Coherence length and vortex filament in the boson-fermion model of superconductivity

R. Friedberg and T. D. Lee
Columbia University, New York, New York 10027

H. C. Ren
Rockefeller University, New York, New York 10021

(Received 6 March 1990)

The boson-fermion model of high-T_c superconductivity is based on the dominance of the "s-channel" reaction $2e \rightarrow \phi \rightarrow 2e$, where ϕ is a local field, representing phenomenologically the pair state. We show that in this model the coherence length ξ can be calculated and is very small, consistent with observation. This, in turn, justifies the local-field approximation. The relevance of Bose-Einstein condensation is discussed; in this theory, the critical temperature can be much higher than that in the BCS theory. We also examine the vortex filament and the critical magnetic fields H_{c1} and H_{c2}.

I. INTRODUCTION

One of the important differences between the recently discovered high-temperature superconductors[1,2] and the usual lower-temperature superconductors is the smallness of the coherence length ξ. For the former,[3,4] ξ is ≈ 10 Å; for the latter, ξ is much longer, typically $\approx 10^4$ Å for type I and $\approx 10^2$ Å for type II. The purpose of this paper is to calculate ξ for high-T_c superconductors, according to the recently proposed (phenomenological) boson-fermion model of superconductivity.[5,6]

A. Review

In order to obtain a perspective on the theoretical model, we begin with a brief review. The observation of such a small coherence length ξ indicates that the pairing between electrons, or holes, is reasonably localized in the coordinate space. Hence, the pair state can be well approximated by a phenomenological local boson field $\phi(\mathbf{r})$, whose mass M is $\approx 2m_e$ and whose elementary charge unit is $2e$, where m_e and e are the mass and charge of an electron. It follows then that the transition

$$2e \rightarrow \phi \rightarrow 2e \tag{1.1}$$

must occur, in which e denotes either an electron or a hole; furthermore, the localization of ϕ implies that phenomena at distances larger than the physical extension of ϕ (which is $<\xi$) are insensitive to the interior of ϕ. Since ξ is of the same order as the scale of a lattice unit cell, it becomes possible to develop a phenomenological theory of superconductivity based *only* on the local character of ϕ.

Of course, physics at large does depend on several overall properties: the spin of ϕ, the stability of an individual ϕ quantum, the isotropicity and homogeneity (or their absence) of the space containing ϕ, and so on. The situation is analogous to that in particle physics: The smallness of the radii of pions, ρ mesons, kaons, etc., makes it possible for us to handle much of the dynamics without any reference to their internal structure, such as quark-antiquark pairs or bag models. Hence, the origin of their formation becomes a problem separate from the description of their mechanics. An important ingredient in this type of phenomenological approach is the selection of the basic interaction Hamiltonian that describes the underlying dominant process. In the usual low-temperature superconductors, the large ξ value makes the corresponding pairing state ϕ too extended and ill-defined in the coordinate space; therefore (1.1) does not play an important role. Instead the BCS theory of superconductivity[7] is based on the emission and absorption of phonons,

$$2e \rightarrow 2e + \text{phonon} \rightarrow 2e \ . \tag{1.2}$$

In the language of particle physics (1.1) is an s-channel process, while (1.2) is t channel. The BCS theory may be called the t-channel theory, and the model that is based on (1.1) the s-channel theory.

The use of a boson field for the superfluidity of liquid He II has had a long history. However, there are some major differences in the following application to (high-temperature) superconductors:

(1) The ϕ quantum is charged, carrying $2e$, while the helium atom is neutral.

(2) We assume each individual ϕ quantum to be *unstable*, with 2ν as its excitation energy.

In any microscopic attempt to construct ϕ out of $2e$, because of the short-range Coulomb repulsion it is very difficult to have ϕ stable. The explicit assumption of instability bypasses this difficulty; it also makes the present boson-fermion model different from the theory of Schafroth[8] and others.

In the rest frame of a single ϕ quantum (in isolation), the decay

$$\phi \rightarrow 2e \tag{1.3}$$

occurs, in which each e carries an energy

$$\frac{k^2}{2m}=\nu \ .$$

Consequently, in a large system, there are macroscopic numbers of both bosons (the ϕ quanta) and fermions (electrons or holes), distributed according to the principles of statistical mechanics.

At temperature $T<T_c$, there is always a macroscopic distribution of zero-momentum bosons coexisting with a Fermi distribution of electrons (or holes). Take the simple example of *zero* temperature: Let ε_F be the Fermi energy. When $\varepsilon_F=\nu$, the decay $\phi\rightarrow 2e$ cannot take place because of the exclusion principle; therefore, the bosons are present. Even when $\varepsilon_F<\nu$, there is still a macroscopic number of (virtual) zero-momentum bosons in the form of a static coherent field amplitude whose source is the fermion pairs. This then leads to the following essential features of the *s*-channel model.

Below the critical temperature T_c the long-range order in the boson field can always be described by its zero-momentum bosonic amplitude B_0, as in the Bose-Einstein condensation (and therefore similar to liquid He II). Because of the transition (1.1), the zero-momentum of the boson in the condensate forces the two e to have equal and opposite momenta, forming a Cooper pair. Therefore, the same long-range-order B_0 also applies to the Cooper pairs of the fermions. Furthermore, as shown in Refs. 5 and 6, the gap energy Δ of the fermion system is related to B_0 by

$$\Delta^2=|gB_0|^2 \ , \tag{1.4}$$

where g is the coupling for $\phi\rightarrow 2e$.

Since in reality ϕ is a composite of $2e$, when the average distance between ϕ quanta becomes less than the diameter of the composite the approximation of treating each ϕ as a single boson breaks down. However, for densities not that high, by representing the $2e$ resonance as an independent ϕ field, we may convert an otherwise strong interaction problem (which forms the resonance and exists at small distances) to one that can be handled by perturbative series in weak coupling (i.e., the residual interaction at relatively larger distances). This enables us to give a systematic analysis of such a theory; it also makes transparent the question of gauge invariance and symmetry breaking.

B. Coherence length

To derive the coherence length ξ, we may either analyze the vibrational spectrum of ϕ, or examine the vortex filament and the critical field H_{c2}. These details will be given in the following sections. Here, we begin with a simple discussion.

Consider the case of a scalar ϕ interacting with an electron (or hole) field ψ through (1.1). Let $\mathbf{A}$ be the *transverse* electromagnetic field. Assume the space to be isotropic and homogeneous. Define the phase-angle variable $\theta(x)$ by

$$\phi(x)=R(x)e^{i\theta(x)} \ , \tag{1.5}$$

with R and θ both Hermitian. Write

$$\psi(x)=\psi'(x)e^{(1/2)i\theta(x)}$$

and

$$\mathbf{V}(x)\equiv \mathbf{A}(x)-(2e)^{-1}\nabla\theta(x) \ . \tag{1.6}$$

At very low temperature we have $R\cong B_0$, the long-range order parameter (chosen to be real). As shown in Ref. 6, the energy spectra for the transverse and longitudinal modes of $\mathbf{V}$ are (in units of $\hbar=c=1$)

$$\omega_t(k)=(\lambda_L^{-2}+k^2)^{1/2} \tag{1.7}$$

and

$$\omega_l(k)=[\lambda_L^{-2}+k^2v^2+(k^2/2M)^2]^{1/2} \ , \tag{1.8}$$

where k is the momentum (or wave number),

$$\lambda_L^{-2}=(2eB_0)^2/M \tag{1.9}$$

is the inverse square of the London length, $e^2=4\pi/137$, v is the "sound" velocity of the boson-fermion system, and M the mass of ϕ. The electron spectrum is of the BCS form

$$E(k)=\left[\left(\frac{k^2}{2m}-\mu\right)^2+\Delta^2\right]^{1/2} \ , \tag{1.10}$$

where Δ is given by (1.4), m is the electron mass, and μ the chemical potential.

Equations (1.7) and (1.8) also follow from general arguments: (i) At zero momentum $k=0$, as in the Higgs mechanism,[9] the energies of the three spin components of the massive vector field $\mathbf{V}$ become the same; i.e., they are all equal to the rest mass m_V, given by

$$m_V=\lambda_L^{-1} \ . \tag{1.11}$$

(ii) When $e=0$, we have $m_V=0$ and the transverse mode is the usual photon with $\omega_t=k$ (since the velocity of light c is 1). On the other hand, the longitudinal mode describes the Goldstone-Nambu boson[10] which, for $e=0$, corresponds to the vibration of ϕ, propagating with the sound velocity $v\ll 1$ (i.e., $\omega_l\rightarrow kv$ as $k\rightarrow 0$). (iii) For very large k, the excitation of ϕ approaches the free boson spectrum $k^2/2M$,

$$\omega_l\rightarrow\frac{k^2}{2M}\quad\text{for } k\gg 2Mv \quad\text{and } (2M/\lambda_L)^{1/2} \ .$$

For $e\neq 0$, the Goldstone-Nambu boson joins with the transverse photon to form a massive vector field $\mathbf{V}$, which leads to the above formulas for ω_l and ω_t, consistent with (i)–(iii).

For the coherence length ξ, we may set $\omega_l(k)=0$ and k becomes complex, which gives a boson amplitude, say $\exp(ikx)$, that decreases exponentially with distance (e.g., along the radius of a vortex filament). The decay rate in x determines ξ. From (1.8), the root

$$k\equiv i\sqrt{2}\mu_{\pm}\quad\text{for } \omega_l(k)=0 \tag{1.12}$$

satisfies

$$\mu_{\pm}^2=(Mv)^2\pm[(Mv)^4-(M/\lambda_L)^2]^{1/2} \ . \tag{1.13}$$

The amplitude $\exp(ikx)$ becomes, then, $\exp(-\sqrt{2}\mu_\pm x)$. To conform to the usual definition (as will also be discussed in Sec. III), the coherence length ξ is given by $[\mathrm{Re}(\mu_-)]^{-1}$, which is always $\geq [\mathrm{Re}(\mu_+)]^{-1}$.

(1) For $v^2 > (M\lambda_L)^{-1}$, μ_+ and μ_- are real and

$$\xi = 1/\mu_- \ . \tag{1.14}$$

(2) For $v^2 < (M\lambda_L)^{-1}$, $\mu_\pm$ are complex with

$$\mu_+ = \mu_-^* = (M/\lambda_L)^{1/2} e^{i\alpha} \ , \tag{1.15}$$

where

$$\cos 2\alpha = M\lambda_L v^2 \tag{1.16}$$

and

$$\sin 2\alpha = [1-(M\lambda_L v^2)^2]^{1/2} \ ; \tag{1.17}$$

correspondingly,

$$\xi = \sqrt{\lambda_L/M}\,\sec\alpha \ . \tag{1.18}$$

A complex $\mu_\pm$ implies the condensate amplitude inside a vortex filament also contains an oscillatory component, which may lead to new observational possibilities.

In the case $v^2 < (M\lambda_L)^{-1}$, according to (1.16) $\cos 2\alpha$ varies from 0 to 1; therefore $\cos\alpha$ is between $1/\sqrt{2}$ and 1. Hence

$$\sqrt{2\lambda_L/M} \geq \xi \geq \sqrt{\lambda_L/M} \ . \tag{1.19}$$

[Recall that $\lambda_L^{-2} = (2eB_0)^2/Mc^2$. The product λ_L times the Compton wavelength $\hbar/Mc$ is independent of c, the velocity of light.] Assume a boson condensate density B_0^2 (at $T \ll T_c$) between 10^{20}–10^{21} cm^{-3}. On account of (1.9), $M \cong 2m_e$ and $e^2/4\pi = \frac{1}{137}$, the London length is

$$\lambda_L \sim \begin{cases} 1200\ \text{Å}, & B_0^2 \sim 10^{21}\ \text{cm}^{-3} \ , \\ 3800\ \text{Å}, & B_0^2 \sim 10^{20}\ \text{cm}^{-3} \ , \end{cases} \tag{1.20}$$

Since the Compton wavelength M^{-1} is $\sim 2\times 10^{-3}$ Å, we see that in case (2), (1.18) and (1.19) give

$$\xi \sim \text{ few Å} \ . \tag{1.21}$$

Case (1) holds only if v is larger than $(M\lambda_L)^{-1/2} \sim 10^{-3}$ times the velocity of light; hence, depending on v, $\xi \sim (Mv)^{-1} \lesssim$ few Å, or $\xi \sim \sqrt{2}v\lambda_L \ll \lambda_L$. In either case, the theory predicts a very small ξ, consistent with experimental observations. Because $\lambda_L \gg \xi$, the boson-fermion model gives, in general, a type-II superconductor.

In case (2), besides complex $\mu_\pm$ there are other interesting consequences, which we shall discuss. As shown in Ref. 6 (and also in the next section), the spectra (1.7), (1.8), and (1.10) are valid at any temperature $T < T_c$. When $T \to T_c -$, the number of bosons in the condensate varies as

$$B_0^2 \propto T_c - T \ . \tag{1.22}$$

Hence, from (1.9) and (1.19),

$$\lambda_L^{-2} \propto T_c - T \tag{1.23}$$

but

$$\xi^{-2} \propto M/\lambda_L \propto (T_c - T)^{1/2} \ . \tag{1.24}$$

As is well known, the critical fields H_{c1} and H_{c2} are given by (see also Sec. IV)

$$H_{c1} \sim \frac{1}{4e\lambda_L^2}\left[\text{const} + \ln\frac{\lambda_L}{\xi}\right] \tag{1.25}$$

and

$$H_{c2} \sim \frac{1}{2e\xi^2} \ . \tag{1.26}$$

Thus, for T below but near T_c,

$$H_{c2} \sim (T_c - T)^{1/2} \tag{1.27}$$

and, neglecting the relatively slow variation of the log term in (1.25),

$$H_{c1} \sim T_c - T \ . \tag{1.28}$$

So far we have not taken into account the Coulomb screening effect on ξ due to the electrons. This will be discussed in Sec. III. As we shall see, it introduces only minor changes for $T \ll T_c$. When $T \to T_c$, its effect may become important; depending on the parameters, (1.24) can be modified to

$$\xi^{-2} \propto T_c - T \ , \tag{1.29}$$

and therefore both H_{c1} and H_{c2} become $\propto T_c - T$, but with $H_{c2}/H_{c1} \gg 1$, as in the low-temperature region.

In order to make the essential features of the model clear, we restrict ourselves in the above discussion and in Secs. II–IV to the idealized situation of an isotropic and homogeneous space. The analysis given here can be readily extended to any lattice space. This will give a somewhat larger coherence length, $\xi \sim 10$ Å, in a more realistic case. The details are planned to be given in a subsequent paper.

C. Bose-Einstein condensation and high T_c

In the boson-fermion model, the long-range order parameter B_0 is due to Bose-Einstein condensation. Consequently, the phase transition can be of statistical origin, in contrast to the usual BCS theory. As we shall see, the critical temperature T_c may then be much higher. Let us first examine the evidence supporting such a picture.

Recently, Uemura *et al.*[11] discovered that in all (high-temperature) cupric superconductors there is a universality law:

$$T_c \propto \rho^*/m^* \ , \tag{1.30}$$

where ρ^* is the number density of superconducting charge carriers, deduced from λ_L^{-2}, and m^* their effective mass; the proportionality constant is the same for all materials, about

$$40\ \text{K to } 4\times 10^{20}\ \text{cm}^{-3}/m_e \ , \tag{1.31}$$

assuming each carrier bears a charge e. In the boson-fermion model, both bosons and fermions contribute to superconductivity. However, according to (1.9), the London length λ_L is determined by the bosonic component. Thus the ρ^* in the muon-spin-rotation (μSR) experiment[11] should be interpreted as due to bosons of charge $2e$; the proportionality constant would then be reduced by a factor 4, and the experimentally determined proportionality constant (1.31) becomes

$$40 \text{ K} \quad \text{to } 10^{20} \text{ cm}^{-3}/m_e \ . \tag{1.32}$$

In these cupric superconductors, the charge carriers concentrate on the two-dimensional CuO_2 plane; their tunneling between these planes gives rise to the three-dimensional character. The average separation c between CuO_2 planes is approximately constant for different materials:

$$c \cong 6 \text{ Å} \ . \tag{1.33}$$

Introducing a two-dimensional density

$$\sigma \equiv \rho^* c \ , \tag{1.34}$$

one may express (1.30) as

$$T_c \propto \sigma / m^* \ . \tag{1.35}$$

In a Bose-Einstein transition, the following two lengths should be of comparable size:

$$d \equiv \sigma^{-1/2} \quad \text{and} \quad \lambda_T \equiv \left[\frac{2\pi\hbar^2}{M k_B T_c} \right]^{1/2} , \tag{1.36}$$

where d is the interparticle distance, λ_T is the thermal wavelength, k_B is the Boltzmann constant, and M the boson mass. Set $T_c = 40$ K and $M = m^*$. Using (1.32) and (1.33), one finds the corresponding two-dimensional density/mass to be

$$\sigma / M = \rho^* c / m^* \cong 6 \times 10^{12} \text{ cm}^{-2}/m_e \ .$$

Hence, in terms of the boson picture, the *experimental* results (1.30)–(1.32) may be stated as $\lambda_T^2 \sigma = 2\pi\hbar^2 \sigma / M k_B T_c \cong 8$; i.e.,

$$\lambda_T / d \cong 2\sqrt{2} \tag{1.37}$$

for all cupric superconductors. (Note that this number is independent of M; i.e., m^*.)

For a two-dimensional boson system, there is no Bose-Einstein condensation; the corresponding values for $(\lambda_T/d)^2$ would be logarithmically ∞. However, the cupric superconductors are three-dimensional structures, made of parallel layers of CuO_2 planes with spacing c. Even without a definite theoretical idea, one may approach the problem heuristically by writing

$$(\lambda_T/d)^2 \cong \text{const} + \ln(c/l) \ , \tag{1.38}$$

where l is a characteristic two-dimensional distance. When the spacing $c \to \infty$, $\lambda_T \to \infty$ and consequently $T_c = 0$ for a two-dimensional system. We recall that for an ideal three-dimensional boson system $\lambda_T/d = (2.612)^{1/3} = 1.377$ and for liquid He II $\lambda_T/d \cong 1.65$. Here, because of the log term in (1.38), it seems reasonable that the ratio λ_T/d for cupric superconductors could be somewhat larger $\cong 2.8$.

In the BCS theory, T_c depends sensitively on the interaction between electrons and phonons (or other excitations). In the Bose-Einstein condensation, T_c is determined by $\lambda_T \sim d$, which is of statistical origin and therefore can be much higher (T_c exists even without interaction). In the boson picture, on account of (1.36), we have

$$(M T_c)^{1/2} d \approx \text{const} \ . \tag{1.39}$$

"Empirically," this product varies by only a factor less than, or ~ 2, from ideal boson to He, and to cupric superconductors. For He, $d \cong 3.58$ Å, $T_c \cong 2.2$ K, and $M \cong 8000 m_e$, whereas $M \sim 2m_e$ for cupric superconductors; i.e., a mass change by a factor ~ 4000. Thus, if one could have smaller d, then T_c would increase accordingly. Of course, d must not be too small; otherwise, the pair states overlap, and the boson approximation breaks down (as in the case of cold superconductors).

II. COULOMB GAUGE

A. Hamiltonian density

In Ref. 6 [and as outlined in (1.7) and (1.8)], the energy spectrum of the massive vector boson $\mathbf{V}$ is obtained by following the standard (relativistic) Higgs mechanism for spontaneous symmetry breaking. In this section, we will repeat the derivation of the same formula by adopting the Coulomb gauge; this approach is more appropriate for a nonrelativistic theory. It will also be useful for the analysis of the Coulomb screening effect on ξ due to the fermions.

The Hamiltonian density in the Coulomb gauge is

$$H = H_A + H_\phi + H_\psi + H_{\text{Coul}} + H_{\text{int}} \ , \tag{2.1}$$

where

$$H_A = \tfrac{1}{2}[\mathbf{E}_{\text{tr}}^2 + (\nabla \times \mathbf{A})^2] \ , \tag{2.2}$$

$$H_\phi = 2\nu_0 \phi^\dagger \phi + \frac{1}{2M}[(\nabla + 2ie\,\mathbf{A})\phi^\dagger] \cdot (\nabla - 2ie\,\mathbf{A})\phi + f^2(\phi^\dagger\phi)^2 \ , \tag{2.3}$$

$$H_\psi = \frac{1}{2m}[(\nabla + ie\,\mathbf{A})\psi^\dagger] \cdot (\nabla - ie\,\mathbf{A})\psi \ , \tag{2.4}$$

$$H_{\text{Coul}} = -\tfrac{1}{2}(\nabla A_0)^2 + e A_0 (2\phi^\dagger\phi + \psi^\dagger\psi - \rho_{\text{ext}}) \ , \tag{2.5}$$

and

$$H_{\text{int}} = g(\phi^\dagger \psi_\uparrow \psi_\downarrow + \psi_\downarrow^\dagger \psi_\uparrow^\dagger \phi) \ , \tag{2.6}$$

in which † denotes the Hermitian conjugate, $\mathbf{A}$ satisfies the transversality condition $\nabla \cdot \mathbf{A} = 0$, $\mathbf{E}_{\text{tr}}$ is the transverse electric field, conjugate to $\mathbf{A}$,

$$\psi = \begin{bmatrix} \psi_\uparrow \\ \psi_\downarrow \end{bmatrix}$$

describes the fermion (electron or hole) field, A_0 is the electrostatic potential, $-e\rho_{\text{ext}}$ is a constant external charge density to keep the whole system electrically neutral, ϕ is the boson field, $2\nu_0$ is its unrenormalized excitation energy related to the renormalized value 2ν by[5,6]

$$2\nu=2\nu_0+\frac{g^2}{2\Omega}\sum_k \mathrm{P}\left[\nu-\frac{k^2}{2m}\right]^{-1}, \tag{2.7}$$

with P denoting the principal value. The coupling f^2 gives a repulsive force between the bosons; for hard-sphere bosons[12] of diameter a,

$$f^2=2\pi a/M\ . \tag{2.8}$$

(The inclusion of such a repulsive force is useful for a later purpose, but not essential to our discussion. For simplicity, we neglect the paramagnetic interaction of the electrons.)

In the Coulomb gauge, the dynamical variables are ψ, ϕ, and $\mathbf{A}$; their conjugate momenta are $i\psi^\dagger$, $i\phi^\dagger$, and $-\mathbf{E}_{\rm tr}$, which satisfy the following equal-time commutation and anticommutation relations:

$$\{\psi(\mathbf{r}),\psi^\dagger(\mathbf{r}')\}=\delta^3(\mathbf{r}-\mathbf{r}')\ , \tag{2.9}$$

$$[\phi(\mathbf{r}),\phi^\dagger(\mathbf{r}')]=\delta^3(\mathbf{r}-\mathbf{r}')\ , \tag{2.10}$$

and

$$[E_{\rm tr}(\mathbf{r})_i, A(\mathbf{r}')_j]=i(\delta_{ij}-\nabla^{-2}\nabla_i\nabla_j)\delta^3(\mathbf{r}-\mathbf{r}')\ , \tag{2.11}$$

where the subscripts i and j vary from 1 to 3 denoting the space components of $\mathbf{E}_{\rm tr}$ and $\mathbf{A}$. The electrostatic potential $A_0(\mathbf{r})$ is regarded as a dependent variable:

$$A_0(\mathbf{r})=e(4\pi)^{-1}\int\frac{J_0(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\ , \tag{2.12}$$

where

$$J_0(\mathbf{r})=2\phi^\dagger\phi+\psi^\dagger\psi-\rho_{\rm ext}\ . \tag{2.13}$$

The neutrality condition

$$\int J_0(\mathbf{r})d^3r=0 \tag{2.14}$$

will be used as a subsidiary relation satisfied by the state vectors.

Expand the field operators in Fourier components inside a volume Ω with periodic boundary conditions:

$$\psi_\sigma(\mathbf{r})=\sum_k \Omega^{-1/2}a_{\mathbf{k},\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.15}$$

and

$$\phi(\mathbf{r})=B_0e^{i\gamma}+\sum_k\Omega^{-1/2}b_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}\ , \tag{2.16}$$

with B_0 and γ both real c numbers, $\sigma=\uparrow$ or $\downarrow$, $\{a_{\mathbf{k},\sigma},a^\dagger_{\mathbf{k}',\sigma'}\}=\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$, and $[b_{\mathbf{k}},b^\dagger_{\mathbf{k}'}]=\delta_{\mathbf{k}\mathbf{k}'}$. The parameter B_0 is the long-range order.

Similarly, J_0 can be written as

$$J_0(\mathbf{r})=\sum_k\Omega^{-1/2}\rho_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}\ , \tag{2.17}$$

where $\rho_{\mathbf{k}}=\rho^\dagger_{-\mathbf{k}}$ and, because of (2.14),

$$\rho_{\mathbf{k}}=0\quad\text{at }\mathbf{k}=0\ . \tag{2.18}$$

In terms of $\rho_{\mathbf{k}}$, the integral of the Coulomb energy density (2.5) becomes

$$\begin{aligned}\int H_{\rm Coul}d^3r&=\sum_k\left[\frac{e^2}{2k^2}\right]\rho_{\mathbf{k}}\rho_{-\mathbf{k}}\\&=\sum_k\frac{2e^2B_0^2}{k^2}(e^{-i\gamma}b_{\mathbf{k}}+e^{i\gamma}b^\dagger_{-\mathbf{k}})\\&\quad\times(e^{-i\gamma}b_{-\mathbf{k}}+e^{i\gamma}b^\dagger_{\mathbf{k}})+\cdots\ ,\end{aligned} \tag{2.19}$$

where $\cdots$ denotes terms cubic and quartic in $a_{\mathbf{k},\sigma},b_{\mathbf{k}}$ and their Hermitian conjugates. On account of (2.18), the $\mathbf{k}=0$ term is absent in the above sum.

Let μ be the chemical potential. The eigenstates of

$$\int d^3r\,\mathcal{H}\equiv-\mu N+\int d^3rH \tag{2.20}$$

with

$$N=\int(2\phi^\dagger\phi+\psi^\dagger\psi)d^3r \tag{2.21}$$

determine the thermodynamical properties of the system. The function $\mathcal{H}$ will be called the "generalized" Hamiltonian density.

B. Fermion system

Substituting (2.15) and (2.16) into (2.1)–(2.6) and (2.20) and (2.21), we collect all the terms in the generalized Hamiltonian that are quadratic in annihilation and creation operators; the fermion part of the quadratic terms in $\int d^3r\mathcal{H}$ is

$$\int d^3r\,\mathcal{H}_f=\sum_k\left[\left[\frac{k^2}{2m}-\mu\right]a^\dagger_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}+gB_0(e^{-i\gamma}a_{\mathbf{k},\uparrow}a_{-\mathbf{k},\downarrow}+e^{i\gamma}a^\dagger_{-\mathbf{k},\downarrow}a^\dagger_{\mathbf{k},\uparrow})\right]\ , \tag{2.22}$$

where the repeated index σ is summed over $\uparrow$ and $\downarrow$.

Let

$$\begin{aligned}\tilde a_{\mathbf{k},\uparrow}&=a_{\mathbf{k},\uparrow}\cos\theta_k-e^{i\gamma}a^\dagger_{-\mathbf{k},\downarrow}\sin\theta_k\ ,\\\tilde a_{-\mathbf{k},\downarrow}&=e^{i\gamma}a^\dagger_{\mathbf{k},\uparrow}\sin\theta_k+a_{-\mathbf{k},\downarrow}\cos\theta_k\ ,\end{aligned} \tag{2.23}$$

$$\begin{aligned}&\sin2\theta_k=gB_0/E_k,\cos2\theta_k=\left[\frac{k^2}{2m}-\mu\right]\Big/E_k\ ,\\&E_k=\left[\left[\frac{k^2}{2m}-\mu\right]^2+g^2B_0^2\right]^{1/2}\ .\end{aligned} \tag{2.24}$$

Evidently $\{\tilde a_{\mathbf{k},\sigma},\tilde a^\dagger_{\mathbf{k}',\sigma'}\}=\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$, and the transformation (2.23) and (2.24) is therefore canonical.[13,14] Correspondingly, (2.22) becomes

$$\int d^3r\,\mathcal{H}_f=\sum_k\left[\frac{k^2}{2m}-\mu-E_k+E_k\tilde a^\dagger_{\mathbf{k},\sigma}\tilde a_{\mathbf{k},\sigma}\right]\ , \tag{2.25}$$

in which the last term gives the fermion spectrum (1.10).

C. Boson spectrum ($g=e=0$)

We first discuss the hypothetical limit $g=e=0$, keeping only the "hard-sphere" interaction $f\neq0$. In this case, $\nu=\nu_0$ and the relevant generalized Hamiltonian density $\mathcal{H}$ can be written as

$$\mathcal{H}=\frac{1}{2M}(\nabla\phi)^\dagger\cdot\nabla\phi+V(\phi^\dagger\phi) , \tag{2.26}$$

where

$$V(\phi^\dagger\phi)=2(\nu_0-\mu)\phi^\dagger\phi+f^2(\phi^\dagger\phi)^2 . \tag{2.27}$$

In the expansion (2.16) of the boson field, the constant B_0 depends on μ; it is determined by requiring V to be a minimum at B_0^2; i.e.,

$$V_0'\equiv\frac{dV(B_0^2)}{dB_0^2}=2(\nu_0-\mu+f^2B_0^2)=0 , \tag{2.28}$$

which gives

$$B_0^2=(\mu-\nu_0)/f^2 . \tag{2.29}$$

Spontaneous symmetry breaking occurs only when $B_0\neq 0$; therefore, $\mu>\nu_0$. The expansion (2.16) can also be written as

$$\phi(\mathbf{r})=B_0e^{i\gamma}+\chi(\mathbf{r}) , \tag{2.30}$$

where

$$\chi(\mathbf{r})=\sum_k\Omega^{-1/2}b_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}} . \tag{2.31}$$

Substituting (2.30) into (2.27) and using (2.28), we have

$$V(\phi^\dagger\phi)=V(B_0^2)+\tfrac{1}{2}V_0''(\phi^\dagger\phi-B_0^2)^2+O(\phi^\dagger\phi-B_0^2)^3 , \tag{2.32}$$

where

$$V_0''\equiv dV_0'/dB_0^2=d^2V(B_0^2)/d(B_0^2)^2 . \tag{2.33}$$

Since

$$\phi^\dagger\phi-B_0^2=B_0(e^{-i\gamma}\chi+e^{i\gamma}\chi^\dagger)+\chi^\dagger\chi , \tag{2.34}$$

neglecting the $|\chi|^3$ term, we find (2.26) to be

$$\mathcal{H}=V(B_0^2)+\frac{1}{2M}(\nabla\chi^\dagger)\cdot\nabla\chi+\tfrac{1}{2}V_0''B_0^2(e^{-i\gamma}\chi+e^{i\gamma}\chi^\dagger)^2 . \tag{2.35}$$

Introduce

$$\tilde{b}_\mathbf{k}=b_\mathbf{k}\cosh\tilde{\theta}_k+e^{2i\gamma}b^\dagger_{-\mathbf{k}}\sinh\tilde{\theta}_k , \tag{2.36}$$

with

$$\sinh 2\tilde{\theta}_k=2\lambda/\omega_k,\quad \cosh 2\tilde{\theta}_k=\left[\frac{k^2}{2M}+2\lambda\right]\Big/\omega_k$$

and

$$\omega_k=\left[\left[\frac{k^2}{2M}\right]^2+\frac{2\lambda k^2}{M}\right]^{1/2} . \tag{2.37}$$

Since $[\tilde{b}_\mathbf{k},\tilde{b}^\dagger_{\mathbf{k}'}]=\delta_{\mathbf{k}\mathbf{k}'}$, the transformation (2.36) is canonical. Setting

$$\lambda=\tfrac{1}{2}V_0''B_0^2 \tag{2.38}$$

and substituting (2.36) into (2.35), we obtain

$$\int d^3r\,\mathcal{H}=\Omega V(B_0^2)+\sum_k\left[\frac{1}{2}\left[\omega_k-\frac{k^2}{2M}\right]+\tilde{b}^\dagger_\mathbf{k}\tilde{b}_\mathbf{k}\omega_k\right] .$$

The last term gives the boson excitation spectrum $\omega_k=[(k^2/2M)^2+k^2v^2]^{1/2}$, which agrees with the well-known formula[12,15] for a system of Bose hard spheres of diameter a; in this case, $f^2=2\pi a/M$, $V_0''=2f^2$, and therefore $v^2=4\pi aB_0^2/M^2$. The above spectrum also agrees with the general conditions (ii) and (iii) mentioned in Sec. I [in the paragraph following (1.11)].

D. Boson spectrum (general case)

In the general case, g,e, and f are all nonzero. We must include the nonlinear boson-boson interaction induced by (2.6), the coupling between the boson and the fermion pairs. The summation over all one-loop fermion graphs is equivalent to the transformation (2.23). Following the steps shown in Ref. 6, we see that the long-range order parameter B_0 is now determined by, instead of (2.28),

$$U_0'\equiv\frac{dU(B_0^2)}{dB_0^2}=0 , \tag{2.39}$$

where

$$U_0'=2[(\nu_0-\mu)+f^2B_0^2]+\Omega^{-1}\sum_k\frac{\partial E_k}{\partial B_0^2}(-1+\tilde{a}^\dagger_{\mathbf{k},\sigma}\tilde{a}_{\mathbf{k},\sigma}) , \tag{2.40}$$

in which the first term with the square bracket is simply the previous V_0' and the second term is obtained by differentiating (2.25) with respect to B_0^2. Hence (2.39) and (2.40) give

$$2(\nu_0-\mu+f^2B_0^2)=\Omega^{-1}\sum_k\frac{g^2}{2E_k}(1-\tilde{a}^\dagger_{\mathbf{k},\sigma}\tilde{a}_{\mathbf{k},\sigma}) , \tag{2.41}$$

which, when $f^2=0$ and with $\tilde{a}^\dagger_{\mathbf{k},\sigma}\tilde{a}_{\mathbf{k},\sigma}$ evaluated for a grand canonical ensemble, reduces to Eq. (4.3) in Ref. 6. As in (2.22), we collect in the generalized Hamiltonian density $\mathcal{H}$ all quadratic terms in the bosonic operator χ; the resulting expression is, in place of (2.26) and (2.32),

$$\mathcal{H}_b=\frac{1}{2M}(\nabla\chi^\dagger)\cdot(\nabla\chi)+\tfrac{1}{2}U_0''B_0^2(e^{-i\gamma}\chi+e^{i\gamma}\chi^\dagger)^2+H_{\mathrm{Coul}} , \tag{2.42}$$

where, as in (2.33),

$$U_0''=dU_0'/dB_0^2 , \tag{2.43}$$

and H_{Coul} is given by (2.19). Set in the present case, in place of (2.38),

$$\lambda=\left[\frac{1}{2}U_0''+\frac{2e^2}{k^2}\right]B_0^2 , \tag{2.44}$$

and use the transformation (2.36) and (2.37). It can be readily verified that

$$\int d^3r\,\mathcal{H}_b=\sum_k\left[\frac{1}{2}\left[\omega_k-\frac{k^2}{2M}\right]+\tilde{b}_{\mathbf{k}}^{\dagger}\tilde{b}_{\mathbf{k}}\omega_k\right], \tag{2.45}$$

where ω_k is given by (2.37); i.e., on account of (2.44),

$$\omega_k=\omega_l(k)\equiv\left[\left[\frac{k^2}{2M}\right]^2+k^2v^2+\lambda_L^{-2}\right]^{1/2}, \tag{2.46}$$

the same as (1.8) with the sound-velocity squared

$$v^2=U_0''B_0^2/M$$
$$=\frac{2B_0^2}{M}\left[f^2+\frac{g^4}{8}\Omega^{-1}\sum_k\frac{1}{E_k^3}(1-\tilde{a}_{\mathbf{k},\sigma}^{\dagger}\tilde{a}_{\mathbf{k},\sigma})\right] \tag{2.47}$$

and the inverse square of the London length

$$\lambda_L^{-2}=4e^2B_0^2/M\ . \tag{2.48}$$

For a grand canonical average at $\beta=(k_BT)^{-1}$, we have

$$\langle\tilde{a}_{\mathbf{k},\uparrow}^{\dagger}\tilde{a}_{\mathbf{k},\uparrow}\rangle=\langle\tilde{a}_{\mathbf{k},\downarrow}^{\dagger}\tilde{a}_{\mathbf{k},\downarrow}\rangle=(1+e^{\beta E_k})^{-1}, \tag{2.49}$$

and therefore

$$\langle 1-\tilde{a}_{\mathbf{k},\sigma}^{\dagger}\tilde{a}_{\mathbf{k},\sigma}\rangle=\tanh\tfrac{1}{2}\beta E_k\ . \tag{2.50}$$

Applying these averages to (2.41), we rederive the same formula [Eq. (4.3) of Ref. 6], that was obtained previously by using the grand partition function directly. The spectrum (2.46) is valid "mechanically" as the energy-level formula; in addition, it can be used thermodynamically with an appropriate ensemble average, as in (2.49) and (2.50). The resulting formulas are valid for any $T<T_c$.

Substituting (2.16) into (2.3) and combining it with (2.2), we see that in $\mathcal{H}$ the quadratic terms in $\mathbf{A}$ and its conjugate momentum are given by

$$\mathcal{H}_A=\frac{1}{2}\left[\mathbf{E}_{\rm tr}^2+(\nabla\times\mathbf{A})^2+\frac{4e^2B_0^2}{M}\mathbf{A}^2\right], \tag{2.51}$$

which gives the spectrum $\omega_t(k)$ of (1.7):

$$\omega_t(k)=(k^2+\lambda_L^{-2})^{1/2}\ . \tag{2.52}$$

E. Remarks

(i) The use of local fields makes it easy to have gauge invariance. The same spectra (1.7)–(1.10) have also been derived in Ref. 6, based on the unitary gauge. These spectra do not explicitly contain the plasma oscillation (which includes fermions, and will be discussed in the next section, together with the Debye length). In this connection, we must differentiate a "microscopic" excitation formula [such as (1.7)–(1.10)] from a "macroscopic" collective mode (such as plasma oscillations, nuclear giant resonances, and the sound vibration of a fermionic system). By putting more quanta in the same bosonic microscopic excitation level, one can always generate a macroscopic collective motion. However, not all macroscopic collective motion can have an obvious microscopic excitation-level realization in the Hamiltonian. A simple example is a *free* fermionic system: Its sound vibration does not explicitly appear in its Hamiltonian, $\sum a_{\mathbf{k},\sigma}^{\dagger}a_{\mathbf{k},\sigma}k^2/2m$.

(ii) In Sec. I C we mention that the London length λ_L and, therefore, also the μSR experiment measure the boson condensate density B_0^2. On the other hand, the stoichiometric analysis determines the total carrier density ρ (in units of e). As may be deduced from the estimate (3.40) in the next section, these two densities can be comparable but *different*, with $4B_0^2<\rho$.

III. VORTEX FILAMENT

We now turn to the problem of a single vortex filament trapped inside a superconductor. In the literature, the same problem has been extensively analyzed based on the Ginzburg-Landau equation.[16] Our method differs in the following way: The Ginzburg-Landau equation describes phenomenologically a *thermodynamical* system, while the boson-fermion model of this paper refers to a *mechanical* system. By applying the ensemble average, we will be able to derive the equivalent of the "Ginzburg-Landau-like" equation; as we shall see, the result is similar, but there are also some essential differences.

A. Basic equations

We begin with the same Hamiltonian density H (2.1)–(2.6), and the same $\mathcal{H}$ introduced in (2.20) and (2.21). In this section, we approximate the electromagnetic potentials $\mathbf{A}$ and A_0 as classical fields. Let r, θ, and z be the cylindrical coordinates. We assume that

$$\mathbf{A}(\mathbf{r})=\hat{\boldsymbol{\theta}}A(r)\ , \tag{3.1}$$

where $\hat{\boldsymbol{\theta}}$ is a unit vector along the direction of the polar angle, $A(r)$ is a c number function which satisfies the boundary condition for a vortex filament

$$A(r)\rightarrow\frac{1}{2er}\quad\text{as } r\rightarrow\infty\ . \tag{3.2}$$

Instead of (2.16) and (2.30), the quantum field operator $\phi(\mathbf{r})$ is written as

$$\phi(\mathbf{r})=B(r)e^{i\theta}+\chi(\mathbf{r})\ , \tag{3.3}$$

where $B(r)$ is a c number function and $\chi(\mathbf{r})$ is the same operator given by (2.31), so that the commutation relation (2.10) holds. We assume the whole space is filled with the superconductor; therefore,

$$B(r)\rightarrow B_0\quad\text{as } r\rightarrow\infty\ , \tag{3.4}$$

where B_0 is determined by (2.41). In (3.3), because θ is the polar angle, we must have

$$B(r)=0\quad\text{at } r=0 \tag{3.5}$$

in order that the operator χ be well defined at the origin. Likewise,

$$A(r)=0\quad\text{at } r=0\ . \tag{3.6}$$

Next, we make the Thomas-Fermi approximation for the fermion field ψ. The details are given in the Appendix. As shown there, the result for the ground state is to replace the following terms in the generalized Hamiltonian density $\mathcal{H}$:

$$H_\psi + H_{\rm int} + (eA_0 - \mu)\psi^\dagger\psi \;, \tag{3.7}$$

where H_ψ and $H_{\rm int}$ are introduced in (2.4) and (2.6), by

$$\mathcal{H}_f(r) = (2\pi)^{-3}\int d^3k[\varepsilon_k(r) - \mu - E_k(r)] \;, \tag{3.8}$$

with

$$\varepsilon_k(r) = \frac{k^2}{2m} + eA_0(r) \tag{3.9}$$

and

$$E_k(r) = [(\varepsilon_k(r) - \mu)^2 + g^2B(r)^2]^{1/2} \;. \tag{3.10}$$

For the ground state we can neglect the boson excitation χ. Thus, on account of (3.1), (3.3), and (3.8), the total generalized Hamiltonian density becomes

$$\mathcal{H}(r) = \frac{1}{2}\left[\frac{1}{r}\frac{d}{dr}(rA)\right]^2 - \frac{1}{2}\left[\frac{dA_0}{dr}\right]^2 - eA_0\rho_{\rm ext} + 2(\nu_0 - \mu + eA_0)B^2 + \frac{1}{2M}\left[\left[\frac{dB}{dr}\right]^2 + \left[\frac{1}{r} - 2eA\right]^2 B^2\right] + f^2B^4 + \mathcal{H}_f \;. \tag{3.11}$$

Setting the variational derivatives of $\int d^3r\,\mathcal{H}$ with respect to $A(r)$, $A_0(r)$, and $B(r)$ zero, we derive

$$-\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rA)\right] = \frac{2eB^2}{M}\left[\frac{1}{r} - 2eA\right] , \tag{3.12}$$

$$-\frac{1}{r}\frac{d}{dr}\left[r\frac{dA_0}{dr}\right] = e(2B^2 - \rho_{\rm ext}) + e\int(2\pi)^{-3}d^3k\left[1 - \frac{\epsilon_k - \mu}{E_k}\right] , \tag{3.13}$$

and

$$-\frac{1}{2M}\left[\frac{1}{r}\frac{d}{dr}\left[r\frac{dB}{dr}\right] - \left[\frac{1}{r} - 2eA\right]^2 B\right] = [-2(f^2B^2 + \nu - \mu + eA_0) + G]B \;, \tag{3.14}$$

where

$$G = -2(\nu_0 - \nu) + g^2(2\pi)^{-3}\int d^3k(2E_k)^{-1} \;. \tag{3.15}$$

By using (2.7) and (3.10), we find [as given by Eq. (5.2) in Ref. 6]

$$G = \left[\frac{g}{\pi}\right]^2\left[\frac{m}{2}\right]^{3/2}(\mu - eA_0)^{1/2} \times\left[-4 + 2\ln\frac{8(\mu - eA_0)}{gB}\right] , \tag{3.16}$$

in which we neglect $(gB/\mu)^2 \ll 1$. Equations (3.12)–(3.14) plus the boundary conditions (3.2), (3.4)–(3.6), and $A_0(r)\to 0$ as $r\to\infty$ determine the shape and fields of the vortex filaments. At infinity, the left-hand sides of (3.12)–(3.14) all tend to zero, so do then the right-hand sides. Hence, (3.12) implies the boundary condition (3.2), (3.13) insures the neutrality condition, and (3.14) determines B_0:

$$f^2B_0^2 + \nu - \mu = \left[\frac{g}{\pi}\right]^2\left[\frac{m}{2}\right]^{3/2}\sqrt{\mu}\left[-2 + \ln\frac{8\mu}{gB_0}\right] . \tag{3.17}$$

Equations (3.12)–(3.15) are valid for the ground state subject to the boundary condition that there is a vortex filament in the system. It is clear that these equations are different from the Ginzburg-Landau equation; of course, both share similar features.

B. Linearized equations

At large distances, we may regard

$$\alpha(r) \equiv \frac{1}{2er} - A(r) \;, \qquad \beta(r) \equiv B_0 - B(r) \;, \tag{3.18}$$

and A_0 as small. Equations (3.11)–(3.14) take on the linearized form

$$-\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\alpha)\right] = -\lambda_L^{-2}\alpha \;, \tag{3.19}$$

$$-\frac{1}{r}\frac{d}{dr}\left[r\frac{dA_0}{dr}\right] = -\lambda_D^{-2}A_0 - 2qB_0\beta \;, \tag{3.20}$$

$$-\frac{1}{r}\frac{d}{dr}\left[r\frac{d\beta}{dr}\right] = -(2Mv)^2\beta + 2qMB_0A_0 \;, \tag{3.21}$$

where, as before,

$$\lambda_L^{-2} = (2eB_0)^2/M \;, \tag{3.22}$$

λ_D is the Debye length,

$$\lambda_D^{-2} = \tfrac{3}{2}e^2\rho_f/\mu \tag{3.23}$$

with $\rho_f = (2m\mu)^{3/2}/(3\pi^2)$ the fermion density, v is the same sound velocity given by (2.47),

$$v^2 = \frac{2B_0^2}{M}\left[f^2 - \frac{1}{2}\frac{\partial G_0}{\partial B_0^2}\right] , \tag{3.24}$$

and the "effective" charge q is

$$q=2e+e\frac{\partial G_0}{\partial\mu}\ , \tag{3.25}$$

where G_0 is a function of μ and B_0^2, given by $G_0=G$ of (3.15) and (3.16) at $A_0=0$ and $B=B_0$; i.e.,

$$G_0=\left[\frac{g}{\pi}\right]^2\left[\frac{m}{2}\right]^{3/2}\mu^{1/2}\left[-4+2\ln\frac{8\mu}{gB_0}\right]. \tag{3.26}$$

For practical applications, $\partial G_0/\partial\mu$ turns out to be rather small, $\ll 1$; hence,

$$q\cong 2e\ . \tag{3.27}$$

Note that in (3.14), $(r^{-1}-2eA)^2B=(2e\alpha)^2B$ is quadratic in α, and therefore absent in the linearized equation (3.21); in addition, we neglect $(gB/\mu)^2\ll 1$ in (3.20).

The solution of (3.19) is

$$\alpha(r)=(2e\lambda_L)^{-1}K_1(r/\lambda_L)\ ; \tag{3.28}$$

correspondingly, the magnetic field is parallel to the z axis (with $\hat{z}$ as its unit vector),

$$\nabla\times\mathbf{A}=\hat{z}(2e\lambda_L^2)^{-1}K_0(r/\lambda_L)\ , \tag{3.29}$$

where, in terms of the Bessel function $J_m(x)$ and the Neumann function $N_m(x)$,

$$K_m(x)=\tfrac{1}{2}\pi i^{m+1}[J_m(ix)+iN_m(ix)]\ .$$

At large r, $K_m(r/\lambda_L)\to(\pi\lambda_L/2r)^{1/2}e^{-r/\lambda_L}$, and the magnetic field has the same distribution as that given by the Ginzburg-Landau equation. However, the matter distribution is different even at large distances.

To derive the exponential decay rate of A_0 and β in space at large distances, we may assume both to vary as $e^{-\sqrt{2}\mu r}$, as $r\to\infty$. In (3.20) and (3.21), we may replace (d/dr) by $-\sqrt{2}\mu$ and neglect $1/\mu r$. This gives two coupled linear homogeneous equations for A_0 and β. On account of (3.27), the eigenvalue μ satisfies

$$\begin{vmatrix}\lambda_D^{-2}-2\mu^2 & 4eB_0\\ -4eMB_0 & (2Mv)^2-2\mu^2\end{vmatrix}=0\ . \tag{3.30}$$

As in (1.12), we may set $k=i\sqrt{2}\mu$, then k satisfies

$$\left[\frac{k^2}{2M}\right]^2+k^2v'^2+(\lambda_L')^{-2}=0\ , \tag{3.31}$$

where

$$v'^2=v^2+(2M\lambda_D)^{-2} \tag{3.32}$$

and

$$(\lambda_L')^{-2}=\lambda_L^{-2}+(v/\lambda_D)^2\ . \tag{3.33}$$

Equation (3.31) is of the same form as $\omega_l(k)=0$ given by (1.12), except for the modification

$$\lambda_L\to\lambda_L' \quad\text{and}\quad v\to v'\ . \tag{3.34}$$

Consequently, the solution $\mu=\mu_\pm$ is given by (1.13), after the same change (3.34); i.e.,

$$\mu_\pm^2=(Mv')^2\pm[(Mv')^4-(M/\lambda_L')^2]^{1/2}\ . \tag{3.35}$$

Without screening, we have $\lambda_D\to\infty$; therefore, $\lambda_L'=\lambda_L$ and $v'=v$, which converts (3.31) into $\omega_l(k)=0$, as given by (1.12) and (1.13).

C. Estimations

For a fermion density $\rho_f=(2m\mu)^{3/2}/3\pi^2\sim 10^{21}\ \text{cm}^{-3}$, and assuming $m=m_e$, the free electron mass, we have $\mu\sim 0.3$ eV and

$$\lambda_D\sim 1\ \text{Å}\ . \tag{3.36}$$

From the observed London length $\lambda_L\sim 10^3$ Å and assuming $M=2m_e$, by using (1.9) we may estimate

$$B_0^2\sim 10^{21}\ \text{cm}^{-3}\ . \tag{3.37}$$

Together with a gap energy $\Delta=gB_0\sim 3\times 10^{-2}$ eV, the coupling constant g can be estimated. We find the dimensionless quantity

$$\bar{g}^2\equiv\left[\frac{g}{\pi}\right]^2\left[\frac{m}{2}\right]^{3/2}\frac{1}{\sqrt{\mu}}\sim 3\times 10^{-3}\ . \tag{3.38}$$

Ignoring the hard-sphere interaction, we may set $f^2=0$ in (3.17) and find

$$\nu-\mu=\bar{g}^2\mu\left[-2+\ln\frac{8\mu}{gB_0}\right]\sim 7\times 10^{-3}\mu\ ; \tag{3.39}$$

from (3.24) and (3.26), we have, in the same approximation $f^2=0$ and in units of $c=1$,

$$v^2=\bar{g}^2\frac{\mu}{M}\sim 10^{-9}\ . \tag{3.40}$$

Thus, even with $\lambda_D\sim 1$ Å we have

$$(v\lambda_L/\lambda_D)^2\sim 10^{-3}\ , \tag{3.41}$$

which can be neglected, and therefore

$$\lambda_L'\cong\lambda_L\ . \tag{3.42}$$

The discussion on the coherence length ξ given in Sec. I depends only on (1.13), and is valid for arbitrary v; hence the replacement of v by v' in (3.34) and (3.35) does not alter any of the previous analysis, nor does it affect the conclusion of a small $\xi\sim$ few Å.

For $\lambda_D\sim 1$ Å, we have $(2M\lambda_D)^{-1}\sim 10^{-3}$ which gives $v'^2\cong(2M\lambda_D)^{-2}\sim 10^{-6}$. Since $(M\lambda_L)^{-1}$ is also $\sim 10^{-6}$, these two are of the same order of magnitude. From (1.14)–(1.21) and depending on the material, we may have either case (1) $v'^2>(M\lambda_L)^{-1}$ or case (2) $v'^2<(M\lambda_L)^{-1}$, exactly the same as in Sec. I. Note that all high-T_c superconductors are ionic crystals. It is questionable whether the type of "free" electron model used here is applicable. In the following, we shall regard λ_D as a phenomenological parameter, which may be greatly modified from (3.23). For a heuristic approach, assume (3.31)–(3.33) remain valid.

Two extreme limits may be of interest.

1. No screening ($\lambda_D \to \infty$)

In this case $v'=v$ and $\lambda_L'=\lambda_L$; we expect, on account of (3.40), $v^2<(M\lambda_L)^{-1}$, and therefore

$$\xi \sim \left[\frac{\lambda_L}{M}\right]^{1/2} \tag{3.43}$$

in accordance with (1.19). At T near (but less than) T_c,

$$\lambda_L^{-2} \sim T_c - T \tag{3.44}$$

but

$$\xi^{-2} \sim (T_c - T)^{1/2} \tag{3.45}$$

as discussed in (1.23) and (1.24).

2. Perfect screening ($\lambda_D \to 0$)

In this case, $v'^2=(2M\lambda_D)^{-2}$ and $(\lambda_L')^{-2}=(v/\lambda_D)^2$. The two roots of (3.25) are $\mu_+^2=2(Mv')^2=(2\lambda_D^2)^{-1}$ and

$$\frac{1}{\xi^2}=\mu_-^2=\tfrac{1}{2}(v'\lambda_L')^{-2}=2(Mv)^2 . \tag{3.46}$$

Because of (3.24) and the boson condensate density $B_0^2 \sim (T_c-T)$ as T is less than but near T_c, $v^2 \sim (T_c - T)$ also; therefore,

$$\xi^{-2} \sim (T_c - T) , \tag{3.47}$$

instead of (3.45). In this case both H_{c1} and H_{c2}, given by (1.25) and (1.26), vary as $T_c - T$ when T is near T_c. But the ratio is very large,

$$\frac{H_{c2}}{H_{c1}} \sim \frac{2\lambda_L^2}{\xi^2}\left[\ln\frac{\lambda_L}{\xi}\right]^{-1} \gg 1 . \tag{3.48}$$

IV. CRITICAL FIELDS

A. H_{c1}

Let $\mathbf{h}$ be the magnetic induction (usually called $\mathbf{B}$) and $\mathbf{H}$ the magnetic field in the Maxwell equations for material. Consider a long cylinder of superconductor with its length L parallel to the z axis. Apply an external field $H\hat{z}$ where $\hat{z}$ is the unit vector $\parallel z$. Since $\nabla \times \mathbf{H}=0$, the field $\mathbf{H}$ is uniform inside the superconductor. For simplicity, consider the special case when $T=0$. The free energy F is defined by

$$F=E-\int d^3r\, \mathbf{h}\cdot\mathbf{H} , \tag{4.1}$$

where E is the total energy. At a fixed $\mathbf{H}$, the function F should be a minimum.

The critical field H_{c1} is determined by the condition that the free energy F_0 with zero vortex filament is equal to the free energy F_1 with one vortex filament inside. For the former, write $E=E_0$; since $\mathbf{h}=0$,

$$F_0=E_0 . \tag{4.2}$$

For the latter, we have $E=E_1$ given by the integral of (3.11), since $\int \mathbf{h}\, d^3r=\hat{z}L\pi/e$; the free energy

$$F_1=E_1-(\pi/e)LH . \tag{4.3}$$

Because $\lambda \gg \xi$, by using (3.18) and (3.28) we find the difference E_1-E_0 to be dominated by the following integral over the region $r>\xi$:

$$E_1-E_0 \cong L\int \frac{1}{2M}B_0^2\left[\frac{1}{r}-2eA\right]^2 d^2r$$

$$\cong \frac{L}{M}\pi B_0^2 \ln(\lambda_L/\xi) , \tag{4.4}$$

on account of $K_1(r/\lambda_L)\to\lambda_L/r$ for $r \ll \lambda_L$. This, together with (1.9) and $F_1=F_0$, gives, as in the usual Ginzburg-Landau case for type-II superconductors,

$$H_{c1} \cong \frac{1}{4e\lambda_L^2}\ln(\lambda_L/\xi) . \tag{4.5}$$

B. H_{c2}

When one steadily increases the external magnetic field H, more vortex filaments appear inside the superconductor, until they reach saturation when the spacing between filaments is $\sim\xi$. To conform to the usual convention we define H_{c2} to be the critical field which, when multiplied by the area $2\pi\xi^2$, gives a product equal to the quantized flux π/e of each filament; i.e.,

$$H_{c2}=(2e\xi^2)^{-1} . \tag{4.6}$$

Because the material inside the vortex filament $r<\xi$ is essentially normal [since $B(r)=0$ at $r=0$], a superconductor saturated with vortex filaments becomes a normal conductor.

Conversely, one may turn the argument around by starting from the normal state under a very large external magnetic field, then decreasing the field steadily until a bubble of superconducting material appears inside the normal state. It is possible that these two procedures may give different results (like the superheated liquid and the supercooled gas in a liquid-gas transition); however, when $T=T_c$, it seems reasonable that they should agree. This alternative method will be discussed in the following section. Although the Ginzburg-Landau equation is not used, our approach follows the standard line.

C. An alternative derivation of H_{c2}

We start from a normal state with a uniform magnetic field $\mathbf{H}$, and at a given temperature T and chemical potential μ. Write

$$\phi(\mathbf{r})=\sum_k \Omega^{-1/2} b_{\mathbf{k}}' e^{i\mathbf{k}\cdot\mathbf{r}} , \tag{4.7}$$

where

$$[b_{\mathbf{k}}', b_{\mathbf{k}'}'^{\dagger}]=\delta_{\mathbf{k}\mathbf{k}'} .$$

Next consider a different expansion of the same operator ϕ:

$$\phi(\mathbf{r})=b(\mathbf{r})+\chi(\mathbf{r}) , \tag{4.8}$$

where $\chi(\mathbf{r})$ satisfies (2.31) with $[b_{\mathbf{k}},b^{\dagger}_{\mathbf{k}'}]=\delta_{\mathbf{k}\mathbf{k}'}$, and $b(\mathbf{r})$ is a complex c number function, confined in space and subject to the boundary condition

$$b(\mathbf{r})=0 \quad \text{as } \mathbf{r}\to\infty \ . \tag{4.9}$$

Let

$$\mathcal{Q}\equiv \mathrm{Tr}\, e^{-\beta\mathcal{H}}$$

be the grand partition function, and p be the pressure, given by

$$p=k_B T\Omega^{-1}\ln\mathcal{Q} \ . \tag{4.10}$$

We shall calculate p as a perturbation series in the coupling g that appears in the interaction (2.6) between ϕ and the fermion pair; however, the product $gb(\mathbf{r})$ is to be taken as a *zeroth* order term. To any *given* order in g^2, the pressure $p(b^2,T,\mu)$ thus computed using (4.8) will be different from that using (4.7). The latter will be denoted by $p(0,T,\mu)$, since the c number function $b(\mathbf{r})$ is zero.

Regard $p(0,T,\mu)$ as the thermodynamic pressure of the normal phase. Under any small variation of b from 0 to an infinitesimal $b(\mathbf{r})$ at fixed T and μ, the normal phase is unstable if the change $\delta p\equiv p(b^2,T,\mu)-p(0,T,\mu)>0$, stable if $\delta p<0$. [Note that Eq. (4.9) insures that the magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{h}$ at infinity are unaltered.]

In the following we shall calculate δp only to the zeroth order in g, but to second order in b and gb. The corresponding differences in the expectation values of the electric current $\mathbf{J}$ and the electric charge J_0 are $O(b^2)$, since the expectation value of χ, at $b=0$, is zero. In the approximation that the electromagnetic potentials $\mathbf{A}$ and A_0 are classical fields (but the matter fields ϕ and ψ remain quantum mechanical), the induced changes $\delta\mathbf{A}$ and δA_0 are likewise $O(b^2)$, because of the Maxwell equations. The validity of the Maxwell equations also insures that the first-order variation of p with respect to $\delta\mathbf{A}$ and δA_0 is zero; hence, the part of δp that is quadratic in $\delta\mathbf{A}$ and δA_0 can be neglected, since it is $O(b^4)$. To $O(b^2)$, the variation δp is entirely due to the matter part, and can be written approximately as

$$\delta p=\Omega^{-1}\int d^3r\left[-\frac{1}{2M}|(\nabla-i2e\,\mathbf{A})b|^2+\left[\frac{\partial p_0}{\partial b_0^2}\right]_{0,T,\mu}|b|^2\right] , \tag{4.11}$$

where $p_0=p_0(b_0^2,T,\mu)$ is the pressure function evaluated by using (4.10) with $\mathbf{H}=0$ and for a b function which is a real *constant* b_0; i.e., in place of (4.8),

$$\phi(\mathbf{r})=b_0+\chi(\mathbf{r}) \ .$$

We note that the first term inside the square brackets in (4.11) is simply the explicit $O(b^2)$ term due to the substitution of (4.8) into the negative of the bosonic kinetic energy

$$-(2M)^{-1}(\nabla+i2e\,\mathbf{A})\phi^{\dagger}\cdot(\nabla-i2e\,\mathbf{A})\phi \ ;$$

there is also an $O(b)$ cross term

$$-(2M)^{-1}(\nabla+i2e\,\mathbf{A})\chi^{\dagger}\cdot(\nabla-i2e\,\mathbf{A})b+\text{H.c.} \tag{4.12}$$

While (4.12) has zero expectation value, in the second order perturbation it would generate an $O(b^2)$ contribution to δp. This contribution is neglected here. Therefore, its validity depends on the thermal excitations at T near T_c being mainly fermionic, not bosonic.

When $T=T_c$ and $\mu=\mu_c$ (the chemical potential at the critical point), the normal state $b_0=0$ is the equilibrium phase with $\mathbf{H}=0$; we have

$$\left[\frac{\partial p_0}{\partial b_0^2}\right]_c\equiv\left[\frac{\partial p_0}{\partial b_0^2}\right]_{0,T_c,\mu_c}=0 \ , \tag{4.13}$$

with the subscript c denoting the critical point. Next, consider a normal phase configuration very near the critical point with μ slightly different from μ_c, T slightly less than T_c, but keeping $\mathbf{H}=0$, so that the normal state, $b_0=0$, is not an equilibrium phase; hence

$$\left[\frac{\partial p_0}{\partial b_0^2}\right]_{0,T,\mu}>0 \ , \tag{4.14}$$

where the three subscripts outside the parentheses indicate that we set $b_0=0$ after the differentiation, and keep T and μ fixed during the differentiation. In order for the normal phase to be stable, we must have $\mathbf{H}\neq 0$. The expression δp in (4.11) refers to the change of p from the state $b_0=0$ and $\mathbf{H}\neq 0$ to one with $b(\mathbf{r})\neq 0$, but keeping the same $\mathbf{H}$. [To $O(b^2)$, the part in p due to $\mathbf{H}\neq 0$ is not changed during this variation, and is therefore absent in (4.11)].

Express the difference between (4.14) and (4.13) in terms of a Taylor series:

$$\left[\frac{\partial p_0}{\partial b_0^2}\right]_{0,T,\mu}=\left[\left[(T-T_c)\frac{\partial}{\partial T}+(\mu-\mu_c)\frac{\partial}{\partial\mu}\right]\frac{\partial p_0}{\partial b_0^2}\right]_{0,T_c,\mu_c} . \tag{4.15}$$

Let b_0 be the value that makes $p_0(b_0^2,T,\mu)$ stationary. Therefore,

$$\left[\frac{\partial p_0}{\partial b_0^2}\right]_{b_0^2,T,\mu}=0 \ . \tag{4.16}$$

Taking the difference between (4.16) and (4.13), we find

$$\left[\frac{\partial^2 p_0}{\partial(b_0^2)^2}\right]_c b_0^2+\left[\frac{\partial^2 p_0}{\partial b_0^2\partial T}\right]_c(T-T_c)+\left[\frac{\partial^2 p_0}{\partial b_0^2\partial\mu}\right]_c(\mu-\mu_c)=0 \ ,$$

which, upon substitution into (4.15), gives

$$\left[\frac{\partial^2 p_0}{\partial b_0^2}\right]_{0,T,\mu}=-\left[\frac{\partial^2 p_0}{\partial(b_0^2)^2}\right]_c b_0^2=Mv^2 \ , \tag{4.17}$$

where v is the sound velocity given by (3.24). Because in a constant local magnetic field $\mathbf{h}=h\hat{z}=(\nabla\times\mathbf{A})$, the lowest eigenvalue of $(2M)^{-1}(\nabla-2ie\,\mathbf{A})^2$ is eh/M, (4.11) satisfies the inequality

$$\delta p \le \Omega^{-1}\int d^3r \left[-\frac{eh}{M}+Mv^2\right]|b|^2 . \tag{4.18}$$

In order to have stability for the normal state, we require $\delta p<0$ which implies

$$h>H_{c2}=(Mv)^2/e . \tag{4.19}$$

In the above derivation, $A_0(\mathbf{r})$ is regarded as a classical field. In the present case, this is equivalent to assuming $\lambda_L=\infty$ for the bosonic excitations: Recall that the derivation of the λ_L^{-2} dependence in $\omega_l(k)$, given by (2.46), rests entirely on the k^{-2} pole term in (2.44), which in turn stems from (2.19). It can be readily verified that if in Sec. II $A_0(\mathbf{r})$ were a classical field, then the k^{-2} pole term would be absent in (2.44), and therefore $\omega_l(k)\to 0$ as $k\to 0$. Since according to (2.46) $\omega_l(k)=[\lambda_L^{-2}+k^2v^2+(k^2/2M)^2]^{1/2}$, the result is equivalent to setting $\lambda_L=\infty$. [On the other hand, in Sec. III since at $T=0$ ϕ is dominated by the c number function $B(r)e^{i\theta}$, the classical approximation of A_0 retains λ_L^{-2} dependence in $B(r)$.]

At the critical point λ_L, ξ, and $1/v$ are all ∞, but according to (3.23) the Debye length λ_D remains finite. For T less than but near T_c, our approximation makes λ_L remain ∞; hence because of (3.32) and (3.33), $(\lambda_L')^{-2}=(v/\lambda_D)^2$ and $v'^2=(2M\lambda_D)^{-2}$. We are then led to the case of perfect screening. In accordance with (3.46),

$$\frac{1}{\xi^2}=2(Mv)^2 . \tag{4.20}$$

Equation (4.19) becomes

$$H_{c2}=(2\xi^2 e)^{-1} ,$$

in agreement with (4.6).

The above alternative derivation holds only for T near T_c. In addition, we assume perfect screening, the neglect of bosonic thermal excitations in (4.12), and the near equality between the magnetic induction h and the magnetic field H in the normal state.

A full discussion of H_c, H_{c1}, and H_{c2} will be given in a subsequent paper. For realistic applications, the isotropic and homogeneous nature of the space must be replaced by the appropriate crystal lattice. This extension can be readily made, and will be discussed elsewhere.

ACKNOWLEDGMENTS

This research was supported in part by the U. S. Department of Energy under Contracts No. DE-AC02-76 ER02271 and DE-AC02-87 ER40325.

APPENDIX

To make the Thomas-Fermi approximation mentioned in Sec. III, we assume the electromagnetic potentials $\mathbf{A}(\mathbf{r})$ and $A_0(\mathbf{r})$ to be classical fields. Divide the entire volume Ω into small cubes of volume l^3, with $l\gg k_F^{-1}$, the inverse of the top Fermi momentum k_F, but $l\ll$ the typical length scale over which $\mathbf{A}$ and A_0 vary. Let

$$\mathbf{r}_n=\text{center of the } n\text{th box} . \tag{A1}$$

A complete orthonormal set of c number functions is $\{f_{n,\mathbf{k}}(\mathbf{r})\}$, where

$$f_{n,\mathbf{k}}(\mathbf{r})=\begin{cases} l^{-3/2}\exp i[\mathbf{k}\cdot\mathbf{r}+eA(\mathbf{r}_n)\cdot(\mathbf{r}-\mathbf{r}_n)], & \text{if } \mathbf{r} \text{ lies inside the box } n ;\\ 0 \quad \text{otherwise} . \end{cases} \tag{A2}$$

Expand the operators $\psi_\uparrow(\mathbf{r})$ and $\psi_\downarrow(\mathbf{r})$ in terms of these functions:

$$\psi_\sigma(\mathbf{r})=\sum_{n,\mathbf{k}} a_{n,\mathbf{k},\sigma} f_{n,\mathbf{k}}(\mathbf{r}) , \tag{A3}$$

where $\sigma=\uparrow$ or $\downarrow$, and the sum extends over all repeated indices n and $\mathbf{k}$. Because of (2.9), we have

$$\{a_{n,\mathbf{k},\sigma}, a^\dagger_{n',\mathbf{k}',\sigma'}\}=\delta_{nn'}\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} . \tag{A4}$$

Consider the generalized Hamiltonian density for the fermion system

$$\mathcal{H}_f\equiv\mathcal{H}_\psi+\mathcal{H}_{\text{int}} , \tag{A5}$$

$$\mathcal{H}_\psi\equiv\frac{1}{2m}[(\nabla+ie\,\mathbf{A})\psi^\dagger] \times(\nabla-ie\,\mathbf{A})\psi-\mu\psi^\dagger\psi-\frac{e}{2m}(\nabla\times\mathbf{A})\cdot\psi^\dagger\boldsymbol{\sigma}\psi , \tag{A6}$$

and

$$\mathcal{H}_{\text{int}}\equiv g(\phi^\dagger\psi_\uparrow\psi_\downarrow+\phi\psi_\downarrow^\dagger\psi_\uparrow^\dagger)+eA_0\psi^\dagger\psi , \tag{A7}$$

where $\boldsymbol{\sigma}$ is the Pauli spin matrix, $\mathbf{A}(\mathbf{r})$ is given by (3.1), and

$$\phi(\mathbf{r})=B(r)e^{i\theta} , \tag{A8}$$

in accordance with (3.3) but with χ neglected [because we are interested here only in the ground state and low-lying fermionic excitation levels, otherwise the operator $\chi(\mathbf{r})$ has to be expanded in terms of a similar set of c number functions (A2), but with a different gauge factor]. Let

$$\mathbf{h}\equiv\nabla\times\mathbf{A}=\hat{z}h(r) \tag{A9}$$

be the magnetic induction parallel to the z axis. By substituting the expansion (A3) into $\mathcal{H}_f$ and integrating over Ω, we find to $O(h)$,

$$\int d^3r\,\mathcal{H}_f = \sum_{n,\mathbf{k},\sigma} a^\dagger_{n,\mathbf{k},\sigma} a_{n,\mathbf{k},\sigma}\left[\frac{k^2}{2m}-\mu+eA_0(r)-\frac{e}{2m}\sigma_z h(r)\right] + g\sum_{n,\mathbf{k}} B(r)(e^{-i\theta}a_{n,\mathbf{k},\uparrow}a_{n,-\mathbf{k},\downarrow}+\text{H.c.})\,, \tag{A10}$$

in which the functions $A_0(r)$, $h(r)$, and $B(r)$ are taken at $\mathbf{r}=\mathbf{r}_n$. In obtaining (A10), we approximate, within each box n,

$$(\nabla - ie\,\mathbf{A})f_{n,\mathbf{k}}(\mathbf{r}) \cong i\mathbf{k}f_{n,\mathbf{k}}(\mathbf{r})\,. \tag{A11}$$

The deviation $\mathbf{A}(\mathbf{r})-\mathbf{A}(\mathbf{r}_n)$ contributes to $\mathcal{H}_f$ the well-known diamagnetic energy which is $O(h^2)$. Define

$$\varepsilon_{n,k,\sigma}=\frac{k^2}{2m}-\mu+eA_0(r)\mp\frac{eh(r)}{2m}\,, \tag{A12}$$

where the upper sign is for $\sigma=\uparrow$ and the lower for $\sigma=\downarrow$. Just as in (2.23), (A10) can be diagonalized by introducing $\tilde{a}_{n,\mathbf{k},\sigma}$ and $\tilde{a}^\dagger_{n,\mathbf{k},\sigma}$:

$$\begin{aligned} a_{n,\mathbf{k},\uparrow} &= \tilde{a}_{n,\mathbf{k},\uparrow}\cos\tilde{\theta}_{n,k}+\tilde{a}^\dagger_{n,-\mathbf{k},\downarrow}\sin\tilde{\theta}_{n,k}e^{i\theta}\,,\\ a_{n,-\mathbf{k},\downarrow} &= -\tilde{a}^\dagger_{n,\mathbf{k},\uparrow}\sin\tilde{\theta}_{n,k}e^{i\theta}+\tilde{a}_{n,-\mathbf{k},\downarrow}\cos\tilde{\theta}_{n,k}\,, \end{aligned} \tag{A13}$$

where

$$\begin{aligned} \sin 2\theta_{n,k} &= gB(r)/E_{n,k}\,,\\ \cos 2\theta_{n,k} &= \bar{\varepsilon}_{n,k}/E_{n,k}\,, \end{aligned} \tag{A14}$$

$$\bar{\varepsilon}_{n,k}=\tfrac{1}{2}(\varepsilon_{n,k,\uparrow}+\varepsilon_{n,k,\downarrow}) = \frac{k^2}{2m}-\mu+eA_0(r)\,, \tag{A15}$$

and

$$E_{n,k}=[\bar{\varepsilon}^{\,2}_{n,k}+g^2B(r)^2]^{1/2}\,. \tag{A16}$$

Since

$$\{\tilde{a}_{n,\mathbf{k},\sigma},\tilde{a}^\dagger_{n',\mathbf{k}',\sigma'}\}=\delta_{nn'}\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}\,, \tag{A17}$$

the transformation is canonical. Correspondingly, (A10) becomes

$$\int d^3r\,\mathcal{H}_f=\sum_{n,\mathbf{k}}\left[\bar{\varepsilon}_{n,k}-E_{n,k}+\tilde{a}^\dagger_{n,\mathbf{k},\uparrow}\tilde{a}_{n,\mathbf{k},\uparrow}\left[E_{n,k}-\frac{eh}{2m}\right]+\tilde{a}^\dagger_{n,\mathbf{k},\downarrow}\tilde{a}_{n,\mathbf{k},\downarrow}\left[E_{n,k}+\frac{eh}{2m}\right]\right]. \tag{A18}$$

Because the $\mathbf{k}$ vector here is defined for the small volume l, we have

$$\begin{aligned} \sum_{\mathbf{k}} &= l^3\int(2\pi)^{-3}d^3k\,,\\ \sum_{n,\mathbf{k}} &= \int d^3r\int(2\pi)^{-3}d^3k\,, \end{aligned} \tag{A19}$$

and therefore, for the ground state, (A18) reduces to (3.8) with $\varepsilon_k(r)-\mu=\bar{\varepsilon}_{n,k}$ and $E_k(r)=E_{n,k}$. Note that unlike (2.4) the $\mathcal{H}_\psi$ of (A6) contains the paramagnetic interaction. The excitation energy $E_{n,k}+(eh\sigma_z/2m)$ has a linear dependence on the magnetic induction h, but to $O(h)$, the ground-state energy has no dependence on the magnetic induction.

[1] J. C. Bednorz and K. A. Müller, Z. Phys. B **64**, 189 (1986).
[2] M. K. Wu, J. R. Ashburn, C. J. Torng, P. H. Hor, R. L. Meng, L. Gao, Z. J. Huang, Y. Q. Wang, and C. W. Chu, Phys. Rev. Lett. **58**, 908 (1987); Z. X. Zhao, L. Chen, Q. Yang, Y. Huang, G. Chen, R. Tang, G. Liu, C. Cui, L. Chen, L. Wang, S. Guo, S. Lin, and J. Bi, Kexue Tongbao **6**, 412 (1987).
[3] P. Chaudhari *et al.*, Phys. Rev. B **36**, 8903 (1987).
[4] T. K. Worthington, W. J. Gallagher, and T. R. Dinger, Phys. Rev. Lett. **59**, 1160 (1987).
[5] T. D. Lee, "*s*-Channel Theory of Superconductivity," in *Symmetry in Nature* (Pisa, Scuola Normale Superiore, 1989), Vol. 2, p. 491; R. Friedberg and T. D. Lee, Phys. Lett. A **138**, 423 (1989).
[6] R. Friedberg and T. D. Lee, Phys. Rev. B **40**, 6745 (1989).
[7] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957); **108**, 1175 (1957).
[8] M. R. Schafroth, Phys. Rev. **100**, 463 (1955).
[9] P. W. Higgs, Phys. Lett. **12**, 132 (1964).
[10] Y. Nambu, Phys. Rev. Lett. **4**, 350 (1960); J. Goldstone, Nuovo Cimento **19**, 154 (1961).
[11] Y. J. Uemura *et al.*, Phys. Rev. Lett. **62**, 2317 (1989).
[12] T. D. Lee, "Bose Liquid" in *Multiparticle Dynamics,* edited by A. Giovanni and W. Kittel (Singapore, World Scientific, 1990), p. 743.
[13] N. N. Bogoliubov, Nuovo Cimento **7**, 6 (1958); **7**, 794 (1958).
[14] J. Valatin, Nuovo Cimento **7**, 843 (1958).
[15] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 547 (1957); T. D. Lee and C. N. Yang, *ibid.* **112**, 559 (1958).
[16] V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20**, 1064 (1950).