of quantum SU(N) antiferromagnets with nearest-neighbor exchange interactions

$$\mathcal{H}=\frac{J}{N}\sum_{\langle i,j\rangle}\hat{S}_\alpha^\beta(i)\hat{S}_\beta^\alpha(j) , \quad (4.1)$$

where $\hat{S}_\alpha^\beta(i)$ are the generators of SU(N). Our main results were for the case of the square lattice, to which we shall restrict our discussion here. A parameter which played a crucial role in our analysis was the integer n_c, the number of columns in the Young tableau of the representation of the "spins" on each lattice site (Fig. 3). For the case of SU(2), we have $n_c=2S$, where S is the usual spin quantum number.

A schematic phase diagram for this model is shown in Fig. 4 as a function of N and n_c. The system exhibits two types of phases: (i) the Néel phase which has a broken spin-rotation symmetry, and (ii) the disordered phase, in which spin-rotation symmetry is unbroken, but, for $n_c\neq0$ (mod 4), discrete lattice symmetries are broken. For large values of N, the boundary between these phases is given by $n_c=\kappa_c N$, with $\kappa_c=0.19$. Previous papers have explored (i) the semiclassical limit[4,6,7,10] $n_c\to\infty$, N fixed, which yields a long-wavelength description based on a nonlinear σ model, and (ii) the fermionic large-N limit,[4,19] $N\to\infty$, n_c fixed, which yields a spin-Peierls ground state with the broken lattice symmetry of Fig. 1(a). Moreover, consistent with predictions in the semiclassical limit,[4,10] the structure of the low-lying states in the fermionic large-N limit were controlled by the value of n_c (mod 4). In this paper we examined the limit $n_c=\kappa N$,

$N \to \infty$ with $\kappa < \kappa_c$ fixed, but arbitrary. The Schwinger boson method of Ref. 8 was used to take this limit. We found that a careful analysis of topological effects yielded ground states with a broken lattice symmetry for all $n_c \neq 0$ (mod 4). The structure of the spin-Peierls order in these states is shown in Fig. 1(a)–1(c).

Even for N finite, our arguments apparently allow us to rule out any intermediate phase between the Néel ordered state and the confining spin-Peierls–valence-bond solid state. Any non-Néel ordered phase will have a gap for spinon excitations, which will couple to gauge fields in qualitatively the way above, producing a gauge theory with action $(1/e^2)F_{\mu\nu}^2$ for some e^2. Instantons will again be present with $1/r$ potential interactions for large separation, as well as Berry phase terms which, because of their "topological" origin, must be robust. Such a plasma has only one phase, the screening phase which confines spinons. For $n_c \neq 0$ (mod 4) this will have spin-Peierls order. It is harder to rule out a coexistence region with both Néel and spin-Peierls order, though we strongly suspect that this will not occur.

An important issue not addressed in this paper is that of the nature of the transition between the spin-Peierls and Néel states. This would require examination of the limit $\kappa \to \kappa_c$ with N fixed, and possibly large. This can be examined by the $D=2+\epsilon$ analysis of the semiclassical nonlinear σ models.[42] However, this approach fails to account for the global topological properties of the order-parameter space and thus cannot be used to include instanton and Berry-phase effects. An alternative is to use the model of N complex scalars coupled to QED obtained in Sec. II: in general, nonlinear terms in the z^α consistent with a global U(N) symmetry should be added to the action (2.14). The transition would then correspond to taking the renormalized mass of the complex scalars to zero. This model was examined by Halperin, Lubensky, and Ma[43] in a $D=4-\epsilon$ analysis and displayed a first-order transition for $N<366$ with a bare electric charge. Instantons and their Berry phases are special to $D=3$: the above analysis is therefore probably not useful for the present problem. Also, the $N=2$ model with $n_c=0$ (mod 4) is the usual $O(3)$ classical Heisenberg model, so at least this case is second order (and the nature of the disordered phase with integer spin excitations is in essential agreement with our results). A $1/N$ expansion of the transition in the complex scalar QED model and models related to the $m>1$ case (Appendix C) was performed by Hikami[44] for arbitrary $2<D<4$: he found a second-order transition. However, a proper extension of this analysis to include Berry-phase effects remains an important open problem.

Finally, we emphasize that our results are for unfrustrated Hamiltonians like (1.1). Weak frustration added to an unfrustrated model with a Néel-ordered ground state already close to the phase boundary could push it across the boundary, giving a transition to a confining phase in the same universality class as our work. Alternatively, or for stronger frustration, the U(1) gauge symmetry may be broken via the Higgs mechanism with or without breaking SU(N) (long-range Néel order). Such a phase will have helical correlations and possibly long-range order[45] and no confinement because instantons must form pairs when the gauge symmetry is broken, just as in the Néel state. The possibilities for frustrated and hole-doped antiferromagnets thus remain very rich.

Note added in proof. We have recently extended the results of this paper to frustrated quantum antiferromagnets [N. Read and S. Sachdev (unpublished)]: The large-N limit was taken using the symplectic groups Sp($2N$). As predicted in this paper, the spin-Peierls phases [Figs. 1(a)–1(c)] are found to persist for finite, but moderate frustration in a square lattice model with nearest-neighbor (J_1) and second-neighbor (J_2) antiferromagnetic interactions. Additional disordered phases are found at large frustration. Also, new numerical and series investigations of the spin-$\frac{1}{2}$ SU(2) version of the same model have been performed by M. P. Gelfand (unpublished) and by R. R. P. Singh and R. Narayan (unpublished). Their results further support the existence of columnar dimer order [Fig. 1(a)] in a disordered phase around $J_2/J_1 \approx 0.5$.

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APPENDIX A: DUALITY TRANSFORMATIONS

Fradkin and Kivelson[22] have recently used an elegant duality transformation, familiar in the lattice gauge theory literature,[46] in their analysis of the quantum dimer description of square lattice antiferromagnets. In this appendix we will review this transformation from a slightly different point of view and discuss its relationship to the results obtained in this paper.

As was noted in the Introduction, the SU(N) antiferromagnet, $\mathcal{H}$ [Eq. (1.1)], becomes equivalent to a quantum dimer model in the limit $N \to \infty$, with $m=1$, and n_c arbitrary. It was shown in Ref. 4 that, in this limit, the low-energy excitations of $\mathcal{H}$ are described by the following quantum dimer Hamiltonian:

$$\frac{H_{\text{eff}}}{J/N} = \sum_{\{n_l\}} |\{n_l\}\rangle \left[\sum_l n_l^2 \right] \langle \{n_l\}| - \sum_{l_1,l_2,l_3,l_4 \in \Box} |\ldots n_{l_1}, n_{l_2}-1, n_{l_3}, n_{l_4}-1, \ldots \rangle 2\sqrt{n_{l_1} n_{l_2} n_{l_3} n_{l_4}} \langle \ldots n_{l_1}-1, n_{l_2}, n_{l_3}-1, n_{l_4}, \ldots | . \quad \text{(A1)}$$

Here the Hilbert space is labeled by integers $\{n_l\}$ on the links of the square lattice, and the second sum extends over all plaquettes. The values of n_l on the four links connected to a site are constrained to sum to n_c:

$$n_{(i,i+\hat{x})}+n_{(i,i-\hat{x})}+n_{(i,i+\hat{y})}+n_{(i,i-\hat{y})}=n_c \ . \tag{A2}$$

We introduce an integer-valued "electric-field"

$$L_j(\mathbf{r})=e^{i\mathbf{Q}\cdot\mathbf{r}}n_{(\mathbf{r},\mathbf{r}+\hat{e}_j)} \ , \tag{A3}$$

where $\mathbf{Q}=(\pi,\pi)$, on the links of the lattice. In terms of the electric field, the constraint takes the simple form

$$\Delta_j L_j = n_c e^{i\mathbf{Q}\cdot\mathbf{r}} \ . \tag{A4}$$

Let $a_j(\mathbf{r})$ be the canonically conjugate "gauge field" (i.e., $[a,L]=i$); the constraint that the eigenvalues of L are integers requires that the wave function be invariant under $a \rightarrow a+2\pi$. The second term in H_{eff} involves terms which are off diagonal in the occupation number representation. The operator e^{ia} leads to similar off-diagonal matrix elements. In the large-n_c limit, we may therefore replace H_{eff} by

$$H'=\frac{J}{N}\sum_{\mathbf{r},j}L_j^2(\mathbf{r})-\frac{4Jn_c^2}{N}\sum_{\mathbf{r}}\cos(\epsilon_{ij}\Delta_i a_j) \ . \tag{A5}$$

The partition function $Z=\text{Tr}\exp(-\beta H')$ can now be evaluated by splitting the time-evolution into infinitesimal time slices of size ϵ, and inserting the identity operator resolved in terms of a basis which diagonalizes $L_j(\mathbf{r})$ between adjacent time slices. As in Refs. 47 and 48 it is convenient to replace $\exp(K\cos[\epsilon_{ij}\Delta_i a_j])$ by the Villain form

$$\sum_k e^{ik[\epsilon_{ij}\Delta_i a_j]}\exp(-k^2/2K) \ .$$

A straightforward analysis now yields

$$Z=\text{Tr}_{\kappa L}[\delta(\Delta_\tau L_i-\epsilon_{ij}\Delta_j k)\delta(\Delta_j L_j-n_c e^{i\mathbf{Q}\cdot\mathbf{r}})\exp(-S')] \ , \tag{A6}$$

where the trace is over integer-valued fields L_i on the links of the square lattice and k on the points of the dual lattice:

$$S'=\sum_{\mathbf{r}\tau j}\left[\frac{1}{2K_1}L_j^2(\mathbf{r},\tau)+\frac{1}{2K_2}k^2(\mathbf{r},\tau)\right] \tag{A7}$$

with $K_1=N/(2\epsilon J)$ and $K_2=Jn_c^2\epsilon/N$. We now reexpress the fields L_j,k in terms of dual-lattice variables designed to satisfy the constraints in the partition function:

$$\begin{aligned} L_i &= \epsilon_{ij}\Delta_j\chi+L_i^0 \ , \\ k &= \Delta_\tau\chi \ . \end{aligned} \tag{A8}$$

Here L_j^0 is the time-independent field

$$L_j^0=\frac{n_c}{4}e^{i\mathbf{Q}\cdot\mathbf{r}}(1,1) \tag{A9}$$

representing the average dimer density on each link. The space-time field χ resides on the dual-lattice points $\mathbf{R}_s$, and its values are restricted to

$$\chi_s=p_s+\frac{n_c}{4}\zeta_s \ , \tag{A10}$$

where the p_s are arbitrary integers and ζ_s takes the fixed values 0,1,2,3 on sublattices W,X,Y,Z (Fig. 7). The same field ζ_s was introduced in Sec. I in the discussion of the Berry phases of the instantons. It is easy to verify that the relations (A8)–(A10) constrain L_i and m to be integer valued and satisfy both constraints in Eq. (A6). Moreover,

$$\begin{aligned} L_i^2 &= (\epsilon_{ij}\Delta_j\chi+L_i^0)^2 \\ &= (\Delta_i\chi)^2+2\epsilon_{ij}\Delta_j\chi L_i^0+\text{const} \ . \end{aligned} \tag{A11}$$

The second term may be integrated by parts and the identity $\epsilon_{ij}\Delta_i L_j^0=0$ shows that it is 0. Inserting Eqs. (A8) and (A11) into Eqs. (A6) and (A7), we find, up to an overall normalization factor,

$$Z=\text{Tr}_\chi\exp\left[-\sum_{s,\tau}\left[\frac{1}{2K_1}(\Delta_i\chi_s)^2+\frac{1}{2K_2}(\Delta_\tau\chi_s)^2\right]\right] \ . \tag{A12}$$

This is the partition function of a discrete Gaussian model. The novel n_c-dependent features arise from the sublattice-dependent restriction (A10) on the values of χ. The field χ can be promoted to a continuous variable by the Poisson summation method:[49]

$$Z=\lim_{E_c\rightarrow 0}\sum_{m_s}\int D\chi\exp\left\{-\sum_{s,\tau}\left[\frac{1}{2K_1}(\Delta_i\chi_s)^2+\frac{1}{2K_2}(\Delta_\tau\chi_s)^2+2\pi i m_s\left[\chi_s-\frac{n_c}{4}\zeta_s\right]+E_c(|m_s|)\right]\right\} \ . \tag{A13}$$

We have introduced above integer "instanton" charges m_s which reside on the vertices of the dual lattice and their core action E_c. If we now take the large-E_c limit, the summation over m_s can be carried out and (after rescaling χ by 2π) the action becomes equivalent to S_{sg} [Eq. (3.66)]. We have thus established the relationship between the results of this paper and Ref. 22. One expects[46,50] that the model behaves similarly for all $0<E_c<\infty$, suggesting that the symmetry-breaking patterns (Fig. 1) apply in the present limit also.

APPENDIX B: HONEYCOMB LATTICE

This appendix sketches the generalization of the results of this paper to another bipartite lattice in two dimensions: the honeycomb lattice. It is now necessary to split the dual triangular lattice into three dual sublattices, X,Y,Z, shown in Fig. 8. The bosonic large-N theory proceeds as in Sec. III with the summation over $\hat{\eta}$ now extending over the three nearest-neighbor vectors. The effective action for the z^α and A_μ is easily shown to have the form S'_{eff} [Eq. (3.15)]. The calculations of the Berry phases of instantons proceeds as in Sec. III A and one obtains an expressions identical to that in Eq. (3.58).

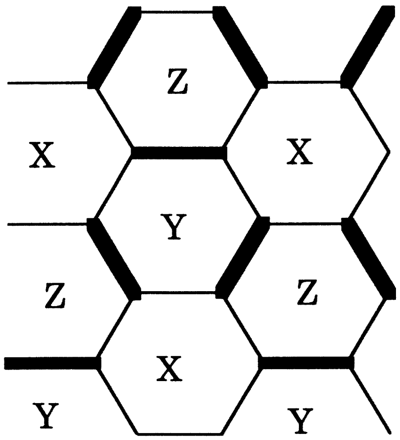

FIG. 8. The dual sublattices X, Y, Z of the honeycomb lattice. The thickness of the links signifies the structure of the spin-Peierls order for $n_c \neq 0$ mod 3; links with equal thickness have equal values of $\langle \hat{S}(i)\hat{S}(i+\hat{\eta})\rangle$.

Evaluating the summation in Eq. (3.58) we find

$$S_B(n_c) = \sum_s i\frac{2n_c\pi}{3}\zeta_s m_s \ , \tag{B1}$$

where the integer $\zeta_s = 0,1,2$ for $\mathbf{R}_s$ on three dual sublattices X, Y, Z (Fig. 8). Apart from the change in the Berry phases, the structure of the sine-Gordon theory of Sec. III B 2 remains unchanged. The state with χ constant is stable only for $n_c = 0$ (mod 3). The rotational symmetry of the lattice is broken for other values of n_c; one of the three possible states for $n_c \neq 0$ (mod 3) is shown in Fig. 8, the other two states can be obtained by a threefold rotation about one of the lattice sites.

APPENDIX C: GAUGE THEORY FOR GENERAL m

This appendix will sketch the generalization of the results of Sec. III to representations with $m > 1$. As noted in Ref. 4, the bosonic representation of the spin-operators now takes the form

$$\begin{aligned} \hat{S}^{\beta}_{\alpha}(i) &= \sum_{a=1}^{m} b^{\dagger}_{\alpha a}(i) b^{\beta a}(i), \quad i \in A \text{ sublattice} \ , \\ \hat{S}^{\beta}_{\alpha}(j) &= -\sum_{a=1}^{m} \bar{b}^{\beta a \dagger}(j)\bar{b}_{\alpha a}(j), \quad j \in B \text{ sublattice} \ , \end{aligned} \tag{C1}$$

with the constraints

$$\begin{aligned} b^{\dagger}_{\alpha a}(i) b^{\alpha b}(i) &= \delta^b_a n_c \ , \\ \bar{b}^{\alpha b \dagger}(j)\bar{b}_{\alpha a}(j) &= \delta^b_a n_c \ , \end{aligned} \tag{C2}$$

on every site i (j) on the A (B) sublattice. The presence of the label a generalizes the local U(1) symmetry of Sec. III to a local U(m) symmetry. The fields Q and λ in the functional integral (3.1) now become $m \times m$ complex matrices (only the λ matrices are Hermitian). We choose these matrices to be diagonal in the mean-field solution; the different m indices then completely decouple and the mean-field solution is identical to that in Sec. III. We parametrize the fluctuations [replacing Eq. (3.9)] as follows:

$$\begin{aligned} Q^a_{i,i+\hat{\eta},b} &= [\bar{Q}\delta^a_c + q^a_c(i)]\{\exp[i\hat{\eta}\cdot B(i)]\}^c_b \ , \\ i\lambda^a_b(i) &= \bar{\lambda}\delta^a_b + iB^a_{\tau,b}(i), \quad \text{for } i \in A \ , \\ i\lambda^a_b(j) &= \bar{\lambda}\delta^a_b - iB^a_{\tau,b}(i) \quad \text{for } j \in B \ , \end{aligned} \tag{C3}$$

where B_μ and $q_{\hat{\eta}}$ are $m \times m$ Hermitian matrices which represent the U(m) gauge fields and the amplitude fluctuations, respectively.

The long-wavelength effective action can be obtained through an analysis very similar to that of Sec. III: the action in Eq. (3.15) is now replaced by

$$S'_{\text{eff}} = \int d^d r \int_0^{c\beta} d\tilde{\tau} \frac{a^{1-d}}{2\sqrt{d}} \left[|(\partial_\mu \delta^a_b - iB^a_{\mu b}) z^{\alpha b}|^2 + \frac{\Delta^2}{c^2}|z^{\alpha a}|^2 \right] . \tag{C4}$$

At distances larger than $\xi = c/(2\Delta)$, the z quanta can be integrated out. It is convenient to express the results in terms of the U(1) and SU(m) components of B_μ:

$$B^a_{\mu b} = \delta^a_b A_\mu + \tilde{A}^a_{\mu,b} \ , \tag{C5}$$

where A_μ is a real U(1) gauge field and $\tilde{A}_\mu$ is a traceless Hermitian SU(m) gauge field. We can also define the U(1) field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the SU(m) field tensor

$$\tilde{F}^a_{\mu\nu b} = \partial_\mu \tilde{A}^a_{\nu b} - \partial_\nu \tilde{A}^a_{\mu b} - i(\tilde{A}^a_{\mu c}\tilde{A}^c_{\nu b} - \tilde{A}^a_{\nu c}\tilde{A}^c_{\mu b}) \ . \tag{C6}$$

These quantities appear in the effective action for the gauge fields [this is the generalization of Eq. (3.16)]

$$S^0_{\text{eff}} = \frac{N}{4e^2}\int d^d r \int_0^{c\beta} d\tilde{\tau} (\text{Tr}\tilde{F}^2_{\mu\nu} + mF^2_{\mu\nu}) \ , \tag{C7}$$

where $e^2 \sim \xi^{d-3}$. Qualitatively new physics now appears that was not present for $m = 1$. The (2+1)-dimensional SU(m) gauge theory is permanently confining[51,52] with electric SU(m) charges experiencing a linear potential at a length scale set by the coupling $Ne^{-2} \sim N\xi$. The z quanta, which carry both U(1) and SU(m) electric charges, will therefore be confined into SU(m) singlets, i.e., either spinon-antispinon pairs (of size $\sim N^{2/3}\xi$) or "baryons" of m spinons, or composites of these, by the confining potential which switchs on at a scale of order $N\xi$. Note that the "baryon"-type object has U(1) charge m.

We now turn to a discussion of instanton effects in the (2+1)-dimensional theory. The elementary instantons[53] are U(m) rotations of

$$B^a_{\mu b} = \delta^a_1 \delta^1_b A^D_\mu \ , \tag{C8}$$

where A^D_μ is the same vector potential of a Dirac monopole discussed in Sec. III A 2 with $\nabla \times \mathbf{A}^D = \mathbf{r}/(2r^3)$. The total U(1) flux associated with this instanton [using Eq. (C5)] is $2\pi/m$. The instanton also carries SU(m) magnetic charges. A Berry-phase calculation for this instanton can be carried out as in Sec. III A 2. Only the $z^{\alpha 1}$ bosons are effected, and the $m = 1$ results therefore remain unchanged. A little more care is required in understanding the interactions between the instantons. As the instantons are spaced much farther apart than the confinement scale $N\xi$, it is not a good approximation to treat the SU(m) gauge fields classically. Strong quantum fluctuations are expected to wipe out any effects of the orientation of the instanton in SU(m) space. No such effects arise in the U(1) component which still leads to a long-range $1/r$ interaction between the instantons. The effective Coulomb gas partition function for the instantons is therefore expected to have the same form as S_m [Eq. (3.63)]. The remaining analysis in Sec. III B remains largely unaffected: the only modification required is that the amplitude mode which couples to the instanton charges is the trace of the matrix q_a^b. Note that, since U(1) charge is now confined, free "baryons" do not appear in the physical spectrum; the basic elementary excitation is the spinon-antispinon confined pair, which transforms as either the singlet or adjoint representation of SU(N).

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