Effective Lagrangian for a system of nonrelativistic fermions in 2+1 dimensions coupled to an electromagnetic field: Application to anyonic superconductors

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We study here, in two spatial dimensions, the effective Lagrangian of nonrelativistic charged fermions in an electromagnetic field. The fermionic integration is performed yielding a one-loop effective action that is evaluated using the inhomogeneity (derivative) expansion technique. The effective Lagrangian involves a Chern-Simons (CS) -like term with a coefficient that is a "staircase" function of B, the magnetic field. We then discuss the application of this effective Lagrangian to a system of anyons, showing that the cancellation of the induced CS term against a CS term included in the beginning to change the fermions to anyons, is favored energetically, together with the expulsion of B from the system of anyons. The cancellation implies the existence of a massless mode. This, together with the fact that $B=0$, implies superconductivity.

I. INTRODUCTION

It is well known that, in a relativistic theory of massless charged fermions in 2+1 dimensions, there exists an anomaly that manifests itself as a Chern-Simons (CS) term involving the external electromagnetic field at the one-loop level.[1] The anomaly is usually thought to arise due to the presence of Dirac fermions. However, recent studies seem to indicate that a CS-like term is also generated in 2+1 dimensions for nonrelativistic fermions subjected to an external electromagnetic field.[2,3] We address this issue in Sec. II by computing the effective Lagrangian of a system of nonrelativistic (2+1)-dimensional fermions in an external electromagnetic field.

We then use this effective Lagrangian to study a system of anyons interacting in 2+1 dimensions with an external electromagnetic field. This system has been of considerable interest recently because of its relevance to high-T_c superconductors,[3] following the pioneering work of Laughlin and co-worker.[4] The system of anyons was formulated and studied extensively by Wilczek and coworkers as a system of charged particles (anyons) in a statistical magnetic field b which is proportional to the anyon density.[5] The system is also equally well described by CS field theory. Therefore, the system is described by a nonrelativistic Schrödinger field interacting with an external electromagnetic field A_μ and a statistical field a_μ with a CS term involving a_μ. If we integrate out the Schrödinger field, we obtain an effective system described by a_μ interacting with A_μ. The effective Lagrangian consists of the original CS term and the terms generated by the fermionic integration as obtained in Sec. II. We analyze this effective theory and find that in the large range of field configurations of a_μ the CS term generated by the fermionic integration precisely cancels with the initial CS term. This was anticipated in the studies by Hosotani and Chakravarty[6] and Banks and Lykken.[7] We shall describe this analysis in Sec. III.

II. COMPUTING THE EFFECTIVE LAGRANGIAN

We present a detailed account of the calculation of the effective Lagrangian for a system of nonrelativistic fermions in 2+1 dimensions coupled to an external electromagnetic field.

The Hamiltonian of the system is assumed to be given by

$$H=\int d\mathbf{x}\left[\frac{1}{2m}|[\nabla-ie\,\mathbf{A}(x)]\psi(x)|^2+e\psi(x)^\dagger\psi(x)A_0(x)\right], \tag{2.1}$$

where A_0 and $\mathbf{A}$ are the external scalar and vector potentials, respectively. We define the partition function by $Z(A)=\mathrm{tr}\exp[\beta(\mu N-H)]$ and the effective action by $W[A]=-\ln Z$, which is a functional of A_ρ. μ is the chemical potential of the system, $\beta=1/k_B$, where k_B is the Boltzmann constant, and T is the absolute temperature of the system.

Using standard path-integral techniques, one may express Z as a path integral over fermionic variables.[8] The integration over the fermionic variables gives the effective action.

The partition function is written as

$$Z(A)=\int D\psi D\bar{\psi}\exp\left[-\int d\mathbf{x}\int_0^\beta d\tau\bar{\psi}(\mathbf{x},\tau)\left[\partial_\tau+\frac{1}{2m}(\mathbf{P}-e\,\mathbf{A})^2+eA_0(x,\tau)-\mu\right]\psi(\mathbf{x},\tau)\right], \tag{2.2}$$

so

$$W[A]=-\mathrm{tr}\ln\left[i\Pi_\tau+\frac{1}{2m}(\Pi_x^2+\Pi_y^2)-\mu\right], \tag{2.3}$$

where

$$A_\tau=iA_0,\quad \Pi_\tau\equiv\hat{p}_\tau-eA_\tau(\hat{x}),\quad \Pi_i\equiv\hat{p}_i-eA_i(\hat{x}),\quad i=x,y\ .$$

The technique we adopt in evaluating $W[A]$ is to compute the current from $W[A]$ and then to functionally integrate the currents over the electromagnetic fields to get the effective action:

$$\langle j_\tau(x)\rangle=\frac{\delta W[A]}{\delta A_\tau(x)}=ie\left\langle x\left|\frac{1}{i\hat{\Pi}_\tau+(1/2m)\hat{\Pi}_i^2-\mu}\right|x\right\rangle, \tag{2.4}$$

and

$$\langle j_k(x)\rangle=\frac{e}{2m}\left\langle x\left|\hat{\Pi}_k\frac{1}{i\hat{\Pi}_\tau+(1/2m)\hat{\Pi}_j^2-\mu}+\frac{1}{i\hat{\Pi}_\tau+(1/2m)\hat{\Pi}_j^2-\mu}\hat{\Pi}_k\right|x\right\rangle. \tag{2.5}$$

To regulate the currents, we use the Pauli-Villars regulator.[9] So

$$\langle j_\tau(x)\rangle=ie\sum_i C_i\left\langle x\left|\frac{1}{i\hat{\Pi}_\tau+(1/2m)\hat{\Pi}_j^2-\mu+M_i}\right|x\right\rangle$$

and

$$\langle j_k(x)\rangle=\frac{e}{2m}\sum_i C_i\left\langle x\left|\hat{\Pi}_k\frac{1}{i\hat{\Pi}_\tau+(1/2m)\hat{\Pi}_j^2-\mu+M_i}+\frac{1}{i\hat{\Pi}_\tau+(1/2m)\hat{\Pi}_j^2-\mu+M_i}\hat{\Pi}_k\right|x\right\rangle, \tag{2.6}$$

where $C_0=1$, $\sum_i C_i=0$, and $M_0=0$. At the end of the calculations, we set $M_i\to\infty$. However, in what follows we shall for simplicity not mention the regularization explicitly. Formally, the current in Eqs. (2.3) and (2.6) may be shown to be conserved in a very straightforward fashion. Accordingly, the effective action is gauge invariant. The effective action in Eq. (2.3) is also invariant under space reflection:

$$W[A]=W[A']\ ,$$

where $A'_\mu=\pm A_\mu(\tau,xy)$; + for $\mu=\tau,y$ and − for $\mu=x$. The currents in Eq. (2.6) are now evaluated using the inhomogeneity expansion technique.[10]

First, we expand $A_\mu(\hat{x})$ around the point x:

$$A_\mu(\hat{x})=A_\mu(x)+(\hat{x}-x)_\nu\partial_\nu A_\mu(x)+\tfrac{1}{2}(\hat{x}-x)_\nu(\hat{x}-x)_\sigma\partial_\nu\partial_\sigma A_\mu(x)+\cdots\ . \tag{2.7}$$

Then we translate x to the origin using the translation operator $\exp[i\hat{p}_\mu x_\mu]$. Further, we make a unitary transformation involving $\hat{x}$ and $\hat{y}$ to express the current in terms of field strengths as far as possible, without violating boundary conditions.

The unitary transformation is given by

$$\hat{U}=\exp\left[ie[\hat{x}_iA_i(x)+\tfrac{1}{2}\hat{x}_i\hat{x}_j\partial_iA_j(x)+\frac{1}{3!}\hat{x}_i\hat{x}_j\hat{x}_k\partial_i\partial_jA_k+\cdots]\right]. \tag{2.8}$$

Notice that the operator $\hat{U}$ does not involve τ explicitly, the reason being that in the imaginary time method the Euclidean time direction is compactified, and hence nontrivial boundary terms will be generated by any transformation involving time explicitly.

The currents are then given by Eq. (2.6) with

$$|x\rangle\to|0\rangle=|x=0\rangle$$

and

$$\hat{\Pi}_\tau\to\hat{p}_\tau-eA_\tau(x)+\hat{\Delta}_\tau,\quad \hat{\Pi}_k\to\hat{\Pi}_k+\hat{\Delta}_k\ .$$

Here,

$$\hat{\Pi}_k=\hat{p}_k-\frac{e}{2}\hat{x}_iF_{ik}(x)\ ,$$

$$\hat{\Delta}_\tau=-e(\hat{\tau}\dot{A}_\tau+\tfrac{1}{2}\hat{\tau}^2\ddot{A}_\tau+\hat{\tau}\hat{x}_i\partial_i\dot{A}_\tau+\tfrac{1}{2}\hat{x}_i\hat{x}_j\partial_i\partial_jA_\tau+\cdots)\ , \tag{2.9}$$

$$\hat{\Delta}_k=-e(\hat{\tau}\dot{A}_k+\tfrac{1}{2}\hat{\tau}^2\ddot{A}_k+\tfrac{1}{3}\hat{x}_i\hat{x}_j\partial_iF_{jk}+\cdots)\ .$$

So the denominator in Eq. (2.6) now looks like

$$i\hat{p}_\tau-ieA_\tau(x)+i\hat{\Delta}_\tau+\hat{H}_0-\mu-\frac{1}{2m}(\hat{\Pi}_i\hat{\Delta}_i+\hat{\Delta}_i\hat{\Pi}_i+\hat{\Delta}_i^2)\ , \tag{2.10}$$

where

$$\hat{H}_0\equiv\frac{1}{2m}(\hat{\Pi}_x^2+\hat{\Pi}_y^2)\ .$$

Performing another unitary transformation,

$$\hat{H}_0=\frac{1}{2m}\{\hat{p}_x^2+[\hat{p}_y-e\hat{x}B(x)]^2\} . \quad (2.11)$$

Since $\hat{H}_0$ is the Hamiltonian of a two-dimensional electron in a uniform magnetic field in the missing z direction, $1/(i\hat{p}_\tau+\hat{H}_0-\mu)$ is exactly soluble, where $\mu' \equiv \mu+ieA_\tau(x)$. We consider the Δ's as perturbations.

Let

$$\hat{P}_0 \equiv i\hat{p}_\tau+\hat{H}_0-\mu' .$$

So

$$\langle j_\tau(x)\rangle = ie\sum_i C_i \left\langle 0 \left| \frac{1}{\hat{P}_0+i\hat{\Delta}_\tau+(1/2m)(\hat{\Pi}_i\hat{\Delta}_i+\hat{\Delta}_i\hat{\Pi}_i+\hat{\Delta}_i^2)+M_i} \right| 0\right\rangle$$

and

$$\langle j_k(x)\rangle = \frac{e}{2m}\sum_i C_i \left\langle 0 \left| \hat{\Pi}_k \frac{1}{\hat{P}_0+i\hat{\Delta}_\tau+(1/2m)(\hat{\Pi}_i\hat{\Delta}_i+\hat{\Delta}_i\hat{\Pi}_i+\hat{\Delta}_i^2)+M_i} + \frac{1}{\hat{P}_0+i\hat{\Delta}_\tau+(1/2m)(\hat{\Pi}_i\hat{\Delta}_i+\hat{\Delta}_i\hat{\Pi}_i+\hat{\Delta}_i^2)+M_i}\hat{\Pi}_k \right| 0\right\rangle . \quad (2.12)$$

Let $eB(x)<0$. Then,

$$[\hat{\Pi}_x,\hat{\Pi}_y]=ieB(x)=-i|eB(x)|=-\frac{i}{l^2} , \quad (2.13)$$

where l is the magnetic length, and $\hat{\Pi}_y$ is like a coordinate, $\hat{\Pi}_x$ being its momentum. For $eB(x)>0$, the roles of $\hat{\Pi}_x$ and $\hat{\Pi}_y$ are reversed. Consider, $eB(x)<0$, and define

$$a\equiv\frac{l}{\sqrt{2}}(\hat{\Pi}_x-i\hat{\Pi}_y) . \quad (2.14a)$$

It can be shown that

$$[a,a^\dagger]=\frac{l^2}{2}(\hat{\Pi}_x-i\hat{\Pi}_y,\hat{\Pi}_x+i\hat{\Pi}_y)=1 .$$

Now, defining guiding center coordinates[11] as

$$\hat{X}=\hat{x}-l^2\hat{\Pi}_y,\quad \hat{Y}=\hat{y}+l^2\hat{\Pi}_x , \quad (2.14b)$$

we have

$$[\hat{X},\hat{\Pi}_i]=[\hat{Y},\hat{\Pi}_i]=0 ,$$
$$[\hat{X},\hat{H}_0]=[\hat{Y},\hat{H}_0]=0 , \quad (2.15)$$
$$[\hat{X},\hat{Y}]=il^2 ,$$

and

$$\hat{\Pi}_x=\frac{1}{\sqrt{2}l}(a+a^\dagger),\quad \hat{\Pi}_y=\frac{i}{\sqrt{2}l}(a-a^\dagger) . \quad (2.16)$$

Hence

$$\hat{X}=\hat{x}-\frac{il}{\sqrt{2}}(a-a^\dagger),\quad \hat{Y}=\hat{y}+\frac{il}{\sqrt{2}}(a+a^\dagger) . \quad (2.17)$$

We work in the basis $|n,X\rangle$, where $|n,X\rangle\equiv|n\rangle\otimes|X\rangle$; $|n\rangle$ is the occupation number basis for the harmonic-oscillator problem, and $|X\rangle$ is the basis where $\hat{X}$ is diagonal:

$$\hat{X}|X\rangle=X|X\rangle,\quad \hat{Y}|X\rangle=il^2\frac{\partial}{\partial X}|X\rangle . \quad (2.18)$$

So the eigenfunctions of $\hat{H}_0$ are given by

$$\langle x,y|n,X\rangle=\frac{1}{\sqrt{2\pi l^3}}e^{iXy/l^2}u_n\left[\frac{x-X}{l}\right] , \quad (2.19)$$

where the u_n's are the standard harmonic-oscillator wave functions. We note that

$$\int dX|\langle x,y|n,X\rangle|^2=\frac{1}{2\pi l^2}=\frac{|eB(x)|}{2\pi} . \quad (2.20)$$

Before proceeding with the actual calculations, we would like to outline a few calculational tricks. Note, if $\hat{O}$ and $\hat{O}'$ consist of $\hat{p}_0$, $\hat{\Pi}_x$, and $\hat{\Pi}_y$,

$$\begin{aligned}\langle 0|\hat{O}\hat{x}\hat{O}'|0\rangle&=\langle 0|\hat{O}(\hat{X}+l^2\hat{\Pi}_y)\hat{O}'|0\rangle\\&=\langle 0|\hat{X}\hat{O}\hat{O}'|0\rangle+l^2\langle 0|\hat{O}\hat{\Pi}_y\hat{O}'|0\rangle\\&=l^2\langle 0|[\hat{O},\hat{\Pi}_y\hat{O}']|0\rangle .\end{aligned} \quad (2.21)$$

Furthermore,

$$\begin{aligned}&\langle 0|\hat{O}\hat{X}\hat{O}'|0\rangle=l^2\langle 0|[\hat{\Pi}_x,\hat{O}]\hat{O}'|0\rangle ,\\&\langle 0|\hat{O}\hat{\tau}\hat{O}'|0\rangle=\langle 0|\hat{O}[\hat{\tau},\hat{O}']|0\rangle .\end{aligned} \quad (2.22)$$

The commutators can be computed using the formulas

$$\begin{aligned}&[\hat{\tau},\hat{p}_\tau]=i,\quad \left[\hat{\tau},\frac{1}{\hat{P}_0}\right]=\frac{1}{\hat{P}_0^2} ,\\&\left[\hat{\Pi}_x,\frac{1}{\hat{P}_0}\right]=\frac{i}{ml^2}\frac{1}{\hat{P}_0}\hat{\Pi}_y\frac{1}{\hat{P}_0} ,\\&\left[\hat{\Pi}_y,\frac{1}{\hat{P}_0}\right]=\frac{-i}{ml^2}\frac{1}{\hat{P}_0}\hat{\Pi}_x\frac{1}{\hat{P}_0} ,\end{aligned} \quad (2.23)$$

etc. Armed with these results, we start the actual computations of the currents (the details of the calculations are given in Appendix A). The perturbations are done using the formula

$$\frac{1}{\hat{A}+\hat{B}}=\frac{1}{\hat{A}}-\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}}+\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}}+\cdots , \quad (2.24)$$

where

$$\hat{A}\equiv\hat{P}_0,\quad \hat{B}\equiv i\Delta_\tau+\frac{1}{2m}(\hat{\Delta}_i\hat{\Pi}_i+\hat{\Pi}_i\hat{\Delta}_i+\hat{\Delta}_i^2)\ , \tag{2.25}$$

$$\begin{aligned}\langle j_\tau(x)\rangle_0&=ie\left\langle 0\left|\frac{1}{P_0}\right|0\right\rangle\\&=\frac{ie}{\beta}\sum_{m,n}\int dX\frac{\langle 0|n,X\rangle\langle X,n|0\rangle}{i\xi_m+E_n-\mu'}\\&=\frac{ie}{\beta}\frac{1}{2\pi l^2}\sum_{m,n}\frac{1}{i\xi_m+E_n-\mu'}\\&=\frac{ie}{2\pi}|eB(x)|\sum_n\frac{1}{1+\exp[\beta(E_n-\mu')]}\ .\end{aligned}$$

The last equality assumes the regularization mentioned above. Here ξ_m are the Matsubara frequencies:

$$\xi_m=(m+\tfrac{1}{2})\frac{2\pi}{\beta},\quad E_n=(n+\tfrac{1}{2})\left|\frac{eB(x)}{m}\right|\ .$$

Up to first order,

$$\langle j_\tau(x)\rangle=\frac{ie}{2\pi}|eB|\gamma(|eB|,A_\tau)-\frac{e^2m}{2\pi|eB|}\left[\gamma+\omega\frac{\partial\gamma}{\partial\omega}\right](\partial_xE_x+\partial_yE_y)\ ,$$

where

$$\gamma(|eB|,A_\tau)\equiv\sum_{n=0}^{\infty}\frac{1}{1+\exp\{\beta[E_n-\mu-ieA_\tau(x)]\}}\ , \tag{2.26}$$

and

$$E_x=F_{\tau x}(x)=\partial_\tau A_x-\partial_x A_\tau,\quad \omega=\left|\frac{eB}{m}\right|\ .$$

It may be shown explicitly that

$$\begin{aligned}\langle j_x(x)\rangle_{(0)}&=\langle j_y(x)\rangle_{(0)}=0\ ,\\ \langle j_x(x)\rangle_{(1)}&=\frac{e}{2m}\left[\left\langle 0\left|\hat{\Pi}_x\frac{1}{\hat{P}_0}\left[-i\hat{\Delta}_\tau-\frac{1}{2m}(\hat{\Pi}_i\hat{\Delta}_i+\hat{\Delta}_i\hat{\Pi}_i+\hat{\Delta}_i^2)\right]\frac{1}{\hat{P}_0}\right|0\right\rangle\right.\\&\quad\left.+\left\langle 0\left|\frac{1}{\hat{P}_0}\left[-i\hat{\Delta}_\tau-\frac{1}{2m}(\hat{\Pi}_i\hat{\Delta}_i+\hat{\Delta}_i\hat{\Pi}_i+\hat{\Delta}_i^2)\right]\frac{1}{\hat{P}_0}\hat{\Pi}_x\right|0\right\rangle\right]\\&=\frac{ie^2}{2\pi}\gamma(|eB(x)|,A_\tau(x))F_{\tau y}(x)-\frac{e^2}{2\pi m}\lambda(|eB(x)|,A_\tau)\partial_yB(x)\ ,\end{aligned} \tag{2.27}$$

where

$$\lambda\equiv\sum_{n=0}^{\infty}(n+\tfrac{1}{2})\frac{1}{1+\exp[\beta(E_n-\mu')]}\ .$$

Similarly,

$$\langle j_y(x)\rangle_{(1)}=-\frac{ie^2}{2\pi}\gamma(|eB(x)|,A_\tau(x))F_{\tau x}(x)+\frac{e^2}{2\pi m}\lambda(|eB(x)|,A_\tau)\partial_xB(x)\ . \tag{2.28}$$

Note that at $T=0$, $\gamma(|eB|,A_\tau)$ is independent of the real part of A_τ and it possesses the property of being a "staircase" function of $(\mu+ieA_\tau)/eB$. Accordingly, $\lambda=\gamma^2/2+\gamma$. For finite T, γ is given by a smoothened out staircase function so that we do not have a simple relation between λ and γ. Up until now we have assumed $eB(x)<0$, but we can check that the same expression holds for $eB(x)>0$. This was expected from parity considerations.

In the approach followed here, the higher-order corrections are given by the higher-order derivatives of the fields. The calculations are straightforward, albeit tedious. Up to the given order of calculations, the current does not appear conserved as it stands, at $T\neq0$. The origin of the problem is that the inhomogeneity expansion does not respect the boundary condition for finite temperature. However, the current is conserved in the case of zero temperature or in the case of finite temperature but static electromagnetic fields. We therefore restrict our attention to these two cases.

It is now straightforward to obtain the effective action at $T=0$. It is given by

$$W[A]=\frac{ie^2}{2\pi}\int d\mathbf{x}\,d\tau\,\epsilon(eB)\gamma A_\tau(x)B(x)+\frac{e^2}{8\pi m}\int d\mathbf{x}\,d\tau\,\gamma^2B^2(x)+\frac{e^2m}{4\pi}\int d\mathbf{x}\,d\tau\frac{\gamma}{|eB|}\mathbf{E}^2(x)\ , \tag{2.29}$$

where

$$\epsilon(eB)=\begin{cases}+1, & \text{for positive } eB\\ -1, & \text{for negative } eB\end{cases}\ .$$

Thus we see that a CS-like term has been generated in the effective action after doing the fermionic integration. This form of effective action may now be used to discuss anyonic superconductivity.

III. ANYONIC SUPERCONDUCTIVITY AND THE EXISTENCE OF MASSLESS EXCITATIONS

In this section, we use the effective Lagrangian computed in Sec. II to discuss anyonic superconductivity. Anyons are quasiparticle excitations of fractional statistics. So we are looking at a system of charged particles of fractional statistics whose Hamiltonian is given by

$$H=\int d\mathbf{x}\left[\frac{1}{2m}|[\nabla-ie(\mathbf{a}+\mathbf{A})]\psi(\mathbf{x})|^2+e\bar{\psi}(\mathbf{x})\psi(\mathbf{x})A_0(\mathbf{x})\right], \quad (3.1)$$

where $\mathbf{a}$ is the solution of

$$b\equiv\partial_x a_y-\partial_y a_x=\frac{e}{\mu_0}\bar{\psi}\psi . \quad (3.2)$$

For simplicity, we hereafter set $e=1$, which can be done by rescaling the gauge fields. According to this Hamiltonian, the charged particles (anyons) are in the statistical magnetic field b, which is proportional to the anyon density. As a consequence, the anyon acquires an extra phase when it goes around other anyons. The phase depends on the parameter μ_0, and the value of μ_0 determines the statistics of the particles.[5] Solving Eq. (3.2) in the Coulomb gauge, we may write the partition function for the system as

$$Z=\int D\psi D\bar{\psi} Da_i\delta(\partial_i a_i)\delta\left[b-\frac{1}{\mu_0}\bar{\psi}\psi\right]\exp(-S) , \quad (3.3)$$

where

$$S=\int d\mathbf{x}\, d\tau\left[\bar{\psi}\left[\partial_\tau-iA_\tau+\frac{1}{2m}[\mathbf{p}-(\mathbf{A}+\mathbf{a})]^2-\mu\right]\psi\right] .$$

Let

$$\delta\left[b-\frac{1}{\mu_0}\bar{\psi}\psi\right]=\int Da_\tau\exp\left[i\int d\tau\, d\mathbf{x}(\bar{\psi}a_\tau\psi-\mu_0 ba_\tau)\right] . \quad (3.4)$$

So the partition function can be written as

$$Z=\int D\psi D\bar{\psi} Da_i Da_\tau\delta(\partial_i a_i)\exp(-S') , \quad (3.5)$$

where

$$S'=\int d\mathbf{x}\, d\tau\left[i\mu_0 ba_\tau+\bar{\psi}\left[i[p_\tau-(a_\tau+A_\tau)]+\frac{1}{2m}[\mathbf{p}-(\mathbf{A}+\mathbf{a})]^2-\mu\right]\psi\right] .$$

We do the fermionic integration to get

$$Z=\int Da_i Da_\tau\delta(\partial_i a_i)\exp\left[-\left[i\mu_0\int d\mathbf{x}\, d\tau\, ba_\tau+W_{\text{eff}}(a+A)\right]\right] . \quad (3.6)$$

$W_{\text{eff}}(a+A)$ has already been computed in Sec. II with A of Sec. II replaced by $(a+A)$ here.

Thus we have an expression for the partition function of the system of anyons. It is interesting to note that

$$\int d\mathbf{x}\, d\tau\, ba_\tau$$

is the full CS term in the Coulomb gauge. The Coulomb gauge condition yields

$$a_i(x)=\epsilon_{ij}\partial_j\phi(x)$$

and

$$b(x)=-\nabla^2\phi(x) ,$$

whence, formally,

$$a_i(x)=-\epsilon_{ij}\frac{\partial_j}{\nabla^2}b(x) . \quad (3.7)$$

Using Eq. (3.7), the functional integral is trivially converted from one over a_i to one over b. It is also worth noting that the signature of $b+B$ is crucial in defining the creation and destruction operators in $\hat{H}_0$. The signature manifests itself in the expression for $\langle j_i(x)\rangle$.

Now, after carrying out the fermionic integration, we get

$$Z=\int Db\, Da_\tau\exp(T_1+T_2+T_3+T_4) , \quad (3.8)$$

where

$$T_1=-i\int d\mathbf{x}\, d\tau\,\mu_0 a_\tau b ,$$

$$T_2=-\frac{i}{2\pi}\int d\mathbf{x}\, d\tau\,\gamma|(b+B)|(a_\tau+A_\tau) ,$$

$$T_3=-\frac{m}{4\pi}\int d\mathbf{x}\, d\tau\frac{\gamma}{|(b+B)|}(\mathbf{e}+\mathbf{E})^2 ,$$

$$T_4=-\frac{1}{8\pi m}\int d\mathbf{x}\, d\tau\,\gamma^2(b+B)^2 ,$$

where γ is given by (2.26) with $B\rightarrow(b+B)$ and $A_\tau\rightarrow(a_\tau+A_\tau)$. It is to be understood that the potential $a_i(x)$ appearing in the statistical electric field has been expressed in terms of $b(x)$. A more explicit expression will be given later when we study the spectrum of the collective excitation. At this point we intend to study two aspects of the system. First, we want to show that the cancellation of the tree level and the induced CS term minimizes the free energy of the system. Second, we want to demonstrate the existence of a massless mode once the cancellation of the CS term is achieved. The exact analysis being quite complicated, we, in what follows, use the fact that the higher derivative terms in the Lagrangian are much smaller compared to the CS term and that they are negligible in the first approximation.

In accordance with the above arguments, we write

$$Z \simeq \int Da_\tau Db \exp\left[-i\int d\mathbf{x}\,d\tau\left[\mu_0 b+\frac{\gamma}{2\pi}|(b+B)|\right]a_\tau\right]$$
$$=\int_{b+B<0} Da_\tau Db \exp\left[-i\int d\mathbf{x}\,d\tau\left[\mu_0 b-\frac{\gamma}{2\pi}(b+B)\right]a_\tau\right]$$
$$+\int_{b+B>0} Da_\tau Db \exp\left[-i\int d\mathbf{x}\,d\tau\left[\mu_0 b+\frac{\gamma}{2\pi}(b+B)\right]a_\tau\right] . \quad (3.9)$$

$B(x)$ being the external field, we first choose it to be positive: $B(x)\geq 0$. Now, since μ_0 is a parameter in the theory, we fix it to be $\mu_0=-N/2\pi$. The case $N=2$ corresponds to that of semions. So

$$Z=\int_{b+B<0} Db\,\delta\left[\frac{N+\gamma}{2\pi}b+\frac{\gamma}{2\pi}B\right]$$
$$+\int_{b+B>0} Db\,\delta\left[\frac{\gamma}{2\pi}B-(N-\gamma)\frac{b}{2\pi}\right] . \quad (3.10)$$

Since $\gamma>0$ and $N>0$,

$$g(b)=\gamma(b+B)+Nb_{|b=b_0}=0$$

does not have a consistent solution for $b+B<0$ and $B>0$.

Thus the first term in the partition function is zero. In the second term let

$$f(b)=(N-\gamma)b-\gamma B . \quad (3.11)$$

If $f(b_0)=0$,

$$(N-\gamma)b_0=\gamma B . \quad (3.12)$$

Defining $\xi=\mu/(b+B)$, we get

$$b=\mu/\xi-B ,$$

and hence

$$(N-\gamma)\left[\frac{\mu}{\xi_0}-B\right]=\gamma B ,$$

or

$$(N-\gamma)\frac{\mu}{\xi_0}=NB . \quad (3.13)$$

In our calculations, at a point x, $B(x)>0$, and μ is fixed. ξ_0 changes due to changes in B.

Also, at $T=0$,

$$\gamma=\sum_{n=0}^{\infty}\Theta\left[\left[n+\frac{1}{2}\right]\frac{1}{m}-\xi\right] . \quad (3.14)$$

For semions, $N=2$, and so

$$\left\{2-\sum_{n=0}^{\infty}\Theta\left[\left[n+\frac{1}{2}\right]\frac{1}{m}-\xi_0\right]\frac{\mu}{\xi_0}\right\}=2B . \quad (3.15)$$

We solve this equation graphically:

$$Z\simeq\int Db\,\delta(f(b)) ,$$

or

$$Z=\sum_{b_0}\prod_{x,\tau}\left|\frac{\partial f}{\partial b}\right|^{-1}_{b=b_0} , \quad (3.16)$$

where b_0 is obtained from (3.15). Also,

$$\frac{\partial f}{\partial b}=(2-\gamma)-\frac{\partial\gamma}{\partial b}(b+B)$$
$$=(2-\gamma)-\sum_{n=0}^{\infty}\delta\left[b+B-\frac{m\mu}{(n+\frac{1}{2})}\right] . \quad (3.17)$$

Now, from the graph, we see that if

$$\frac{2\mu m}{5}<b_0+B<\frac{2\mu m}{3} ,$$

then $\gamma=2$. Equation (3.15) has a solution provided $B=0$. In this case, $|\partial f/\partial b|^{-1}=\infty$ and gives a large contribution to Z. If, however, b_0 lies somewhere else, a solution to (3.15) exists even if $B\neq 0$. In this case $\gamma\neq 2$ so that $|\partial f/\partial b|_{b=b_0}$ is finite.

A similar analysis may be carried out for $B(x)\leq 0$. This is done in Appendix C. The conclusion reached, however, is the same. We conclude, therefore, that the free energy has a strong minimum at $B(x)=0$.

With this in mind let us look at the partition function more carefully. Z gets its maximum contribution from the region of the $b(x)$ integration that yields $N=\gamma$. So, for $N=2$,

$$\frac{2\mu m}{5}<b_0+B<\frac{2\mu m}{3} , \quad (3.18)$$

and

$$Z=\int\int Da_\tau Db \exp(A_1+A_2+A_3+A_4+A_5+A_6) , \quad (3.19)$$

where

$$A_1=-\frac{i}{\pi}\int dx[(b+B)A_\tau+Ba_\tau]\ ,$$

$$A_2=-\frac{1}{2\pi m}\int dx(b+B)^2\ ,$$

$$A_3=-\frac{m}{2\pi}\int dx\frac{1}{|(b+B)|}[(D_ib)^2+(\partial_i a_\tau)^2+2\partial_i a_\tau\epsilon_{ij}D_jb]\ ,$$

$$A_4=\int dx\,\mathbf{E}^2\ ,$$

$$A_5=-2\int dx\,\epsilon_{ij}D_jb(x)E_i\ ,$$

$$A_6=-2\int dx\,\partial_i a_\tau E_i\ ,$$

and

$$D_i=\frac{\partial_\tau\partial_i}{\nabla^2}\ .$$

But from an order of magnitude estimate, we note that $A_1 \gg A_2, A_3, A_4, A_5, A_6$. Also, in the low-momentum regime, $(\partial_i a_\tau)^2 \ll (D_ib)^2$. So even though we may drop $(\partial_i a_\tau)^2$ when doing the a_τ integration, we may not drop $(D_ib)^2$ in the b integration. So

$$Z=\delta(B)\int Db\,\exp(P+Q+R)\ , \tag{3.20}$$

where

$$P=-\frac{i}{\pi}\int dx\,bA_\tau\ ,$$

$$Q=-\frac{1}{2\pi m}\int dx\,b^2\ ,$$

$$R=\frac{m}{2\pi}\int dx\frac{1}{|B|}[(D_ib)^2+\mathbf{E}^2-2\epsilon_{ij}D_jb(x)E_i]\ .$$

This indicates that $B(x)=0$, which is the Meissner effect. Further, in doing the $b(x)$ integration we would like to do a saddle-point calculation. Let

$$h(b)=P+Q+R\ . \tag{3.21}$$

For the extremum,

$$0=-\frac{i}{\pi}\nabla^2A_\tau-\frac{1}{\pi m}\nabla^2b_0-\frac{m}{\pi|b_0|}\partial_\tau^2b_0+\frac{m}{\pi|b_0|}\epsilon_{ij}\partial_\tau\partial_jE_i\ . \tag{3.22}$$

Since A_τ is the external scalar potential, $\nabla^2A_\tau=0$. $\mathbf{E}$, the external electric field, is time independent. So $b_0=$const is a solution of Eq. (3.22). b_0 lies in the range

$$\frac{2\mu m}{5}\le b_0\le\frac{2\mu m}{3}\ .$$

Further,

$$\frac{\delta^2h(b)}{\delta b(x)\delta b(y)}=-\frac{1}{\pi}\left[\frac{1}{m}+\frac{m}{|b_0|}\partial_\tau^2\nabla^{-2}\right]\delta(x-y)\ . \tag{3.23}$$

So

$$h(b)\simeq h(b_0)-\int dx\,\eta(x)\left[\frac{1}{\pi m}+\frac{m}{\pi|b_0|}\partial_\tau^2\nabla^{-2}\right]\eta(x)\ , \tag{3.24}$$

and

$$Z\simeq\delta(B)e^{h(b_0)}\int D\eta\,e^{S[\eta]}\ ,$$

where

$$S[\eta]\equiv-\frac{1}{\pi m}\int dx\,\eta\left[1+\frac{m^2}{|b_0|}\partial_\tau^2\nabla^{-2}\right]\eta\ . \tag{3.25}$$

Thus, if we rescale as

$$\frac{1}{(\pi m)^{1/2}}\frac{1}{|p|}\tilde\eta(\mathbf{p},p_0)\equiv\tilde\xi(\mathbf{p},p_0)\ ,$$

where as $\beta\to\infty$, $(1/\beta)\sum_n\to(1/2\pi)\int dp_0$, we see that the propagator of the $\xi(x,\tau)$ field is

$$\left[p^2+\left[\frac{m}{(b_0)^{1/2}}\right]^2p_0^2\right]^{-1}\ , \tag{3.26}$$

which is the propagator for a massless excitation. So the fluctuation around a constant background is a propagating massless mode. With the above arguments, we have shown, at least at $T=0$, that the anyon gas is a superfluid that expels the external applied magnetic field.

IV. CONCLUSION

We computed the effective action using the inhomogeneity expansion method. The perturbations are carried out about a local vacuum consisting of filled Landau levels, because of which a CS-like term appears in the nonrelativistic Schrödinger field theory. This result agrees with that of Abuelsaood.[2] However, unlike in Ref. 2, the fields here are space-time dependent. The calculated space-time action was then used to study the low-energy behavior of anyons. Free-energy considerations revealed that the cancellation of induced and tree-level CS terms is favored and the external magnetic field is screened (Meissner effect), indicating that the system is a superconductor. Once cancellation is achieved, the higher derivative terms in this effective action describe the density fluctuations of anyons, exhibiting a massless mode.

It is worth pointing out at this point that the current as computed in the text is not conserved at $T\neq0$. The cause of this problem lies in the inhomogeneity expansion method, which does not respect the boundary condition for finite temperature. For the special case of $T=0$ and the static electromagnetic fields at finite temperature, the current is conserved. Accordingly, only in this case is the effective Lagrangian (3.23) gauge invariant. Because of this, we are hesistant to draw any conclusions regarding the case of finite temperature. We feel that more careful analysis should be undertaken as regards this question.

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APPENDIX A: GUIDING CENTER COORDINATES AS A MEANS OF DOING HIGHER-ORDER CALCULATIONS

Typically, a calculation for $\langle j_\mu(x)\rangle^{(1)}$ involves a matrix element of the form

$$\left\langle 0 \left| \hat{\Pi}_x \frac{1}{\hat{P}_0} \hat{x}\hat{\tau} \frac{1}{\hat{P}_0} \right| 0 \right\rangle . \tag{A1}$$

It is then convenient to choose a basis where $\hat{H}_0$ is diagonal. To deal with the spatial operators, however, we extend the Hilbert space of $\hat{H}_0$ by taking the direct product of the harmonic-oscillator basis with the eigenbasis of an operator that commutes with H_0:

$$\hat{P}_0 \equiv i(\hat{p}_\tau - eA_\tau) + \frac{1}{2m}(\hat{\Pi}_x^2 + \hat{\Pi}_y^2) - \mu ,$$

where

$$\begin{aligned} &\hat{\Pi}_x = \hat{p}_x, \quad \hat{\Pi}_y = \hat{p}_y - eB(x)\hat{x} , \\ &H_0 \equiv \frac{1}{2m}(\hat{\Pi}_x^2 + \Pi_y^2) . \end{aligned} \tag{A2}$$

The operators X and Y are defined in Sec. II:

$$\hat{X} \equiv \hat{x} - l^2\hat{\Pi}_y ,$$
$$\hat{Y} \equiv \hat{y} + l^2\hat{\Pi}_x ,$$

where

$$l^2 \equiv \frac{1}{|eB(x)|} , \tag{A3}$$

and

$$\begin{aligned} &[\hat{X},\hat{\Pi}_i] = [\hat{Y},\hat{\Pi}_i] = 0 , \\ &[\hat{X},\hat{Y}] = 2il^2 , \\ &[\hat{X},\hat{H}_0] = [\hat{Y},\hat{H}_0] = 0 . \end{aligned}$$

So we may choose a basis that is diagonal for $\hat{H}_0$ and $\hat{X}$. Thus the extended Hilbert space would be

$$|n,X\rangle \equiv |n\rangle \otimes |X\rangle ,$$

where

$$\begin{aligned} &\hat{H}_0|n\rangle = (n+\tfrac{1}{2})\frac{|eB(x)|}{m}|n\rangle , \\ &\hat{X}|X\rangle = X|X\rangle , \\ &\hat{Y}|X\rangle = il^2\frac{\partial}{\partial X}|X\rangle . \end{aligned} \tag{A4}$$

But

$$\begin{aligned} &\left[\hat{\tau}, \frac{1}{P_0}\right] = \frac{1}{P_0}[P_0,\hat{\tau}]\frac{1}{P_0} = \frac{1}{P_0^2} , \\ &\hat{x} = \hat{X} + l^2\hat{\Pi}_y, \quad \hat{\tau}|0\rangle = 0 . \end{aligned} \tag{A5}$$

So

$$\begin{aligned} \left\langle 0 \left| \Pi_x \frac{1}{P_0}\hat{x}\hat{\tau}\frac{1}{P_0} \right| 0 \right\rangle &= \left\langle 0 \left| \hat{\Pi}_x \frac{1}{P_0}\hat{x}\frac{1}{P_0^2} \right| 0 \right\rangle \\ &= \left\langle 0 \left| \hat{\Pi}_x \frac{1}{P_0}(\hat{X} + l^2\hat{\Pi}_y)\frac{1}{P_0^2} \right| 0 \right\rangle \\ &= \left\langle 0 \left| \hat{\Pi}_x \frac{1}{P_0}\hat{X}\frac{1}{P_0^2} \right| 0 \right\rangle + l^2\left\langle 0 \left| \hat{\Pi}_x \frac{1}{P_0}\hat{\Pi}_y\frac{1}{P_0^2} \right| 0 \right\rangle \\ &= -l^2\left\langle 0 \left| \hat{\Pi}_y\hat{\Pi}_x \frac{1}{P_0}\frac{1}{P_0^2} \right| 0 \right\rangle + l^2\left\langle 0 \left| \hat{\Pi}_x \frac{1}{P_0}\hat{\Pi}_y\frac{1}{P_0^2} \right| 0 \right\rangle \\ &= l^2\left\langle 0 \left| \left[\hat{\Pi}_x\frac{1}{P_0},\Pi_y\right]\frac{1}{P_0^2} \right| 0 \right\rangle \\ &= l^2\frac{i}{ml^2}\left\langle 0 \left| \hat{\Pi}_x\frac{1}{P_0}\hat{\Pi}_x\frac{1}{P_0}\frac{1}{P_0^2} \right| 0 \right\rangle - l^2\frac{i}{l^2}\left\langle 0 \left| \frac{1}{P_0^3} \right| 0 \right\rangle \\ &= \frac{i}{m}\left\langle 0 \left| \hat{\Pi}_x\frac{1}{P_0}\hat{\Pi}_x\frac{1}{P_0^3} \right| 0 \right\rangle - i\left\langle 0 \left| \frac{1}{P_0^3} \right| 0 \right\rangle . \end{aligned} \tag{A6}$$

Hence Eq. (A1) reduces to

$$\left\langle 0 \left| \hat{\Pi}_x \frac{1}{P_0}\hat{x}\hat{\tau}\frac{1}{P_0} \right| 0 \right\rangle = \frac{i}{m}\left[\left\langle 0 \left| \hat{\Pi}_x\frac{1}{P_0}\hat{\Pi}_x\frac{1}{P_0^3} \right| 0 \right\rangle - m\left\langle 0 \left| \frac{1}{P_0^2} \right| 0 \right\rangle\right] . \tag{A7}$$

Now there are no guiding center coordinates in Eq. (A7). Recalling that

$$\int dX \langle X,n|n'X\rangle = \frac{1}{2\pi l^2}\delta_{n,n'} ,$$

$$\left\langle 0\left|\Pi_x \frac{1}{P_0}\hat{x}\hat{\tau}\frac{1}{P_0}\right|0\right\rangle = \frac{i}{2\pi l^2 m}\frac{1}{\beta}\left[\sum_m\sum_n \langle m|\left\langle n\left|\hat{\Pi}_x\frac{1}{P_0}\hat{\Pi}_x\frac{1}{P_0^3}\right|n\right\rangle|m\rangle - m\sum_{m,n}\langle m|\left\langle n\left|\frac{1}{P_0^3}\right|n\right\rangle|m\rangle\right]$$

$$= \frac{i}{2\pi l^2 m\beta}\left[\sum_{m,n'}\frac{\langle n|\hat{\Pi}_x|n'\rangle\langle n'|\hat{\Pi}_x|n\rangle}{(i\xi_m+E_n-\mu)^3(i\xi_m+E_n'-\mu)} - m\sum_{m,n}\frac{1}{(i\xi_m+E_n-\mu)^3}\right] , \quad \text{(A8)}$$

where $\xi_m=(2m+1)(\pi/\beta)$ and $E_n=(n+\frac{1}{2})(1/ml^2)$. So, from Eq. (A8), defining $\Gamma_n=1/(i\xi_m+E_n-\mu)$, and recalling $\hat{\Pi}_x=(1/\sqrt{2}l)(a+a^\dagger)$,

$$\left\langle 0\left|\hat{\Pi}_x \frac{1}{P_0}\hat{x}\hat{\tau}\frac{1}{P_0}\right|0\right\rangle = \frac{i}{2\pi l^2 m\beta}\left[\sum_{m,n,n'}\frac{1}{2l^2}\left[\frac{1}{\Gamma_n^3\Gamma_{n'}}[(n+1)\delta_{n',n+1}+n\delta_{n',n-1}]\right] - m\sum_{m,n}\frac{1}{\Gamma_n^3}\right] . \quad \text{(A9)}$$

From (A9),

$$\left\langle 0\left|\hat{\Pi}_x \frac{1}{P_0}\hat{x}\hat{\tau}\frac{1}{P_0}\right|0\right\rangle = \frac{i}{2\pi l^2 m\beta}\left[\frac{1}{2l^2}\left[\sum_{m,n}\frac{n+1}{\Gamma_n^3\Gamma_{n+1}}+\frac{n}{\Gamma_n^3\Gamma_{n-1}}\right] - \frac{1}{2}m\frac{\partial^2}{\partial\mu^2}\sum_{m,n}\frac{1}{\Gamma_n}\right] . \quad \text{(A10)}$$

Further, from Appendix B,

$$\sum_{m,n}\frac{1}{\Gamma_n} = \beta\sum_n\frac{1}{e^{\beta(E_n-\mu)}+1} = \beta\gamma ,$$

and

$$\sum_{m,n}\frac{n+1}{\Gamma_n^3\Gamma_{n+1}} + \sum_{m,n}\frac{n}{\Gamma_n^3\Gamma_{n-1}} = \frac{\beta}{2\omega}\frac{\partial^2}{\partial\mu^2}\gamma + \frac{2\beta}{\omega^3}\left[1+\omega\frac{\partial}{\partial\omega}\right]\gamma . \quad \text{(A11)}$$

From Eqs. (A10) and (A11),

$$\left\langle 0\left|\hat{\Pi}_x \frac{1}{\hat{P}_0}\hat{x}\hat{\tau}\frac{1}{\hat{P}_0}\right|0\right\rangle = \frac{i}{2\pi l^2 m\beta}\left\{\frac{m\omega}{2}\left[\frac{\beta}{2\omega}\frac{\partial^2}{\partial\mu^2}\gamma + \frac{2\beta}{\omega^3}\left[1+\omega\frac{\partial}{\partial\omega}\right]\gamma\right] - m\frac{\beta}{2}\frac{\partial^2}{\partial\mu^2}\gamma\right\}$$

$$= \frac{i}{2\pi l^2 m\beta}\left[\frac{2\beta}{\omega^3}\left[1+\omega\frac{\partial}{\partial\omega}\right]\gamma - \frac{m\beta}{4}\frac{\partial^2}{\partial\mu^2}\gamma\right] . \quad \text{(A12)}$$

Thus

$$\left\langle 0\left|\hat{\Pi}_x \frac{1}{\hat{P}_0}\hat{x}\hat{\tau}\frac{1}{\hat{P}_0}\right|0\right\rangle = \frac{i}{2\pi l^2 m}\left[\frac{2}{\omega^3}\left[1+\omega\frac{\partial}{\partial\omega}\right]\gamma - \frac{m}{4}\frac{\partial^2}{\partial\mu^2}\gamma\right] . \quad \text{(A13)}$$

This the technique for computing the matrix elements that arise in the perturbation expansion.

APPENDIX B: ON MATSUBARA SUMS

In our calculations complicated Matsubara sums played a big role. An algorithm has been developed to handle those sums.

First of all, we establish the notation

$$\Gamma_n = i\xi_m + E_n - \mu, \quad \text{where } \xi_m = (2m+1)\frac{\pi}{\beta} ,$$

and

$$E_n = \left[n+\frac{1}{2}\right]\frac{1}{ml^2}, \quad \omega = \frac{1}{ml^2} ,$$

$$\sum_{m,n}\frac{1}{\Gamma_n} = \beta\sum_n\frac{1}{e^{\beta(E_n-\mu)}+1} = \beta\gamma(\omega,\mu) . \quad \text{(B1)}$$

This the fundamental sum. We look at

$$C_1=\sum_m \frac{n+1}{\Gamma_n^2\Gamma_{n+1}}+\frac{n}{\Gamma_n^2\Gamma_{n-1}}=\sum_m \frac{(n+1)}{\Gamma_n\omega}\left[\frac{1}{\Gamma_n}-\frac{1}{\Gamma_{n-1}}\right]+\frac{n}{\omega\Gamma_n}\left[\frac{1}{\Gamma_{n-1}}-\frac{1}{\Gamma_n}\right]$$

$$=\frac{1}{\omega}\sum_m \frac{1}{\Gamma_n^2}-\frac{(n+1)}{\omega^2}\sum_m\left[\frac{1}{\Gamma_n}-\frac{1}{\Gamma_{n+1}}\right]+\frac{n}{\omega^2}\sum_m\left[\frac{1}{\Gamma_{n-1}}-\frac{1}{\Gamma_n}\right],$$

$$\sum_n C_1=\frac{\beta}{\omega}\frac{\partial}{\partial\mu}\gamma(\omega,\mu)-\frac{\beta}{\omega^2}\gamma(\omega,\mu)+\frac{\beta}{\omega^2}\gamma(\omega,\mu)=\frac{\beta}{\omega}\frac{\partial}{\partial\mu}\gamma(\omega,\mu)\ .$$

So

$$\sum_{m,n}\frac{n+1}{\Gamma_n^2\Gamma_{n+1}}+\frac{n}{\Gamma_n^2\Gamma_{n-1}}=\frac{\beta}{\omega}\frac{\partial}{\partial\mu}\gamma(\omega,\mu)\ . \qquad \text{(B2)}$$

Now,

$$C_2=\sum_m \frac{n+1}{\Gamma_n^2\Gamma_{n+1}}-\frac{n}{\Gamma_n^2\Gamma_{n-1}}=\frac{(n+1)}{\omega\Gamma_n}\sum_m\left[\frac{1}{\Gamma_n}-\frac{1}{\Gamma_{n+1}}\right]-\frac{n}{\omega\Gamma_n}\sum_m\left[\frac{1}{\Gamma_{n-1}}-\frac{1}{\Gamma_n}\right]$$

$$=\frac{2(n+\frac{1}{2})}{\omega}\sum_m\frac{1}{\Gamma_n^2}-\frac{n+1}{\omega^2}\sum_m\left[\frac{1}{\Gamma_n}-\frac{1}{\Gamma_{n+1}}\right]-\frac{n}{\omega^2}\sum_m\left[\frac{1}{\Gamma_{n-1}}-\frac{1}{\Gamma_n}\right]$$

$$=-\frac{2}{\omega}\frac{\partial}{\partial\omega}\sum_m\frac{1}{\Gamma_n}-\frac{2}{\omega^2}\sum_m\frac{1}{\Gamma_n}\ .$$

Hence

$$\sum_n C_2=-\frac{2}{\omega^2}\beta\left[1+\omega\frac{\partial}{\partial\omega}\right]\gamma(\omega,\mu)\ . \qquad \text{(B3)}$$

Further,

$$C_3=\sum_m\frac{n+1}{\Gamma_n^3\Gamma_{n+1}}+\sum_m\frac{n}{\Gamma_n^3\Gamma_{n-1}}$$

$$=\frac{n+1}{\omega\Gamma_n^2}\sum_m\left[\frac{1}{\Gamma_n}-\frac{1}{\Gamma_{n+1}}\right]+\frac{n}{\omega\Gamma_n^2}\sum_m\left[\frac{1}{\Gamma_{n-1}}-\frac{1}{\Gamma_n}\right]$$

$$=\sum_m\frac{1}{\omega\Gamma_n^3}-\frac{n+1}{\omega^2\Gamma_n}\sum_m\left[\frac{1}{\Gamma_n}-\frac{1}{\Gamma_{n+1}}\right]+\frac{n}{\omega^2\Gamma_n}\sum_m\left[\frac{1}{\Gamma_{n-1}}-\frac{1}{\Gamma_n}\right]$$

$$=\frac{1}{\omega}\sum_m\frac{1}{\Gamma_n^3}-\frac{2(n+\frac{1}{2})}{\omega^2}\sum_m\frac{1}{\Gamma_n^2}+\frac{n+1}{\omega^3}\sum_m\left[\frac{1}{\Gamma_n}-\frac{1}{\Gamma_{n+1}}\right]+\frac{n}{\omega^3}\sum_m\left[\frac{1}{\Gamma_{n-1}}-\frac{1}{\Gamma_n}\right]\ .$$

So

$$\sum_n C_3=\frac{\beta}{2\omega}\frac{\partial^2}{\partial\mu^2}\gamma(\omega,\mu)+\frac{2\beta}{\omega^2}\frac{\partial}{\partial\omega}\gamma(\omega,\mu)+\frac{2}{\omega^3}\beta\gamma(\omega,\mu)$$

$$=\frac{\beta}{2\omega}\frac{\partial^2}{\partial\mu^2}\gamma(\omega,\mu)+\frac{2\beta}{\omega^3}\left[1+\frac{\omega\partial}{\partial\omega}\right]\gamma(\omega,\mu)\ , \qquad \text{(B4)}$$

$$C_4=\sum_m\frac{n+1}{\Gamma_n^3\Gamma_{n+1}}-\sum_m\frac{n}{\Gamma_n^3\Gamma_{n-1}}=\frac{n+1}{\omega\Gamma_n^2}\sum_m\left[\frac{1}{\Gamma_n}-\frac{1}{\Gamma_{n+1}}\right]-\frac{n}{\omega\Gamma_n^2}\sum_m\left[\frac{1}{\Gamma_{n-1}}-\frac{1}{\Gamma_n}\right]$$

$$=\sum_m\frac{2(n+\frac{1}{2})}{\omega\Gamma_n^3}-\frac{n+1}{\omega^2\Gamma_n}\sum_m\left[\frac{1}{\Gamma_n}-\frac{1}{\Gamma_{n+1}}\right]-\frac{n}{\omega^2\Gamma_n}\sum_m\left[\frac{1}{\Gamma_{n-1}}-\frac{1}{\Gamma_n}\right]$$

$$=\frac{2}{\omega}(n+\tfrac{1}{2})\sum_m\frac{1}{\Gamma_n^3}-\frac{1}{\omega^2}\sum_m\frac{1}{\Gamma_n^2}+\frac{n+1}{\omega^3}\sum_m\left[\frac{1}{\Gamma_n}-\frac{1}{\Gamma_{n+1}}\right]-\frac{n}{\omega^3}\sum_m\left[\frac{1}{\Gamma_{n-1}}-\frac{1}{\Gamma_n}\right]\ .$$

Therefore,

$$\sum_n C_4 = -\frac{\beta}{\omega}\frac{\partial^2}{\partial\mu\,\partial\omega}\gamma(\omega,\mu) - \frac{\beta}{\omega^2}\frac{\partial}{\partial\mu}\gamma(\omega,\mu) = -\frac{\beta}{\omega^2}\frac{\partial}{\partial\mu}\left[1+\omega\frac{\partial}{\partial\omega}\right]\gamma(\omega,\mu)\ . \tag{B5}$$

These are the results that we need in our calculations.

APPENDIX C: MINIMIZATION OF THE FREE ENERGY IN THE CASE OF SEMIONS ($N=2$) DUE TO THE CANCELLATION OF THE INITIAL AND THE INDUCED CS TERMS AND THE MEISSNER EFFECT

From Sec. III the partition function at $T=0$ for the system of anyons is

$$Z=\int\int Da_i\, Da_\tau\, \delta(\partial_i a_i)\exp(A+C+D+E)\ , \tag{C1}$$

where

$$A=-\frac{i}{2\pi}\int\int dx\, d\tau (N-N_c)b(a_\tau+A_\tau)\ ,$$

$$C=-\frac{i}{2\pi}\int\int dx\, d\tau\, NB(a_\tau+A_\tau)\ ,$$

$$D=-\frac{m}{4\pi}\int\int dx\, d\tau\frac{N}{|(b+B)|}(E+e)^2\ ,$$

$$E=-\frac{1}{8\pi m}\int\int dx\, d\tau\, N^2(b+B)^2\ ,$$

and

$$N=\gamma=\lim_{\beta\to\infty}\sum_{n=0}^{\infty}\frac{1}{\exp\eta+1}\ ,\quad N_c=2\pi\mu_0\ ,$$

where $\eta=\beta(E_n-\mu')$ and $\mu'=\mu-i(A_\tau+a_\tau)$. For real a_τ and A_τ, there is no contribution from $i(A_\tau+a_\tau)$ at $T=0$. Now the dominant term comes from the term with one derivative since we are interested in the low-momentum regime. Therefore, consider

$$\frac{i}{2\pi}\int dx\int d\tau[(N-N_c)B+NB]a_\tau\ .$$

So, even though there is a quadratic term in a_τ, we ignore it as a first approximation and look at

$$\int Da_\tau \exp\left[i\int dx\int d\tau[(N-N_c)b+NB]a_\tau\right]$$
$$=\delta((N-N_c)b+NB)\ . \tag{C2}$$

This δ function, so far as the integration over $a_i(x)$, or, equivalently, over $b(x)$ is concerned behaves like $\delta[f(b)]$, where $f(b)\equiv(N_c-N)b-NB$, where

$$N=\sum_{n=1}^{\infty}\Theta\left[\left[n+\frac{1}{2}\right]-\frac{\mu}{\omega}\right]\ . \tag{C3}$$

Now

$$\int Db\,\delta(f(b))=\sum_{b_0}\int Db\,\delta(b-b_0)\frac{1}{|\delta f/\delta b|}\ ,$$

where $f(b_0)=0$;

$$\frac{\delta f}{\delta b}=(N_c-N)-\frac{\delta N}{\delta b}(b+B)\ . \tag{C4}$$

b_0 is obtained as follows:

$$0=(N_c-N_0)b_0-N_0B\ , \tag{C5}$$

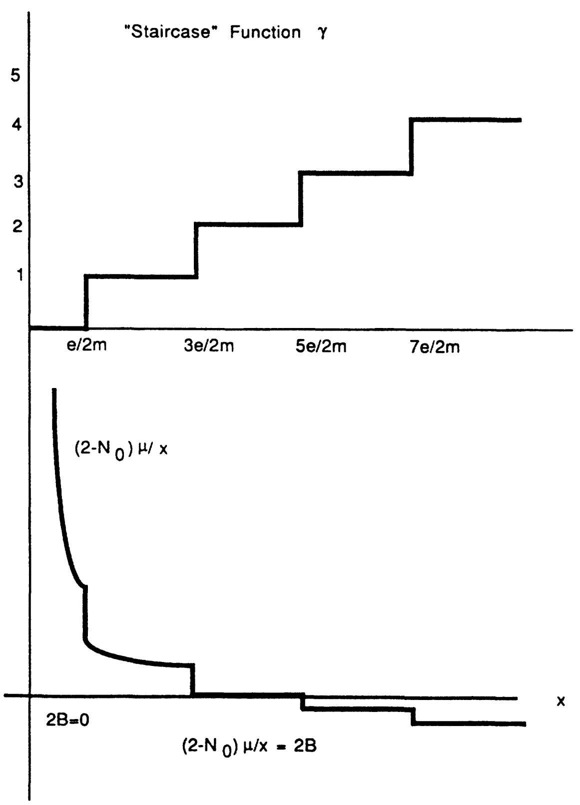

FIG. 1. The staircase function $\gamma(T=0)$ as a function of $x=\mu/(b+B)$. It is also shown that the solution to $(2-N_0)(\mu/x)=2B$ for $B=0$ occurs for $3e/2m\le x\le 5e/2m$. $N_0\equiv\gamma(T=0)$.

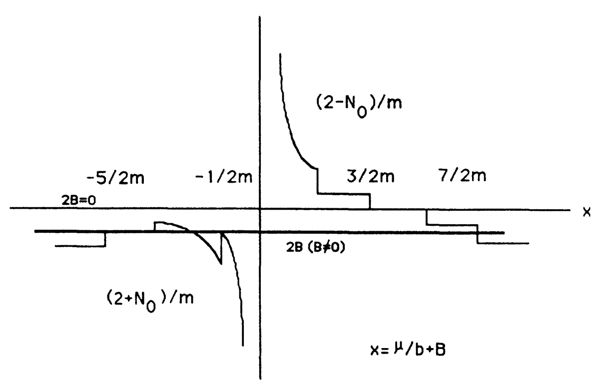

FIG. 2. It is shown that $(2+N_0)(\mu/x)=2B$ does not have a solution for $B=0$. $(2-N_0)(\mu/x)=2B$ does have a solution which corresponds to an absolute minimum of the free energy of the system.

where

$$N_0 = \sum_{n=1}^{\infty} \Theta\left[\left(n+\frac{1}{2}\right) - \frac{\mu}{\omega_0}\right],$$

where

$$\omega_0 \equiv \frac{|(b_0+B)|}{m} .$$

Define

$$x \equiv \frac{\mu}{|b_0+B|} . \tag{C6}$$

If b and B are positive,

$$b_0 = \frac{\mu}{x} - B .$$

So, from (C5) and (C6),

$$(N_c - N_0)\left[\frac{\mu}{x} - B\right] = N_0 B . \tag{C7}$$

So

$$(N_c - N_0)\frac{\mu}{x} = N_c B . \tag{C8}$$

This equation is solved graphically as N_0 depends on x in a nontrivial fashion. Let $N_c = 2$. N_0 is a "staircase" function of x as shown in Fig. 1:

$$N_0 = \sum_{n=1}^{\infty} \Theta\left[\left(n+\frac{1}{2}\right)\frac{1}{m} - x\right] .$$

$(N_c - N_0)$ is a modified staircase function of x. Now, in (C8), $(N_c - N_0)$ is multiplied by μ/x, which is an inverse function of x. For a given μ, μ/x is a rectangular hyperbola when plotted as a function of x. From the above graph it is obvious that for $N_c = N_0$, (C8) will have a solution if $B(x) \to 0$.

Now, from (C4), if $N_0 = N_c$, $\delta N/\delta b = 0$,

$$\int Db\, \delta(f(b)) = \sum_{b_0} \int Db\, \delta(b - b_0)\frac{1}{|N_0 - N_c|}$$

$$\sim \sum_{b_0} \frac{1}{|N_0 - N_c|} .$$

Now $Z \sim \int Db\, \delta(f(b))$. So

$$F \equiv -\ln Z \sim -\ln\frac{1}{|N_0 - N_c|} = \ln|N_0 - N_c| , \tag{C9}$$

which $\to -\infty$ as $N_0 \to N_c$. So the free energy is minimized for $N_0 = N_c$. But if $N_0 = N_c$, we have argued that a solution $b = b_0$ exists if $B = 0$. This is the Meissner effect. In the above analysis we started from $B \geq 0$ and obtain a minimization of the free energy for $B = 0$. The same holds even for $B \leq 0$ as the discussion below shows. As in Eq. (3.9), we write

$$\begin{aligned} Z &\simeq \int Da_\tau\, Db \exp\left[-i\int d\mathbf{x}\, d\tau \left[\mu_0 b + \frac{\gamma}{2\pi}|b+B|\right] a_\tau\right] \\ &= \int_{b+B<0,B\leq 0} Da_\tau\, Db \exp\left[-i\int d\mathbf{x}\, d\tau \left[\mu_0 b + \frac{\gamma}{2\pi}|b+B|\right] a_\tau\right] \\ &\quad + \int_{b+B>0,B\leq 0} Da_\tau Db \exp\left[-i\int d\mathbf{x}\, d\tau \left[\mu_0 b + \frac{\gamma}{2\pi}|b+B|\right] a_\tau\right] . \end{aligned} \tag{C10}$$

Let $\mu_0 = -N/2\pi$. Thus

$$\begin{aligned} Z = &\int_{b+B<0,B\leq 0} Db\, \delta(Nb + \gamma(b+B)) \\ &+ \int_{b+B>0,B\leq 0} Db\, \delta(-Nb + \gamma(b+B)) . \end{aligned} \tag{C11}$$

Let

$$\rho(b) \equiv Nb + \gamma(b+B) , \tag{C12}$$

$$\sigma(b) \equiv -Nb + \gamma(b+B) . \tag{C13}$$

Now we require the zeros of $\rho(b)$ and $\sigma(b)$. From (C12),

$$Nb_0 + \gamma(b_0+B) = 0 . \tag{C14}$$

Let $x \equiv \mu/(b_0+B)$. If $b_0+B<0$, $x<0$. If $b_0+B>0$, $x>0$.

So, from (C14), $(N+\gamma)(\mu/x - B) = -\gamma B$. Or we have

$$(N+\gamma)\frac{\mu}{x} = NB . \tag{C15}$$

Here $x<0$. Again, from (C13),

$$(N-\gamma)\frac{\mu}{x} = NB . \tag{C16}$$

Here $x>0$. In (C15) let $B<0$ be such that the solution exists when $\gamma = 1$. A solution for (C16) also exists simultaneously as is seen from Fig. 2. For these solutions,

$$\begin{aligned} Z_1 \simeq &\sum_{b_0} \int Db\, \delta(b-b_0)\left|\frac{\partial\rho}{\partial b}\right|^{-1}_{b=b_0} \\ &+ \sum_{b_0} \int Db\, \delta(b-b_0)\left|\frac{\partial\sigma}{\partial b}\right|^{-1}_{b=b_0} . \end{aligned} \tag{C17}$$

If, however, $B=0$, a solution for (C15) does not exist, but a solution for (C16) exists with $N=\gamma$. Now

$$\frac{\partial\sigma}{\partial b} = (\gamma - N) . \tag{C18}$$

So

$$Z_2 \simeq \sum_{b_0} \int Db\, \delta(b-b_0)|\gamma-N|^{-1} . \tag{C19}$$

So the free energy $F_1 \equiv -\ln Z_1 \gg F_2 \equiv -\ln Z_2$. Thus an absolute minimum of F is attained if $B=0$, even if we start from $B \leq 0$. The above is illustrated in Fig. 2. Now, with this rather crude estimate as guideline, we may proceed to evaluate Z more carefully as has been done in Sec. III.

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