Superfluid kinetic equation approach to the dynamics of the ^{3}He A-B phase boundary

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The dynamics of the A-B phase boundary is studied using a nonequilibrium theory inspired by the microscopic approach to flux flow in type-II superconductors, namely a generalized two-fluid model consisting of coupled dynamical equations for the superfluid order parameter and the quasiparticle fluid. The interface mobility is obtained to lowest order in the front velocity in three different dynamical regimes: the gapless, hydrodynamic, and ballistic. Experiments have so far only been performed in the ballistic regime, and in this regime we find that, if only Andreev scattering processes are accounted for in the interface mobility, then the theoretical predictions for the terminal velocity of the planar interface are too big by a factor ~ 2. From this we conclude that there may be other important contributions to the interface mobility in the ballistic regime, and we discuss a few possibilities.

I. INTRODUCTION

The A and B phases are the stable bulk superfluid states of liquid ^{3}He in zero magnetic field for temperatures in the millikelvin region.[1,2] The phase boundary between the two bulk phases is a distortion in the ^{3}He superfluid order parameter—a generalization of the textural inhomogeneities that can exist in the bulk phases alone—and the dynamics of the phase front is in many ways similar to the dynamics of Abrikosov vortices in type-II superconductors in the mixed state.[3–5] The propagation of the A-B phase boundary is also an example of a first-order phase transition taking place under highly nonequilibrium, dissipative conditions, and as such has obvious similarities with the motion of domain (bubble) walls between false and true vacuum states in the early universe.[6]

The high-temperature A phase can be substantially supercooled below the thermodynamic (first-order) $A \rightarrow B$ transition temperature, T_{AB}. Over the majority of the pressure (P)–temperature (T) phase diagram the supercooling is so great that one says that the liquid is *hypercooled* at a temperature $T < T_{AB}$, which roughly means that the latent heat evolved per unit volume, L, in the $A \rightarrow B$ transition is incapable of heating the phase boundary region back to T_{AB} from T; i.e., $L/C_p < T_{AB} - T$, where C_p is the specific heat per unit volume of the liquid. In contrast to what happens in conventional supercooled transitions, the flow of latent heat does not play a dominant role in the dynamics of hypercooled transitions; instead, some microscopic mechanisms control the speed of front propagation[7–11] and govern the front's stability properties.[12,13] Here we are interested in the hypercooled dynamics of the topologically stable planar phase boundary between the bulk A and B phases, which (to use common parlance) is a kink solution to an appropriate nonlinear field equation. The path traced out in the 18-dimensional order-parameter configuration space by the planar A-B interface takes the system out of any one degenerate manifold, and it is found that the dynamics of this type of inhomogeneity in the liquid is inherently nonlinear and dissipative.

The dynamics of the A-B interface is interesting both from the general point of view of the dynamics of first-order phase transitions and as a testing ground for the well-developed theories of superfluid dynamics. The microscopic nonequilibrium theories for Fermi superfluids are generalized time-dependent mean-field theories and are one of the few examples of quantitatively accurate mean-field theories for strongly interacting many-body systems such as liquid ^{3}He. In this paper we use the microscopic superfluid kinetic equation formalism, which in the semiclassical limit is a generalized two-fluid model consisting of coupled dynamical equations for the superfluid order parameter and the quasiparticle fluid, to calculate the mobility of the planar A-B interface that may form and subsequently propagate when the stable B phase nucleates in the hypercooled metastable A phase. Since (for T not too close to T_c) the A-B interface width d is believed to be of the same order of magnitude as the superfluid coherence length ξ_0, the semiclassical approximation is not at first sight appropriate. Thus we might expect that the semiclassical method that we adopt here must be treated as a *model* calculation, whose results ought to agree with those of a more appropriate quasiclassical calculation only in the limit $\xi_0/d \ll 1$. As we shall discuss below, remarkably, the semiclassical and quasiclassical results for the part of the interface mobility due to Andreev processes appear to be in exact agreement.

The organization and principal results of the paper are as follows. In Sec. II we describe what we mean by an A-B interface and give an outline of the different levels of theoretical description appropriate for treating its dynamics. We then briefly discuss the five different dynamical regimes (diffusion limited, gapless, hydrodynamic, ballistic, and pair-breaking) for the motion of the interface and describe how the dynamics is governed by different physical processes depending on the location in the pressure-temperature phase diagram.

In Sec. III we review the previous theoretical work on the interface dynamics in the ballistic regime (the only regime to be studied experimentally so far), where the quasiparticle mean free path is much greater than the interface width, and show that the two previous computations of the part of the interface mobility due to the Andreev scattering of thermal quasiparticles (the quasiclassical Green's function calculation of Kopnin[11] and the semiphenomenological calculation of Leggett and Yip[7,9] are in disagreement.

In Sec. IV we use the superfluid kinetic equations to study the dynamics of the *A-B* interface in the gapless, hydrodynamic, and ballistic regimes. In each regime we attempt to obtain an expression for the mobility of the planar interface, which can then be used to predict the interface's terminal velocity. After explaining why a semiclassical calculation of the interface mobility in the ballistic regime should be able to decide between the two previous conflicting results, we show that the semiclassical superfluid kinetic equations lead to a prediction for the part of the mobility due to Andreev processes that is in exact agreement with the full quasiclassical calculation; we find this agreement somewhat puzzling, for the interface width $d \sim \xi$, and the semiclassical approximation is not expected to be quantitatively accurate (unfortunately we have not been able to find a transparent physical explanation for this agreement).

We show in Sec. V that if only the Andreev processes are accounted for in the interface mobility, then the theoretical predictions for the terminal velocity of the planar interface overestimate the experimental data by a factor ~ 2. This leads us to conclude that contributions to the interface mobility not previously accounted for may play an important role; in particular, we demonstrate that some of the quasiparticle states with momenta in or close to the plane of the interface (*grazing angle states*), which are neglected in the Andreev contribution to the mobility, seem to give rise to a contribution that is of the same order of magnitude as the Andreev contribution. (Since we have not been able to obtain a full quantitative prediction for the grazing angle state's contribution to the mobility, we are not in a position to make a comparison with the data.)

Finally, in Sec. VI, we present a conclusion and summary and point out that the superfluid kinetic equation formalism ought to be useful in calculating the interface mobility in the low-temperature pair-breaking regime and in obtaining a quantitative estimate of the effective inertial mass for the interface, two quantities for which at present there are only rough estimates.

Even without the agreement between our semiclassical calculation and the seemingly more appropriate quasiclassical one, the semiclassical approach is interesting in its own right as a model calculation for the dynamics of the *A-B* interface, especially since it gives us insight into what is happening locally in the region of the order-parameter domain wall, where both the gap and the quasiparticle (QP) distribution function are changing on a length scale $\sim d$ (such a local description did not arise in the previous investigations of the interface mobility). Furthermore, our kinetic equation approach allows us to treat quasiparticle collisions in a simple manner and also handle in a unified and physically transparent way the hydrodynamic and gapless regimes for the motion of the interface (which were not previously discussed). In the ballistic regime it also allows us to estimate in a simple way a part of the seemingly important contribution from the quasiparticle states not accounted for in the Andreev part of the interface mobility. One of the shortcomings of the method, however, is that it seems to be difficult to handle in a general way the nonunitary superfluid states that appear in the interface region.

The dynamics of domain walls and first-order phase fronts is often modeled by a time-dependent Ginzburg-Landau (TDGL) equation. Here we show that a generalized TDGL description does hold, but only in the gapless and hydrodynamic regimes, which make up only a tiny portion of the available phase diagram (or parameter space). Over the major portion of the available phase diagram a TDGL description completely breaks down, and this breakdown leads to an interesting and unconventional type of *kink* dynamics. The study of the dynamics of nonlinear distortions like the *A-B* interface as well as of defects such as vortices is naturally the next step in understanding nonequilibrium phenomena in superfluid ^{3}He (cf. Ref. 14), and this study has only recently begun. As we will see below, the dynamics of the *A-B* interface provides a valuable probe of kinetic processes in the liquid that may otherwise be difficult to study.

II. NONEQUILIBRIUM SUPERFLUIDITY AND A-B INTERFACE DYNAMICS

The characteristic atomic momentum and frequency scales associated with the ^{3}He liquid are p_F, the Fermi momentum, and $\epsilon_F \equiv (p_F^2/2m^*) \sim$ Fermi energy ~ 1 K, where m^* is the quasiparticle (QP) effective mass (we use $\hbar = k_B = 1$ throughout, and energy is measured in temperature units). The Cooper pairing in the superfluid phases of ^{3}He is believed to take place in a p-wave ($L = 1$, odd angular momentum), spin-triplet state, and for sufficiently slow time and space variations (slow on the microscopic time, ϵ_F^{-1}, and length, p_F^{-1}, scales) the order parameter or gap matrix can conveniently be written as

$$\hat{\Delta}_{\alpha\beta}(\hat{\mathbf{p}},\mathbf{r},t) = i(\hat{\sigma}_a \hat{\sigma}_2)_{\alpha\beta} \Delta_a(\hat{\mathbf{p}},\mathbf{r},t) , \tag{1}$$

with $\Delta_a(\hat{\mathbf{p}},\mathbf{r},t)$ the components of $\boldsymbol{\Delta}(\hat{\mathbf{p}},\mathbf{r},t)$, which is a (generally complex) spin vector.[1]

For $L = 1$ pairing, $\Delta(\hat{\mathbf{p}})$ can be expanded in $L = 1$ spherical harmonics or more conveniently in terms of the components of the unit wave vector $\hat{\mathbf{p}}$:

$$\Delta_a(\hat{\mathbf{p}},\mathbf{r},t) = \Delta_{ai}(\mathbf{r},t)\hat{p}_i , \tag{2}$$

where we have introduced the complex bivector order parameter $\Delta_{ai}(\mathbf{r},t)$. The order-parameter configuration space can then be regarded as an 18-dimensional real Euclidean space.

The stable bulk phases in zero magnetic field are the anisotropic A phase, which has

$$\Delta_{ai}^A = \sqrt{\tfrac{3}{2}} \Delta_A \hat{d}_a (\hat{\mathbf{w}}_1 + \hat{\mathbf{w}}_2)_i , \tag{3}$$

where Δ_A is the rms value of the energy gap, and the pseudoisotropic B phase, which has

$$\Delta_{ai}^{B}=\Delta_B R_{ai}(\theta,\hat{\omega})e^{i\phi} , \tag{4}$$

where $\hat{\mathbf{w}}_1$ and $\hat{\mathbf{w}}_2$ are a pair of real orthogonal unit vectors in real space that describe the orbital part of the Cooper-pair wave function, $\hat{\mathbf{d}}$ is a unit vector in spin space that describes the spin part of the wave function, ϕ is a phase, and $R_{ai}(\theta,\hat{\omega})$ is an orthogonal rotation matrix that describes a rotation through an angle θ about the axis defined by $\hat{\omega}$. The B phase has an isotropic energy gap Δ_B, while the A phase has an isotropic one with nodes at $\pm\hat{l}$,

$$\Delta_A(\hat{\mathbf{p}})\equiv|\Delta^A(\hat{\mathbf{p}})|=\sqrt{\tfrac{3}{2}}\Delta_A[1-(\hat{\mathbf{p}}\cdot\hat{l})^2]^{1/2} ,$$

where $\hat{l}=\hat{\mathbf{w}}_1\times\hat{\mathbf{w}}_2$ is the A-phase liquid-crystal–like anisotropy vector along which the angular momenta of the Cooper pairs point. Both the A and B phases separately form distinct, topologically inequivalent, degenerate manifolds of states with states in each manifold differing only by the choice of orientation for the symmetry-breaking variables ($\hat{\mathbf{d}}$, $\hat{\mathbf{w}}_1$, etc). A static A-B interface can exist at the thermodynamic first-order phase transition temperature $T_{AB}(P,H)$, and from the above considerations we see that the planar phase boundary is simply a path in configuration space $\Delta_{ai}^{AB}(x)$ parametrized by a coordinate normal to the interface, say x, connecting A and B phase states having particular orientations, i.e.,

$$\Delta_{ai}^{AB}(x)=\begin{cases}\Delta_{ai}^{A} & \text{for } x\to+\infty\\ \Delta_{ai}^{B} & \text{for } x\to-\infty\end{cases} . \tag{5}$$

If we let $\Delta_0=\Delta_0(T)$ denote the typical equilibrium gap magnitude at a temperature T, then in the Ginzburg-Landau (GL) regime where $|\Delta_0|\ll T$, the thickness of the phase boundary is $d(T)\sim\xi(T)$, where

$$\xi(T)\equiv\xi_0\left[1-\frac{T}{T_c}\right]^{-1/2} , \tag{6}$$

is the temperature-dependent coherence length (usually the natural scale for spatial variations in the superfluid order parameter) and

$$\xi_0=\left[\frac{7\zeta(3)}{48\pi^2}\right]^{1/2}\left[\frac{v_F}{T_c}\right]\sim 100 \tag{7}$$

in Å. (T_c is the second-order superfluid transition temperature and $v_F=p_F/m^*$ is the Fermi velocity.) At low T we expect $d\sim\xi_0$.

A. Levels of description for superfluid dynamics

When $T\neq T_{AB}$, unless the phase boundary is pinned, there is no equilibrium state with the phase boundary in the system, and to understand the dynamics of the interface we must use a nonequilibrium theory. In particular, we must allow for the possibility that the fluid of normal excitations may be out of equilibrium with the nonequilibrium condensate degrees of freedom.

There are four basic levels of theoretical description for (Fermi) superfluid dynamics. We briefly summarize the different levels so that we can recall the different methods that can be (or have been) used to study the A-B interface dynamics and keep firmly in mind the formal domain of validity of each. At the lowest level of interest in condensed matter phenomena, there is the *microscopic Green's function formalism,*[15] which can be used to describe the dynamics (with a characteristic frequency ω and wave vector q) right down to atomic frequencies ($\sim\varepsilon_F$) and wave vectors ($\sim p_F$). The full microscopic formalism is extremely *heavy,* and since most phenomena of interest take place on time and length scales much slower than atomic, it is desirable to derive more tractable dynamical theories that are valid on these slower scales.

In the superfluid state there are new quantum length and time scales, specified by the coherence length $\xi_0\sim v_F/T_c\gg p_F^{-1}$ and the inverse gap frequency $\Delta^{-1}\gtrsim T_c^{-1}\gg\varepsilon_F^{-1}$, which are considerably slower than the atomic scales. At these quantum length and time scales the appropriate reduced descriptions are the *quasiclassical Green's function* theory[16] and the *matrix kinetic equations.*[17] At this superfluid quantum level the superfluid order parameter and the fluid of quasiparticles (QP'S) are strongly coupled, interconversion (pair-breaking) processes can be important, and both the order parameter and the QP's can both be far from equilibrium.

In the *semiclassical or Boltzmann limit,* $\omega\ll\Delta$ and $q\xi,(\Delta_0\tau)^{-1}\ll 1$($\tau$ is the inelastic QP collision time), where the dynamics is slow on the scale of the quantum length and time scales, both the quasiclassical Green's-function (GF) theory and the matrix kinetic equations simplify substantially, leading to generalized two-fluid models consisting of a fluid of Fermionic excitations and a superfluid condensate whose behavior is governed by coupled dynamical equations. The condensate order parameter is close to local equilibrium and obeys hydrodynamiclike equations, while the QP fluid is described by a distribution function. The semiclassical kinetic equations are written out and discussed in greater detail in Sec. IV where we use them to study the QP response to the nonlinear, dissipative motion of the A-B interface.

If in addition to the semiclassical approximations the QP mean free path (MFP) $l\ll q^{-1}$ and mean free time (MFT) $\tau\ll\omega^{-1}$, then the system is never very far from local equilibrium and a *hydrodynamic* description is possible. We will show below that there is a part of the pressure-temperature phase diagram where the response of the QP fluid to the motion of the interface can be treated hydrodynamically.

B. Dynamical regimes for the motion of the interface

The effective force driving the A-B interface when $T\neq T_{AB}$ is ΔG_{AB}, the difference in Gibbs free-energy density between the two bulk phases.[7–9] For the A-B transition propagating at a velocity v, we can identify five different dynamical regimes in the P-T phase diagram (see Fig. 1). In each regime the dynamics is dominated by different physical processes. We briefly describe them in the order in which they occur with decreasing tempera-

ture. The inequality $v \ll v_F$ is well satisfied over the whole phase diagram (perhaps becoming marginal in the low-T pair-breaking regime[7-9]), and this inequality plays a crucial role in determining the various dynamical regimes outlined below. The important length scales entering the problem are the superfluid coherence lengths ξ_0 and $\xi(T)$, the thickness of the phase boundary d, and the QP MFP $l = v_F \tau$. The important time scales are the collision time τ, the characteristic time associated with the motion of the interface d/v, and the characteristic Cooper-pair adjustment time Δ_0^{-1}. The five dynamical regimes are the following:

(i) *Diffusion-limited regime,* which exists for T very close to T_{AB} when the liquid is not hypercooled (see Appendix A) and the latent heat from the transition limits the speed and governs the stability of the phase front. We will not investigate this regime at all, since it is important only for a very small part of the phase diagram and is already a well studied problem in the context of other first-order phase transitions.[18]

(ii) *Gapless regime,* which exists only for T very close to T_c [which for the A-B interface dynamics means in practise only near the polycritical point (PCP), since the B phase cannot be superheated very much at higher pressures] in the region where the liquid is hypercooled and $\tau\Delta_0 \ll 1$. Here the energy gap is smeared out by quasiparticle collisions, and the liquid behaves as a gapless superfluid with the dynamics governed by a simple time-dependent Ginzburg-Landau (TDGL) equation.

(iii) *Hydrodynamic regime* (or local approximation), which occurs when the liquid is hypercooled, $\tau\Delta_0 \gg 1$, and the MFP $l \ll d$. In this regime the quasiparticle fluid is always close to being in local equilibrium, and the dynamics of the order parameter can be described by a generalized time-dependent Ginzburg-Landau (TDGL) equation.

(iv) *Ballistic regime,* which occurs when $l \gg d$ and Cooper pairbreaking processes are not important (i.e., when $d/v \gg |\Delta_0|^{-1}$). In this regime the dynamics of the front depends strongly on the transmission and Andreev[19] reflection of ballistic thermal QP's, and this picture should be valid for roughly $T/T_c \gtrsim 0.5$.[7-9] Although the ratio $\tau/(d/v)$ does not play an important role for most QP states as long as $v/v_F \ll 1$, it becomes important when the QP states with a small group velocity in the x direction (normal to the plane of the interface) are taken into account. These are the QP states with energies near the local gap edge and/or momenta making a small angle with respect to the plane of the interface *(grazing angles).* It will turn out that the QP states with small group velocity in the x direction can be treated hydrodynamically provided $\tau/(d/v) \ll 1$ (i.e., $v/v_F \ll d/l$), which will be seen to hold provided the temperature is not too low. We will then find that the grazing angle QP states, although generally small in number, seem to make an important contribution to the interface mobility, even in the *extreme ballistic limit* (EBL), $l \gg d$.

(v) *Pair-breaking regime,* which occurs when the velocity of the interface increases and the equilibrium number density of thermal QP's decreases to the point where the dissipative Cooper pair-breaking processes induced by the time-dependent pair potential begin to dominate over the dissipation due to the processes discussed above in the hydrodynamic and ballistic regimes. This crossover should occur roughly in the temperature range $T/T_c \lesssim 0.5$.[7-9]

In Secs. IV E, IV D, and IV C we study the dynamics of the A-B interface in the gapless, hydrodynamic, and ballistic regimes. The gapless and hydrodynamic regimes have been discussed in the context of flux flow in dirty type-II superconductors or pure ones very near to T_c [where the inequality $l \ll \xi(T) \sim$ *vortex core diameter* hold],[3] and the ballistic regime, where $l(T) \gg \xi(T)$, has been investigated for the flux flow problem in *superpure* superconductors.[4,5] The work on the dynamics of the A-B interface should also be helpful in theoretical studies of the dynamics of both singular and continuous vortices in ^{3}He,[20] a subject that has barely been touched on until now.[21]

So far experiments on the dynamics of the A-B front have only been conducted in the ballistic regime,[7,10] and we will therefore focus on the *superfluid kinetic approach* to the dynamics in this regime. To study the hydrodynamic and gapless regime requires working with small magnetic fields very close to the PCP in order to satisfy the appropriate constraints (see above) and still be in a re-

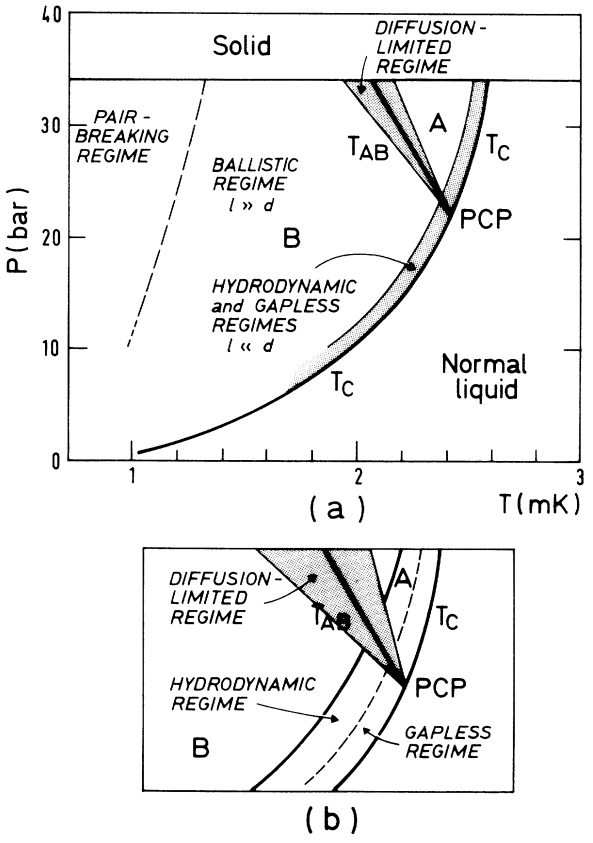

FIG. 1. A schematic diagram of the various dynamical regimes for the motion of the A-B interface in the P-T phase diagram (not to scale).

gion of the phase diagram where the liquid is hypercooled, and it is not yet clear whether the experiments can be carried out. (Perhaps the $B \rightarrow A$ transition can be studied near the PCP where the B phase can be superheated.)

III. REVIEW OF PREVIOUS WORK ON THE BALLISTIC REGIME

Leggett and Yip (LY) (Refs. 7–9) have used a semiphenomenological QP *scattering method* to study the dynamics of the A-B phase front in the temperature regime where the Andreev scattering of QP's off of the propagating kink in the order parameter provides an important contribution to the frictional force on the moving interface. In their approach the A-B front is treated as a *black box* QP scattering potential characterized by transmission and reflection coefficients, T and R. By assuming that the incoming QP's are in thermal equilibrium in the rest frame of the cell walls (which play the role of a *thermostat*) and then explicitly calculating the momentum transfer from a steadily moving planar A-B phase front to the QP fluid, LY find an expression for the friction coefficient $\Gamma_{\rm and}$ (inverse of the mobility) due to Andreev scattering processes to lowest order in v/v_F, where v is the front velocity. They found that the frictional force on the moving interface is a factor $(\Delta_0/\varepsilon_F)^2$ smaller than the value one would find if the QP's were scattered in the normal way rather than by the Andreev mechanism. Their expression for $\Gamma_{\rm and}$ explicitly contains the coefficients T and R and is defined so that the power dissipated per unit area during the motion of the planar interface is $\Gamma_{\rm and} v^2$. T and R must be obtained from a separate and relatively difficult calculation.[22] The terminal velocity of the phase front, $v_{AB} = \Delta G_{AB}/\Gamma$, is then determined by equating the power dissipated per unit area with the power per unit area, $v\Delta G_{AB}$, released during the transition from the metastable A to the stable B phase. (We use v to denote an arbitrary steady interface velocity and v_{AB} to denote the interface *terminal velocity* obtained by solving $\Gamma v^2 = v\Delta G_{AB}$.)

Since the A-B front is a rather sharp distortion in the superfluid with a thickness $d \sim \xi$,[23,24] we might expect that quantum Andreev (over the barrier) QP scattering off of the kink in the order parameter should be important, and it seems unlikely that one can get away with a semiclassical *ballistic* description of the QP response to the motion of the front. Naively, a semiclassical description should apply only when $d \gg \xi$ (cf. Ref. 25). Kopnin,[11] however, has used the same microscopic nonequilibrium GF formalism used for the vortex problem in a superpure superconductor[5] to calculate $\Gamma_{\rm and}$ to lowest order in v/v_F, and within a quasiclassical approximation, $d \gg p_F^{-1}$, but with *apparently* no other approximations, he finds that, due to a unitarity principle that accounts for the conservation of current, *the T and R coefficients drop out of the final expression.* While Kopnin's quasiclassical calculation of $\Gamma_{\rm and}$ requires the inequality $p_F^{-1}/d \ll 1$, it makes no use of the ratio ξ_0/d, and therefore *implies that the semiclassical and quasiclassical results for this quantity should be the same;* we find below that this is indeed the case.

The Andreev friction coefficient, or inverse of the interface mobility, $\Gamma_{\rm and}$, is composed of contributions from the individual QP states specified by an energy E and a momentum direction $\hat{\mathbf{p}}$, so that $\Gamma_{\rm and}$ has the form of a sum over all contributing QP states. In general it is possible to write

$$\Gamma_{\rm and} = 2v_F^{-1} N(0) \int \frac{d\hat{\Omega}}{4\pi} \frac{1}{|\hat{p}_x|} H(\Delta_{\max}(\hat{\mathbf{p}}), \Delta_{\min}(\hat{\mathbf{p}})) \ , \tag{8}$$

with

$$\Delta_{\max}(\hat{\mathbf{p}}) \equiv \max\{\Delta_B, \Delta_A(\hat{\mathbf{p}})\} \ ,$$

$$\Delta_{\min}(\hat{\mathbf{p}}) \equiv \min\{\Delta_B, \Delta_A(\hat{\mathbf{p}})\}$$

[$N(0)$ is the single spin density of states.] After some manipulation we find that Kopnin's result [Eq. (33) of Ref. 11] for H can be put in the form

$$H^K = 2\int_{\Delta_{\min}}^{\Delta_{\max}} dE[-f_0'(E)][E^2 - |\Delta_{\min}(\hat{\mathbf{p}})|^2] + \int_{\Delta_{\max}}^{\infty} dE[-f_0'(E)]\{[E^2 - |\Delta_{\max}(\hat{\mathbf{p}})^2]^{1/2} - [E^2 - |\Delta_{\min}(\hat{\mathbf{p}})|^2]^{1/2}\}^2 \ , \tag{9}$$

which is non-negative; f_0 is the Fermi function and $f_0' = (\partial f_0/\partial E) \leq 0$.

The LY result for H, $H^{\rm LY}$ [see Eq. (4) of Ref. 7], explicitly contains the transmission coefficient and therefore appears to be incompatible with Eq. (9). In evaluating their expression numerically, however, LY made a semiclassical ballistic approximation by setting the transmission coefficient $T = \Theta(E - \Delta_{\max}(\hat{\mathbf{p}}))$, where Θ is the step function (since transmission processes give rise to a negative contribution to the friction coefficient, this approximation provides a lower bound on $\Gamma_{\rm and}$ and hence an upper bound on v_{AB}). If we integrate H^K by parts we find

$$H^K(\Delta_{\min}, \Delta_{\max}) = H^{\rm LY}(\Delta_{\min}, \Delta_{\max}) - (|\Delta_{\max}|^2 - |\Delta_{\min}|^2) f_0(\Delta_{\max}) \ , \tag{10}$$

and therefore the two results for H differ by a simple term. Thus if Kopnin is correct, then LY *overestimate* $\Gamma_{\rm and}$. [The expressions for $\Gamma_{\rm and}$ naively diverge if the integral over $\hat{p}_x$ is not cut off at some (small) value. The issue of the physically correct choice of cutoff is addressed in Sec. V and Appendix C.]

Here we tackle the problem in a completely different way, using the semiclassical *superfluid kinetic equations,* which are strictly speaking only valid in the semiclassical

limit, $\omega/\Delta_0, \xi q \ll 1$ and $\tau\Delta_0 \gg 1$ (for the moving *A-B* interface we expect that $\omega \sim v/d$ and $q \sim d^{-1}$). The method we use here appears to be similar in spirit to, but somewhat different in detail from, the kinetic equation approach to the motion of vortices in superpure superconductors (see especially Ref. 26). Even though our semiclassical approach to the dynamics of the *A-B* interface is not *a priori* justifiable (since $d \sim \xi$), as a model calculation it can certainly be used to decide between the semiclassical approximation to the LY result for Γ_{and} and Kopnin's quasiclassical result (which should obviously remain valid in the semiclassical limit).

Using a simple relaxation-time approximation (RTA) for the collision integral, we exactly solve the nonlinear kinetic equation for arbitrary MFP l and MFT τ to find the nonequilibrium QP distribution induced by the planar *A-B* interface moving at a steady velocity v. Employing v/v_F as a small expansion parameter, we find the nonequilibrium QP distribution to first order in v/v_F and then directly calculate the irreversible (nonadiabatic) rate of energy transfer (per unit area), $\dot{\mathcal{E}}_{\text{irr}}$, from the moving phase front to the QP fluid and identify the friction coefficient Γ from $\dot{\mathcal{E}}_{\text{irr}} = \Gamma v^2$, where the dot denotes the time derivative d/dt.

In the extreme ballistic limit $[l(T) \gg d(T)]$, which holds over the major portion of the experimentally accessible part of the phase diagram (see Fig. 1), the vast majority of the QP states in the interfacial region (where the energy transfer actually takes place) do not feel the effects of collisions, and we can set the collision integral to zero. Assuming that the gap does not have a maximum in the region of the interface and using the collisionless solutions to the kinetic equations to calculate Γ_{and} to lowest order in v/v_F, *we find complete agreement with Kopnin's GF calculation*, which implies that the semiclassical and quasiclassical results are in exact agreement (at least to lowest order in v/v_F). Because we are working strictly at the semiclassical level we cannot within our theory independently assess the accuracy of our semiclassical results; this can only be done by reexamining the derivation of the semiclassical kinetic equation from the lower-level quasiclassical quantum kinetic equations to see why the expected corrections to the semiclassical results do not appear in Γ_{and} to lowest order in v/v_F, but this has not yet been carried out.

The agreement between Kopnin's quasiclassical and our semiclassical calculation of Γ_{and} strongly implies that the LY result misses something. It now appears that this is in fact the case, and it is likely that the discrepancy comes about from LY's use of an incorrect semiclassical approximation for the transmission coefficient when the interface is in motion.[27] For a planar interface moving with a velocity $\mathbf{v}$ the physically correct semiclassical approximation for the QP transmission coefficient is still a step function, but with the energies measured in the rest frame of the interface. With this correction LY's semiclassical result appears to agree with Kopnin's result,[27] although at the time of writing this has not yet been completely cleared up. Even if this modification brings LY's semiclassical result into agreement with Kopnin's, there still remains the discrepancy between LY's general result (with a nontrivial transmission coefficient) and Kopnin's result.

IV. SUPERFLUID KINETIC EQUATIONS

To study the dynamics of the *A-B* front using kinetic equations, we make some simplifying assumptions along the lines of those adopted by LY.[7–9] The cell walls are taken to establish a laboratory frame with a fixed temperature T (a thermostat) and the QP fluid far (distances $\gg l$) from the interface region is taken to be at rest and in thermal equilibrium with respect to these cell walls. This approximation should be valid due to the large viscosity between the cell walls and the normal component as compared with the mutual friction between the QP's and the moving *A-B* interface.

A. Excitation (momentum) representation

In a uniform, global—possibly metastable—equilibrium state with unitary order parameter $\hat{\Delta}_{\alpha\beta}(\hat{\mathbf{p}})$, the distribution $\nu_{\mathbf{p}\sigma}$ of normal excitations (Bogoliubov QP's) in phase space is given by the Fermi function

$$\nu_{\mathbf{p}}^{(0)} = \frac{1}{e^{\beta E_{\mathbf{p}}^{(0)}} + 1} , \tag{11}$$

with the scalar energy $E_{\mathbf{p}}^{(0)} = (\varepsilon_p^2 + |\hat{\Delta}(\hat{\mathbf{p}})|^2)^{1/2}$, where $\varepsilon_p \approx v_F(p - p_F)$ (unless we are dealing explicitly with spin-dependent phenomena, the QP spin indices will be suppressed). In those nonequilibrium situations where a semiclassical theory is valid, a generalized two-fluid type of model makes sense (see Sec. II A) even when the liquid is far from global and local equilibrium, and it is possible to define a *local equilibrium* distribution function, $\nu_{\mathbf{p}}^{\text{LE}}(\mathbf{r},t)$, towards which a nonequilibrium distribution, $\nu_{\mathbf{p}}(\mathbf{r},t)$, relaxes via inelastic QP-wall and QP-QP collisions. The QP distribution function $\nu_{\mathbf{p}}(\mathbf{r},t)$ gives the phase-space density of Bogoliubov excitations with momentum $\mathbf{p}$ at the space-time point $(\mathbf{r},t)$. $\nu_{\mathbf{p}}^{\text{LE}}(\mathbf{r},t)$ has the form of a local Fermi function

$$\nu_{\mathbf{p}}^{\text{LE}}(\mathbf{r},t) = \frac{1}{e^{\beta^{\text{LE}}(\mathbf{r},t) E_{\mathbf{p}}(\mathbf{r},t)} + 1} , \tag{12}$$

with a local temperature $T^{\text{LE}}(\mathbf{r},t) = 1/\beta^{\text{LE}}(\mathbf{r},t)$ and a local energy $E_{\mathbf{p}}(\mathbf{r},t)$ that depends on, among other things, the instantaneous local order parameter $\hat{\Delta}_{\alpha\beta}(\hat{\mathbf{p}},\mathbf{r},t)$, the local chemical potential shift $\delta\mu(\mathbf{r},t)$, the local normal velocity $\mathbf{v}_n(\mathbf{r},t)$, the local superfluid velocity $\mathbf{v}_s(\mathbf{r},t)$, and Fermi-liquid (mean-field) terms due to the QP interactions.[17,28,29] Since they do not appear to play an important role in the problems considered here, we will completely ignore Fermi-liquid corrections throughout this work.

Because of the strong viscous coupling between the QP fluid and the cell walls and the extreme hypercooling of the *A-B* transition, we take $T^{\text{LE}} \approx T$, the temperature of both the cell walls and the metastable *A* phase before the interface appeared on the scene and also assume that the local, instantaneous QP energy is given by

$$[\hat{E}_{\rm p}(\mathbf{r},t)]_{\alpha\beta}\approx[\varepsilon_p^2+|\hat{\Delta}(\hat{\mathbf{p}},\mathbf{r},t)|^2]^{1/2}_{\alpha\beta}\ , \tag{13}$$

which is in general a 2×2 spin matrix. Although this approximation only takes into account the modification of the local excitation spectrum by the instantaneous gap and neglects the other modifications discussed above (cf. Ref. 26), which can be important in other situations, we believe it captures the most important physics of the present problem, namely the *direct* coupling between the nonequilibrium collective field (the superfluid order parameter) and the Fermionic QP fluid.

For an arbitrary spin-triplet (complex) vector order parameter $\mathbf{\Delta}(\hat{\mathbf{p}},\mathbf{r},t)$ the instantaneous gap spin matrix is given by

$$(|\hat{\Delta}(\hat{\mathbf{p}},\mathbf{r},t)|^2)_{\alpha\beta}=[|\mathbf{\Delta}(\hat{\mathbf{p}},\mathbf{r},t)|^2\delta_{\alpha\beta}+i\hat{\sigma}_{\alpha\beta}\cdot(\mathbf{\Delta}\times\mathbf{\Delta}^*)] \tag{14}$$

and is only proportional to $\delta_{\alpha\beta}$ for unitary states, which have $(\mathbf{\Delta}\times\mathbf{\Delta}^*)=0$. Unfortunately, although the bulk A and B phases are unitary, the superfluid states within the A-B phase boundary region have been found to be nonunitary,[23,24,30] and this complicates the analysis considerably because we are then forced to use a 2×2 spin matrix representation for the QP distribution function. In order to simplify the analysis, we will at first neglect the nonunitary contributions to the local QP energy, so that the approximate QP energy gets further simplified to

$$E_{\rm p}(\mathbf{r},t)\approx[\varepsilon_p^2+|\mathbf{\Delta}(\hat{\mathbf{p}},\mathbf{r},t)|^2]^{1/2}\ , \tag{15}$$

and the QP energy and distribution function become scalar quantities. Since with the above approximations our final result for $\Gamma_{\rm and}$ is completely independent of the detailed structure of the interface itself (which is what Kopnin also concluded without making any unitary or semiclassical approximations), we believe our result is valid—within the semiclassical approximation at least—for general nonunitary states within the interface region, but we have not yet been able to prove it. We will show, however, how to generalize our kinetic equations calculations to allow for a special variational set of Kaul-Kleinert (KK) (Ref. 30) nonunitary configurations for the A-B interface, and we get the same result for $\Gamma_{\rm and}$ that we got making the unitary assumption.

A steadily moving kink in the superfluid order parameter will induce in the rest frame of the kink a steady-state nonequilibrium distribution of QP's. We measure the nonequilibrium QP distribution by its *deviation from local equilibrium*

$$\delta\nu_{\rm p}(\mathbf{r},t)=\nu_{\rm p}(\mathbf{r},t)-\nu_{\rm p}^{\rm LE}(\mathbf{r},t)\ , \tag{16}$$

(and not from global equilibrium). In the semiclassical approximation and for nonmagnetic phenomena and unitary order parameters, $\nu_{\rm p}$ is a scalar distribution function that satisfies the nonlinear kinetic equation (KE),[26,28]

$$\partial_t\nu_{\rm p}+\nabla_{\rm r}\nu_{\rm p}\cdot\nabla_{\rm p}E_{\rm p}-\nabla_{\rm r}E_{\rm p}\cdot\nabla_{\rm p}\nu_{\rm p}=I[\delta\nu_{\rm p}]\ , \tag{17}$$

where the QP group velocity $\dot{\mathbf{r}}=\mathbf{v}_{\rm p}(\mathbf{r},t)=\nabla_{\rm p}E_{\rm p}(\mathbf{r},t)$ and force $\dot{\mathbf{p}}=\mathbf{F}=-\nabla_{\rm r}E_{\rm p}(\mathbf{r},t)$ are in general space and time dependent with the QP energy $E_{\rm p}(\mathbf{r},t)$ acting as a Hamiltonian for the QP motion; the collision integral $I(\sim-\delta\nu_{\rm p}/\tau)$ vanishes when $\nu_{\rm p}=\nu_{\rm p}^{\rm LE}(\mathbf{r},t)$ and conserves QP momentum and energy (but not QP number in the superfluid state). It is usually found that the semiclassical KE method gives physically reasonable—if not exact—results right down to its lower limit of validity, so it is definitely a worthwhile method of studying the dynamics of the A-B interface, but its formal limitations ought to be kept in mind (see Sec. II). [For magnetic phenomena and nonunitary order parameters, the QP distribution function becomes a spin matrix $(\hat{\nu}_{\rm p})_{\alpha\beta}$.] Throughout this work we will consistently neglect terms of order $\Delta_0/\varepsilon_F\sim10^{-3}$ and assume particle-hole symmetry.

With the above approximations, we have a generalized two-fluid theory with a superfluid condensate [described by an order-parameter collective field $\hat{\Delta}_{\alpha\beta}(\mathbf{r},t)$] and a QP fluid [described by the distribution function $\nu_{\rm p}(\mathbf{r},t)$] obeying coupled dynamical equations: the KE [Eq. (17)] plus a self-consistency equation ("gap equation") for the order-parameter generalized to take into account deviations from equilibrium. (Actually a third *continuity* equation, involving the dynamics of the condensate phase variable and enforcing mass conservation, is needed to fully specify the dynamics, but we will not go into further details here.[17,26,28])

There are kinetic terms in the generalized gap equation that will give rise at lowest order to an effective inertial mass for the moving interface (cf. Refs. 7 and 9), and higher-order (very likely nonanalytic) terms are expected to account for the Cooper pair breaking induced by the time-dependent gap, although such terms are really outside the scope of the semiclassical analysis. The details of how the general dynamical order-parameter equation can lead to an effective interfacial mass and dissipation through QP relaxation and pair breaking have not yet been fully worked out for the moving A-B phase front problem, but they have been worked out by others for the dynamics of the A-phase $\hat{l}$ vector,[31] and we are now attempting to use the A phase results as a template for the more difficult interface dynamics.

In a general nonequilibrium setting, the coupled dynamical equations must be solved self-consistently, and this is exceedingly difficult. Luckily, however, for the moving interface problem we shall be able to solve the equations iteratively in powers of the small quantity v/v_F.

The basic idea now is to calculate the rate of energy transfer from the steadily moving planar A-B interface to the QP fluid when the interface is moving at a velocity $v\ll v_F$ in the, say, $+x$ direction. The total energy density in the QP fluid is

$$\varepsilon_{\rm QP}(\mathbf{r},t)=\sum_{\mathbf{p}\sigma}E_{\rm p}(\mathbf{r},t)\nu_{\rm p}(\mathbf{r},t)\ , \tag{18}$$

and for the planar symmetry of the one-dimensional (1D) A-B kink (where all quantities can only depend on x) the total QP energy density per unit interface area is

$$\mathcal{E}_{\rm tot}^{\rm QP}=\int dx\sum_{\mathbf{p}\sigma}E_{\rm p}(x,t)\nu_{\rm p}(x,t)\ . \tag{19}$$

Then it is straightforward to show[28] that the time rate of change of $\mathcal{E}_{\rm tot}^{\rm QP}$ is

$$\dot{\mathcal{E}}_{\rm tot}^{\rm QP}=\int dx \sum_{p\sigma}(\nu_p^{\rm LE}\partial_t E_p+\delta\nu_p\partial_t E_p) . \tag{20}$$

For a planar interface moving in the $+x$ direction at a steady velocity v, the QP energy gap assumes the functional form

$$|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|=|\Delta(\hat{\mathbf{p}},u)| , \tag{21}$$

where $u\equiv x-vt$, for in the steady state all space- and time-dependent quantities can depend on x and t only through u. With this functional dependence in mind, we now discuss the two contributions to the time rate of change of $\dot{\mathcal{E}}_{\rm tot}^{\rm QP}$ [Eq. (20)].

Using our approximation Eq. (15) for the local QP energy, we find that if the gap depends on time then the QP energy is not conserved:

$$\begin{aligned}\partial_t E_p(u)&=\frac{|\Delta(\hat{\mathbf{p}},u)|}{E_p(u)}\partial_t|\Delta| \\ &=-v[\partial_u|\Delta(\hat{\mathbf{p}},u)|]\frac{|\Delta(\hat{\mathbf{p}},u)|}{E_p(u)} ,\end{aligned} \tag{22}$$

and $\partial_t E_p(u)\propto v$. The first term in Eq. (20),

$$\dot{\mathcal{E}}_{\rm adiab}\equiv\int dx \sum_{p\sigma}\nu_p^{\rm LE}\partial_t E_p\propto v , \tag{23}$$

depends only on the local equilibrium distribution function, and therefore this term must be identified with the adiabatic (reversible) energy transfer from the moving phase front to the QP fluid; this adiabatic energy transfer is proportional to v and has nothing to do with irreversible energy dissipation, which can only occur when a nonequilibrium QP distribution deviates from and eventually relaxes towards local equilibrium.

The second term in Eq. 20,

$$\dot{\mathcal{E}}_{\rm irr}^{\rm QP}\equiv\int dx \sum_{p\sigma}\delta\nu_p(u)\partial_t E_p(u) , \tag{24}$$

depends explicitly on the deviation from local equilibrium and therefore must be identified as the *irreversible* energy transferred to and finally dissipated in the QP fluid due to the motion of the interface. Since to lowest nonvanishing order both $\partial_t E_p(u)$ and $\delta\nu_p(u)\propto v$, we see that to lowest nonvanishing order $\dot{\mathcal{E}}_{\rm irr}^{\rm QP}\propto v^2$, and the coefficient of proportionality in this relation is just the friction coefficient Γ we seek.

In order to compute $\dot{\mathcal{E}}_{\rm irr}^{\rm QP}$ (Eq. 24) in the steady state, we must solve the KE (17) for $\delta\nu_p(u)$, given the gap $|\Delta(\hat{\mathbf{p}},u)|$ with the boundary conditions

$$|\Delta(\hat{\mathbf{p}},u)|\to\begin{cases}\Delta_A(\hat{\mathbf{p}}), & \text{the } A\text{-phase gap for } u\to+\infty \\ \Delta_B, & \text{the } B\text{-phase gap for } u\to-\infty .\end{cases}$$

We first insert $\nu_p=\nu_p^{\rm LE}+\delta\nu_p$ into Eq. (17), and then after using the property that in the steady state all quantities can depend on x and t only through the combination $u=x-vt$ and using a simple RTA for the collision integral, we find that Eq. (17) reduces to the nonlinear KE

$$-v\partial_u\delta\nu_p(u)+v_p^x(u)\partial_u\delta\nu_p(u)-[\partial_u|\Delta(\hat{\mathbf{p}},u)|]\frac{|\Delta(\hat{\mathbf{p}},u)|}{E_p(u)}\left[\frac{\partial\delta\nu_p}{\partial p_x}\right]+\frac{\delta\nu_p}{\tau}=v[\partial_u|\Delta(\hat{\mathbf{p}},u)|]\frac{|\Delta(\hat{\mathbf{p}},u)|}{E_p(u)}\left[\frac{\partial\nu_p^{\rm LE}(u)}{\partial E}\right] , \tag{25}$$

where the right-hand side (RHS) plays the role of a driving term and $v_p^x(u)=[\varepsilon_p/E_p(u)]v_F\hat{p}_x$ is the x component of the local u-dependent QP group velocity $\mathbf{v}_p(u)=\nabla_p E_p(u)$. In deriving Eq. (25) we have used the identity

$$\nabla_r\nu_p^{\rm LE}\cdot\nabla_p E_p-\nabla_p\nu_p^{\rm LE}\cdot\nabla_r E_p=0 .$$

The first two terms on the left-hand side (LHS) of Eq. (25) are *streaming* terms (the first term comes from the explicit time dependence of the distribution function, the second from the spatial dependence), the third term is an effective force on the QP's due to the inhomogeneous order-parameter field, and the fourth term is a RTA to the collision integral. In the region of the interface the gap is changing from the A to the B phase on a length scale $\sim d$, and in this *domain wall* or *core* region we expect by dimensional arguments that the ratios of the four terms on the LHS of Eq. (25) go roughly like v/v_F: 1: 1: d/l, which allows us to get an idea of the relative importance of the various terms (these ratios are only valid for $\hat{p}_x$ not too small and for QP energies not too close to the gap edge, see below); in particular we see that in the EBL where $d/l\ll 1$ only the second (spatial dependence) and the third (effective force) terms are important as long as $v/v_F\ll 1$.

When the interface is in motion the superfluid order parameter should assume the nonequilibrium form

$$\hat{\Delta}_{\alpha\beta}(\hat{\mathbf{p}},\mathbf{r},t)=\hat{\Delta}_{\alpha\beta}^{(0)}(\hat{\mathbf{p}},u)+\hat{\Delta}_{\alpha\beta}^{(1)}(\hat{\mathbf{p}},u) . \tag{26}$$

where $\hat{\Delta}_{\alpha\beta}^{(0)}(\hat{\mathbf{p}},u)$ is a static order-parameter configuration (not necessarily the minimum-energy one though, see Refs. 7–9 and Sec. V) and $\hat{\Delta}_{\alpha\beta}^{(1)}(\hat{\mathbf{p}},u)$ is a correction proportional to v. Equation (25) clearly shows that we only need the static configuration $\hat{\Delta}_{\alpha\beta}^{(0)}(\hat{\mathbf{p}},u)$ in the driving term on the RHS of the KE to get $\delta\nu_p(u)$ to first order in v/v_F. Although it is beyond the scope of the present work, we can see at least in principle how the coupled superfluid dynamical equations can be solved iteratively in powers of v/v_F for the moving interface problem: The first-order nonequilibrium QP distribution function can in turn be substituted into the generalized gap equation to get the first-order correction to the interface configuration when it is in motion, and so on.

The RTA is really only justifiable in the limit $T\to T_c$,[32] but it makes no difference here at least for the part of the interface mobility due to Andreev scattering, since in the EBL the actual details of the inelastic QP relaxation do not matter at all for these processes; we simply need the *existence* of a relaxation mechanism to set the time scale $\sim\tau$ for QP relaxation and to help fix the right boundary

conditions on the nonequilibrium distribution. (In other words, in the EBL τ will not appear in the final expression for $\Gamma_{\rm and}$.)

The above KE contains nonlinear terms involving products of the *gap distortion* $\sim[\partial_u|\Delta(\hat{\mathbf{p}},u)|]$ with the deviation from local equilibrium distribution function $\delta\nu_{\rm p}(u)$; thus the KE is nonlinear in the perturbation in the gap associated with the moving interface. It can be shown that the *linearized* version of the KE [Eq. (25)] can describe the transmission of QP's across the interface, but cannot account for Andreev reflection processes, which are nonperturbative in nature;[33] thus we must solve the nonlinear KE, but due to the presence of the third term on the LHS (the force term), Eq. (25) is not in a convenient form to be solved iteratively in the small quantity v/v_F. As we shall see below, this problem can be remedied by going from the *excitation or momentum representation* for the QP distribution function used above ($\nu_{\rm p}$) to the *particle or energy representation* distribution function $f_\pm(\hat{\mathbf{p}},\varepsilon;\mathbf{r},t)$ (with ε of both signs), which gives the phase-space density of excitations with energy ε and momentum direction $\hat{\mathbf{p}}$ at $(\mathbf{r},t)$. [The energy variable, ε, appearing in the particle representation distribution function $f_\pm(\varepsilon)$ is a completely independent variable, is not connected with the momentum $\mathbf{p}$ by a dispersion relation, and is distinct from and should not be confused with the energies $E_{\rm p}$ and ε_p appearing in the excitation representation.] We have introduced the excitation representation KE because it will be useful when we try to deal with certain nonunitary forms for the *A-B* interface and when we discuss the hydrodynamic regime in Sec. IV D.

B. Particle (energy) representation

For static *inhomogeneous* order-parameter fields the energy representation is more convenient to use because it incorporates a wiser choice of quantum numbers for the excitations: ε and to an excellent approximation (valid to within small terms of order Δ/ε_F) the momentum direction $\hat{\mathbf{p}}$ are both conserved during the ballistic propagation of QP wave packets even when the gap varies with position, while the momentum $\mathbf{p}$ is not conserved. As we will see the advantages of the energy representation carry over to situations with space- and time-dependent order parameters, even though the QP energy is not then in general conserved.

For nonmagnetic phenomena and unitary superfluid order parameters the dynamics of the fluid of excitations can be described by scalar distribution functions $f_\pm(\hat{\mathbf{p}},\varepsilon;\mathbf{r},t)$, which obey Boltzmann-like KE's that can either be obtained directly from the KE in the excitation representation Eq. (17) by making the change of variables $\mathbf{p}\rightarrow(\varepsilon,\hat{\mathbf{p}})$ or be derived in a systematic manner from the full nonequilibrium quasiclassical GF formalism[16,34] (for magnetic phenomena and nonunitary superfluid order parameters, vector distribution functions must also be introduced). To account for the twofold particle-hole space that plays such an important role in Fermi superfluids, there exist two branches of the excitation spectrum, f_+ the distribution function for the *particlelike* branch and f_- the distribution function for the *holelike* branch. Within the semiclassical approximation and neglecting nonunitary contributions to the energy gap, the appropriate KE's are[16,34]

$$\partial_t f_\pm(\hat{\mathbf{p}},\varepsilon;\mathbf{r},t)+\mathbf{v}_\pm(\hat{\mathbf{p}},\varepsilon;\mathbf{r},t)\cdot\nabla_{\mathbf{r}} f_\pm + [\partial_t|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|]\frac{|\Delta|}{\varepsilon}\left[\frac{\partial f_\pm}{\partial\varepsilon}\right]=I_\pm[\delta f_\pm]\ , \tag{27}$$

where

$$\mathbf{v}_\pm(\hat{\mathbf{p}},\varepsilon;\mathbf{r},t)\equiv\pm\frac{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|^2]^{1/2}}{|\varepsilon|}v_F\hat{\mathbf{p}} \tag{28}$$

is the effective local excitation group velocity; thus particlelike excitations travel parallel to their momentum and holelike ones travel antiparallel. $I_\pm$ is a collision integral that vanishes when $\delta f_\pm\equiv 0$, where the *deviation from equilibrium* distribution functions are defined by

$$\delta f_\pm(\hat{\mathbf{p}},\varepsilon;\mathbf{r},t)\equiv f_\pm(\hat{\mathbf{p}},\varepsilon;\mathbf{r},t)-f_0(\varepsilon) \tag{29}$$

with

$$f_0(\varepsilon)=\frac{1}{e^{\beta\varepsilon}+1} \tag{30}$$

the equilibrium Fermi distribution function.

The above KE [Eq. (27)] must be supplemented by the boundary conditions:[16]

$$(1)\quad \delta f_+(\hat{\mathbf{p}},\varepsilon;\mathbf{r},t)\equiv 0\quad\text{for } \varepsilon<|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|\ , \tag{31a}$$

which simply states that for the set of excitations whose distribution is governed by the semiclassical KE, which does not include any bound levels, there can be no excitations with energies below the local gap;

$$(2)\quad \delta f_+(\hat{\mathbf{p}},\varepsilon=|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|;\mathbf{r},t) =\delta f_-(\hat{\mathbf{p}},\varepsilon=|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|;\mathbf{r},t)\ , \tag{31b}$$

which states that the two branches of the distribution function must coincide at the point where they join [the bottom of the energy band, $\varepsilon=|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|$].

Boundary condition (2) states that branch conversion can take place at $(\mathbf{r},t)$ for those excitations with energy $\varepsilon=|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|$; for example, an incoming localized particlelike wave packet with energy ε can, after reaching a point in space where $\varepsilon=|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|$, be converted to a hole by undergoing Andreev reflection, provided there are no allowed states available to it along the original trajectory. Thus we have the important result that *ballistic Andreev reflection processes that may take place in the presence of inhomogeneous order-parameter fields are described in the semiclassical KE formalism by an appropriate boundary condition on $\delta f_\pm$ as a function of energy in real space.*[16]

It can also be shown that the relationship between the particle and excitation pictures is simply[16]

$$\nu_{\rm p}(\mathbf{r},t)=\begin{cases} f_+(\hat{\mathbf{p}},\varepsilon=E_{\rm p}(\mathbf{r},t),\mathbf{r};t) & \text{for } \varepsilon_p>0 \\ f_-(\hat{\mathbf{p}},\varepsilon=E_{\rm p}(\mathbf{r},t),\mathbf{r};t) & \text{for } \varepsilon_p<0\ , \end{cases} \tag{32}$$

where for our problem $E_{\rm p}(\mathbf{r},t)$ is defined in Eq. (15).

For a planar interface moving steadily at a velocity v in

the $+x$ direction, Eq. (27) reduces to the KE

$$-v\partial_u \delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u)+v_{\pm}^{x}(\hat{\mathbf{p}},\varepsilon;u)\partial_u \delta f_{\pm}+\frac{\delta f_{\pm}}{\tau} =v[\partial_u|\Delta(\hat{\mathbf{p}},u)|]\frac{|\Delta|}{\varepsilon}\left[\frac{\partial f_{\pm}}{\partial\varepsilon}\right], \quad (33)$$

where

$$v_{\pm}^{x}(\hat{\mathbf{p}},\varepsilon;u)\equiv\pm\frac{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}}{|\varepsilon|}v_F\hat{p}_x \quad (34)$$

is the x component of the excitation group velocity, and we have again used a RTA for the collision integral.

In going to the energy representation KE we have eliminated the troublesome force term that appeared in the momentum representation KE [Eq. (25)], but using

$$\left[\frac{\partial f_{\pm}}{\partial\varepsilon}\right]=f_0'+\left[\frac{\partial\delta f_{\pm}}{\partial\varepsilon}\right],$$

where $f_0'\equiv(\partial f_0/\partial\varepsilon)$, we see that the *driving term* on the RHS of Eq. (33) now depends on the distribution function $\delta f_{\pm}$ itself. With this complication the above KE is in general difficult to solve, but as we shall shortly see the KE is in a form suitable for being solved in powers of the small quantity v/v_F; first we recast the full KE (33) in the form of an integral equation:

$$\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u)=\chi_{\pm}(\hat{\mathbf{p}},\varepsilon;u)\left[C_{\pm}(\hat{\mathbf{p}},\varepsilon)+\frac{v}{2\varepsilon}\int_{u_0}^{u}du'\left[f_0'+\frac{\partial\delta f_{\pm}}{\partial\varepsilon}(\hat{\mathbf{p}},\varepsilon;u')\right]\chi_{\pm}^{-1}(\hat{\mathbf{p}},\varepsilon;u')\frac{\partial_{u'}[|\Delta(\hat{\mathbf{p}},u')|^2]}{v_{\pm}^{x}(\hat{\mathbf{p}},\varepsilon;u')-v}\right], \quad (35)$$

where $C_{\pm}(\hat{\mathbf{p}},\varepsilon)$ is an integration constant independent of u and

$$\chi_{\pm}(\hat{\mathbf{p}},\varepsilon;u)\equiv\exp\left[-\frac{\tau_{\pm}(\hat{\mathbf{p}},\varepsilon;u)}{\tau}\right]. \quad (36)$$

Here

$$\tau_{\pm}(\hat{\mathbf{p}},\varepsilon;u)\equiv\int_{u_0}^{u}du'\frac{1}{v_{\pm}^{x}(\hat{\mathbf{p}},\varepsilon;u')-v} \quad (37)$$

is, loosely speaking, an effective local *ballistic propagation time* in the sense that if $\tau_{\pm}(\hat{\mathbf{p}},\varepsilon;u)\ll\tau$, then the QP state $(\hat{\mathbf{p}},\varepsilon)$ at u can be treated as being in the collisionless limit [i.e., the collision term $\sim\tau^{-1}$ in Eq. (33) can be neglected]. In the EBL ($l\gg d$) such ballistic states predominate in the interface region. That the above integral equation is equivalent to the KE [Eq. (33)] can be checked by direct substitution, and this integral equation will be important when it comes to fixing the (Andreev) boundary condition (2) Eq. (31b) (see below).

In the EBL there are essentially three distinct spatial regions to be considered: the *core* region of the phase boundary where the order parameter is changing on the length scale $\sim d$, the *intermediate* regions (distances from the core region $\gg d$, but $<l$) where the order parameter and QP distribution functions are out of (local) equilibrium but still essentially constant, and the *distant* regions (distances $>l$) where the order parameter is essentially constant (very nearly assuming its bulk forms) and because of inelastic collisions the nonequilibrium QP distribution is relaxing on a length scale l. In the important core region, where energy transfer between the QP's and the interface takes place, dimensional arguments show that the first three terms on the LHS of the KE (33) have roughly the relative ratios, v/v_F:1:d/l. Thus for $d/l\ll1$ and as long as $v/v_F\ll1$ only the second term, which is the streaming term that arises from gradients in the distribution function, is important in the EBL. On the other hand in the extreme hydrodynamic limit (EHL) ($l\ll d$) only the third (collision) term is important (see Sec. IV D). The above considerations imply that as along as $v/v_F\ll1$, the ratio $\tau/(d/v)$, which comes from the ratio of the first and third terms in the KE, does not play a dominant role in determining the various dynamical regimes for the motion of the *A-B* interface (see Fig. 1). The ratio $\tau/(d/v)$ will, however, play an important role in understanding how to cut off a formally divergent integral over QP momentum direction in the expression for the Andreev friction coefficient, Γ_{and}, and in attempting to estimate the contribution to the friction coefficient from the QP states with an x-component group velocity $|v_{\pm}^{x}(\hat{\mathbf{p}},\varepsilon;u)|\lesssim v$ [these are the QP states with $\hat{p}_x\ll1$ (grazing angles) and/or $\varepsilon\to|\Delta(\hat{\mathbf{p}},u)|$].

In the distant regions dimensional arguments show that the first three terms on the LHS of the KE have roughly the relative ratios, v/v_F:1:1, and thus the second and third terms must be kept, giving the exponential relaxation of the QP distribution to equilibrium on a length scale l. In general, depending on which of the three terms on the LHS of the KE (33) dominates, the physics will be very different [just which term does dominate depends critically on the QP state $(\varepsilon,\hat{\mathbf{p}})$ in question and the value of u]: (1) In the situation where the first term dominates, $v\gg|v_{\pm}^{x}(\hat{\mathbf{p}},\varepsilon;u)|, d/\tau$ and the QP's can be treated in a collisionless, *sudden approximation*, where the QP's have time neither to propagate nor collide as the interface sweeps by. (2) If the second term dominates then the QP's move ballistically and get transmitted across or Andreev reflected by the interface as it passes. Since this is the case for the vast majority of the QP states in the EBL under consideration in this section, this situation is discussed in great detail below. (3) If the third term dominates then the QP's are never far from local equilibrium and we have the hydrodynamic regime discussed in Sec. IV D. As we shall discuss in Sec. V, for some grazing angle QP states normal scattering processes may dominate, but such processes are not accounted for in the superfluid KE formalism.

Since both the driving term on the RHS of Eq. (33) and $\delta f_{\pm}$ are $\sim O(v)$, to get $\delta f_{\pm}$ to first in v we can replace $\partial f_{\pm}/\partial\varepsilon$ by f_0' to arrive at a relatively simple KE formally linearized in v/v_F:

$$v_{\pm}^{x}(\hat{\mathbf{p}},\varepsilon;u)\partial_u \delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u)+\frac{\delta f_{\pm}}{\tau} = v[\partial_u|\Delta(\hat{\mathbf{p}},u)|]\frac{|\Delta|}{\varepsilon}f_0' \ . \quad (38)$$

As discussed earlier it is also sufficient to use the static gap configuration as a function of u in the KE [see Eq. (26)].

The idea now is to solve Eq. (38) for $\delta f_{\pm}$ and then use Eqs. (24) and (32) to calculate $\dot{\mathcal{E}}_{\rm irr}^{\rm QP}$. [Since we will only need $\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u)$ for $\varepsilon>0$, we drop the absolute value operation on ε in the expression for $v_{\pm}^{x}(\hat{\mathbf{p}},\varepsilon;u)$, Eq. (34). Upon comparing Eqs. (25) and (33) we see that, indeed, the energy representation is the natural choice for the problem at hand.

Since Eq. (38) is a simple first-order ordinary differential equation in u, we can immediately write down the general solution in terms of a quadrature:

$$\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u)=\tilde{\chi}_{\pm}(\hat{\mathbf{p}},\varepsilon;u)\left[C_{\pm}(\hat{\mathbf{p}},\varepsilon)\pm\frac{v}{v_F}\frac{f_0'}{2\hat{p}_x}\int_{u_0}^{u}du'\tilde{\chi}_{\pm}^{-1}(\hat{\mathbf{p}},\varepsilon;u')\frac{\partial_{u'}[|\Delta(\hat{\mathbf{p}},u')|^2]}{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u')|^2]^{1/2}}\right], \quad (39)$$

where $\tilde{\chi}_{\pm}(\hat{\mathbf{p}},\varepsilon;u)$ is $\chi_{\pm}(\hat{\mathbf{p}},\varepsilon;u)$ [Eq. (36)] with $v=0$. This general solution is rather complicated and unwieldy and generally cannot be simplified further. However, since it is valid for arbitrary MFP l and MFT τ, it can handle both the EBL considered in this section and the EHL (extreme hydrodynamic limit) considered in Sec. IV D, and in both these limits the general solution simplifies enormously. In the EBL it not only contains the appropriate behavior of the distribution function in the core region, but the exponential relaxation in the distant region as well.

We conveniently choose the origin of the u axis to lie somewhere in the interface region (perhaps where the gradient in the gap is largest, for example, although the precise definition is not important). For the interface moving with respect to the lab frame fixed by the cell walls, we must supplement boundary conditions (1) and (2) [Eqs. (31)] with a third that says that the QP distribution must relax to equilibrium far ($|u|\gg l$) from the interface region. The boundary conditions for our problem are then

(1) $\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u)\equiv 0$ for $\varepsilon<|\Delta(\hat{\mathbf{p}},u)|$,

(2) $\delta f_{+}(\hat{\mathbf{p}},\varepsilon=|\Delta(\hat{\mathbf{p}},u)|;u)=\delta f_{-}(\hat{\mathbf{p}},\varepsilon=|\Delta(\hat{\mathbf{p}},u)|;u)$,

(3) $\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u)\to 0$ as $|u|\to\infty$.

Since the semiclassical KE formalism cannot handle any bound levels that might be localized near a minimum in the gap located in the interface region [states with energies $\varepsilon<\Delta_{\rm min}(\hat{\mathbf{p}})$], we supplement boundary condition (1) with (1′) $\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u)\equiv 0$ for $\varepsilon<\Delta_{\rm min}(\hat{\mathbf{p}})$.

Boundary condition (3) above and simple physical considerations[7] (essentially causality arguments) also lead us to the important constraint that the *incoming* excitations (in single-particle language these are the excitations incident on the interface from both the A and B phases) must be in thermal equilibrium with the cell walls outside the interface core region; in other words,

$$\delta f_{+}(\hat{\mathbf{p}},\varepsilon;u)\equiv 0 \ \text{ for } \hat{p}_x<0 \text{ and } u\gg +d \ ,$$

$$\delta f_{+}(\hat{\mathbf{p}},\varepsilon;u)\equiv 0 \ \text{ for } \hat{p}_x>0 \text{ and } u\ll -d \ ,$$

$$\delta f_{-}(\hat{\mathbf{p}},\varepsilon;u)\equiv 0 \ \text{ for } \hat{p}_x>0 \text{ and } u\gg +d \ ,$$

$$\delta f_{-}(\hat{\mathbf{p}},\varepsilon;u)\equiv 0 \ \text{ for } \hat{p}_x<0 \text{ and } u\ll -d \ .$$

These constraints will be important below when it comes to finding the solutions to the KE that obey the appropriate boundary conditions.

Using Eq. (32), the irreversible power absorbed and then dissipated by the QP fluid, $\dot{\mathcal{E}}_{\rm irr}^{\rm QP}$ [Eq. (24)], can be written, after some manipulation, as

$$\dot{\mathcal{E}}_{\rm irr}^{\rm QP}=-vN(0)\int\frac{d\hat{\Omega}}{4\pi}\int_{-\infty}^{+\infty}du\int_{-\infty}^{+\infty}d\varepsilon_p\frac{\partial_u[|\Delta(\hat{\mathbf{p}},u)|^2]}{[\varepsilon_p^2+|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}}\delta\nu_{\rm p}(u)$$

$$=-2vN(0)\int\frac{d\hat{\Omega}}{4\pi}\int_{-\infty}^{+\infty}du\int_{|\Delta(\hat{\mathbf{p}},u)}^{+\infty}d\varepsilon\frac{\partial_u[|\Delta(\hat{\mathbf{p}},u)|^2]}{2[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}}[\delta f_{+}(\hat{\mathbf{p}},\varepsilon;u)+\delta f_{-}(\hat{\mathbf{p}},\varepsilon;u)] \ . \quad (40)$$

We must find the solutions Eq. (39) that satisfy boundary conditions (1)–(3), then substitute these solutions into Eq. (40), and finally extract the friction coefficient $\Gamma\equiv\dot{\mathcal{E}}_{\rm irr}^{\rm QP}/v^2$.

C. Andreev friction coefficient $\Gamma_{\rm and}$

We now concentrate on obtaining the Andreev coefficient. Since the integrand in Eq. (40) is essentially

nonzero only in the interface *core region* ($|u| \lesssim d$) where the gap strongly depends on u, to get $\dot{\mathcal{E}}_{\text{irr}}^{\text{QP}}$ we only need the form of $\delta f_{\pm}$ valid in this region. In the EBL the vast majority of the QP states $(\hat{\mathbf{p}},\varepsilon)$ in the core region do not feel the effects of collisions and can be treated in the ballistic (collisionless) approximation. Quantitatively we define the *ballistic* states at a given u by the condition that $\tau_{\pm}(\hat{\mathbf{p}},\varepsilon;u) \ll \tau$ or in other words $\tilde{\chi}_{\pm}(\hat{\mathbf{p}},\varepsilon;u) \approx 1$. In general the relatively small number of states that do not satisfy this ballistic condition (i.e., those states with $\hat{p}_x \to 0$ and/or $\varepsilon \to$ local gap edge) can be difficult to handle; the situation is complicated because besides the semiclassical breakdowns of the ballistic solutions, there is also a quantum breakdown due to the fact that the conditions for Andreev scattering breakdown for grazing angle QP's. A detailed discussion of this breakdown will be postponed until Sec. V (see also Appendix C), where we will also attempt to estimate the contribution to the friction coefficient from some of the QP states that cannot be described by the simple ballistic solutions; we will find that in the region of the phase diagram where $v/v_F \ll d/l$ (i.e., for not too low T, see Sec. V below) at least some of the grazing angle QP states can be treated hydrodynamically even in the EBL, and a rough estimate shows that they seem to give rise to a contribution to the interface mobility on par with the Andreev contribution.

For the ballistic states the collision integral in the KE (38) can be dropped and $\tilde{\chi}_{\pm}(\hat{\mathbf{p}},\varepsilon;u)$ can be set equal to 1 in the general solution Eq. (39). We then find the following simple result for the ballistic states:

$$\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u) \approx C_{\pm}(\hat{\mathbf{p}},\varepsilon) \mp \frac{v}{v_F}\frac{f_0'}{\hat{p}_x}\{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}-[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u_0)|^2]^{1/2}\}\ , \tag{41}$$

where we are free to choose u_0 and $C_{\pm}(\hat{\mathbf{p}},\varepsilon)$ is to be fixed by the boundary conditions.

From now on in this section we neglect all QP states $(\hat{\mathbf{p}},\varepsilon)$ in the core region for which Eq. (41) is *not* a good approximation to the exact solution of the full KE (33) [or, equivalently, the integral equation (35)]. The above solution [Eq. (41)] will break down when any one of the terms in the KE neglected in the derivation of the ballistic solutions becomes comparable in magnitude to the term that was kept (the second term on the LHS of the KE). We find that the first (time derivative) term on the LHS of Eq. (33) can be neglected roughly when

$$(\hat{p}_x/\varepsilon)[\varepsilon^2-|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|^2]^{1/2} \gg v/v_F\ ,$$

the term dropped in the driving term on the RHS can be neglected roughly when

$$(1/\varepsilon)[\varepsilon^2-|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|^2]^{1/2} \gg v/v_F\ ,$$

and the third (collision) term can only be neglected when roughly $\hat{p}_x \gg d/l$. These breakdowns of Eq. (41) naturally provide semiclassical cutoffs for a divergent integral over $\hat{p}_x$ that will arise in the expression for $\dot{\mathcal{E}}_{\text{irr}}^{\text{QP}}$ [Eq. (40)]. In Sec. V and Appendix C we present a discussion of the cutoff procedure and show that there is also a quantum cutoff that at not too low temperatures takes precedence over the semiclassical cutoffs described above. Even though Eq. (41) formally diverges as $\hat{p}_x \to 0$, the exact solution to the KE remains finite in this limit; the divergence is just a signal that the states with $\hat{p}_x$ too small can never be in the ballistic camp, and therefore the solution Eq. (41) is not at all a good approximation for them. For the present purposes, the Andreev friction coefficient Γ_{and} will be loosely defined as the friction coefficient that is due to the QP states for which the above ballistic solutions are a good approximation to the exact solutions. In summary, we find that in the EBL ($l \gg d$) and for small front velocity ($v/v_F \ll 1$), Eq. (41) is an excellent approximate solution to the nonlinear KE (33) for the vast majority of the QP states in the important core region of the interface.

Since the approximate ballistic solutions [Eq. (41)] are rather simple, we can readily write down the appropriate solutions obeying the boundary conditions (1)–(3). In order not to complicate the analysis unduly, we will make the simplifying assumption that the gap $|\Delta(\hat{\mathbf{p}},u)|$ is either monotonic or has a minimum within the phase boundary region, but no maximum. With this assumption $\Delta_{\max}(\hat{\mathbf{p}})$ is the true maximum value of the gap anywhere in the system. Even though this is not the most general case, it does seem to correspond[8,9] to the variational[30] and numerical[23] results for the configuration of the order parameter in the boundary region.

We first consider the set of QP states with energies above the gap maximum, $\varepsilon > \Delta_{\max}(\hat{\mathbf{p}})$, which in the ballistic one-particle language consists of the QP's that can be transmitted across a *static* interface. The appropriate ballistic solutions [Eq. (41)]—valid for $|u \ll l$—satisfying the boundary conditions (1)–(3) and the constraint that the incoming excitations are in equilibrium with the cell walls are

$$\delta f_{+}(\hat{\mathbf{p}},\varepsilon;u) = \begin{cases} -\dfrac{v}{v_F}\dfrac{f_0'}{\hat{p}_x}\{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}-[\varepsilon^2-|\Delta_{\max}(\hat{\mathbf{p}})|^2]^{1/2}\} & \text{for } \hat{p}_x > 0 \\ -\dfrac{v}{v_F}\dfrac{f_0'}{\hat{p}_x}\{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}-[\varepsilon^2-|\Delta_{\min}(\hat{\mathbf{p}})|^2]^{1/2}\} & \text{for } \hat{p}_x < 0 \end{cases} \tag{42}$$

and

$$\delta f_-(\hat{p},\varepsilon;u)=\begin{cases}\dfrac{v}{v_F}\dfrac{f_0'}{\hat{p}_x}\{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}-[\varepsilon^2-|\Delta_{\min}(\hat{\mathbf{p}})|^2]^{1/2}\} & \text{for } \hat{p}_x>0\\ \dfrac{v}{v_F}\dfrac{f_0'}{\hat{p}_x}\{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}-[\varepsilon^2-|\Delta_{\max}(\hat{\mathbf{p}})|^2]^{1/2}\} & \text{for } \hat{p}_x<0\ .\end{cases} \tag{43}$$

What is happening is physically reasonable and quite simple (see Fig. 2): since in the EBL collisions are rare in the interface core region, the A-phase QP distribution is effectively frozen as the interface advances, but the local equilibrium distribution is modified from the global A-phase equilibrium one to a new local one characterized by the local instantaneous value of the gap $|\Delta(\hat{\mathbf{p}},\mathbf{r},t)|$. Since the two distributions are not in general equal, the moving kink in the gap creates a distribution of QP's out of local equilibrium, and these nonequilibrium excitations then propagate ballistically far from the interface core region until they are effectively damped by QP collisions at large distances ($\gtrsim l \gg d$) from where they were born. Thus in the collisionless situation that occurs near the core region, the deviation from the local-equilibrium QP distribution is finite both because the actual occupancies of the local QP states change through ballistic QP motion and also because the local equilibrium distribution changes as the kink in the gap advances.

We next consider the QP states with $\Delta_{\min}(\hat{\mathbf{p}})\le\varepsilon\le\Delta_{\max}(\hat{\mathbf{p}})$ and $|u|\ll l$. [As we discussed earlier there may be QP states with $\varepsilon<\Delta_{\min}(\hat{\mathbf{p}})$ localized near a minimum in the gap if one happens to exist in the interface region, but these bound states are outside the scope of the KE formalism.] With our earlier assumptions on the behavior of the gap $|\Delta(\hat{\mathbf{p}},u)|$ (either monotonic or has a minimum, but no maximum, in the boundary region), it is clear that for QP energies $\Delta_{\min}(\hat{\mathbf{p}})\le\varepsilon\le\Delta_{\max}(\hat{\mathbf{p}})$, there is a unique value of u, $u_0(\varepsilon)$, for which $|\Delta(\hat{\mathbf{p}},u)|=\varepsilon$ (see Fig. 3, below). Since we are free to do so, we choose $u_0=u_0(\varepsilon)$ for $\varepsilon<\Delta_{\max}(\hat{\mathbf{p}})$ in the integral equation (35). Then with this choice the ballistic solutions [Eq. (41)] for $\varepsilon<\Delta_{\max}(\hat{\mathbf{p}})$ reduce to

$$\delta f_\pm(\hat{\mathbf{p}},\varepsilon;u)\approx C_\pm(\hat{\mathbf{p}},\varepsilon)\mp\frac{v}{v_F}\frac{f_0'}{\hat{p}_x}[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}\ , \tag{44}$$

where $C_\pm(\hat{\mathbf{p}},\varepsilon)$ must be fixed by the boundary conditions. For $\varepsilon\le\Delta_{\max}(\hat{\mathbf{p}})$, boundary conditions (1) and (2) [Eqs. (31)] imply that

$$\delta f_\pm(\hat{\mathbf{p}},\varepsilon;u)\equiv 0 \quad \text{for } u<u_0(\varepsilon)\ , \tag{45}$$

and

$$\delta f_+(\hat{\mathbf{p}},\varepsilon=|\Delta(\hat{\mathbf{p}},u_0(\varepsilon))|;u_0(\varepsilon)) = \delta f_-(\hat{\mathbf{p}},\varepsilon=|\Delta(\hat{\mathbf{p}},u_0(\varepsilon))|;u_0(\varepsilon))\ , \tag{46}$$

where in writing Eq. (45) we have assumed without loss of generality (valid to lowest order in v/v_F) that $\Delta_{\max}(\hat{\mathbf{p}})=|\Delta(\hat{\mathbf{p}},-\infty)|$.

In applying boundary condition Eq. (46) we have to be careful because we found earlier that the ballistic solu-

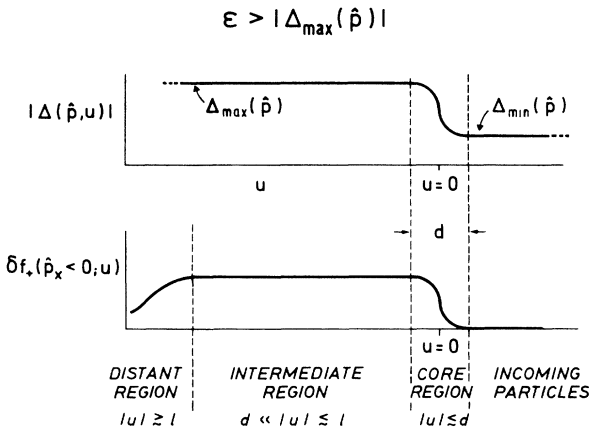

FIG. 2. Schematic diagram of the nonequilibrium QP distribution induced by the moving A-B interface for energies $\varepsilon>\Delta_{\max}(\hat{\mathbf{p}})$ in the ballistic regime.

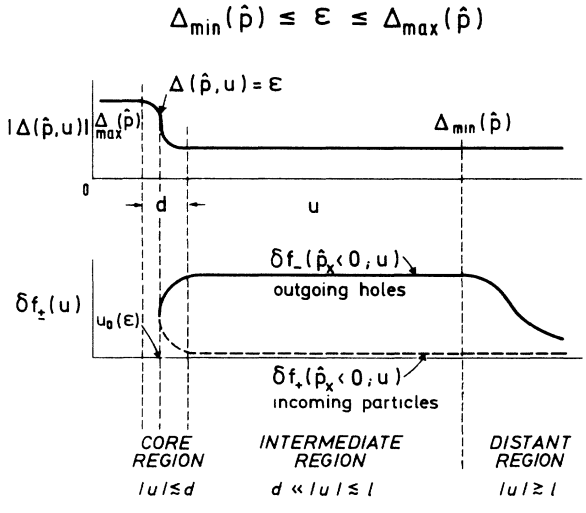

FIG. 3. Schematic diagram of the nonequilibrium QP distribution induced by the moving A-B interface for energies $\Delta_{\min}(\hat{\mathbf{p}})\le\varepsilon\le\Delta_{\max}(\hat{\mathbf{p}})$ in the ballistic regime. The QP's are Andreev reflected at $u_0(\varepsilon)$.

tions Eq. (41) break down for states with energies too close to the gap edge ($\varepsilon \to |\Delta(\hat{\mathbf{p}},u)|$). Luckily, with our judicious choice of u_0 we can work directly with the *exact* integral equations (35) to find

$$\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon=|\Delta(\hat{\mathbf{p}},u_0(\varepsilon))|;u_0(\varepsilon)) = C_{\pm}(\mathbf{p},\varepsilon=|\Delta(\hat{\mathbf{p}},u_0(\varepsilon))|) , \quad (47)$$

and then Eq. (46) yields $C_+(\hat{\mathbf{p}},\varepsilon)=C_-(\hat{\mathbf{p}},\varepsilon)\equiv C(\hat{\mathbf{p}},\varepsilon)$ for $\Delta_{\min}(\hat{\mathbf{p}}) \le \varepsilon \le \Delta_{\max}(\hat{\mathbf{p}})$. Thus even though the ballistic solutions Eq. (41) break down *near* the gap edge they take on the correct values, $C_{\pm}(\mathbf{p},\varepsilon=|\Delta(\hat{\mathbf{p}},u)|)$, *right* at the gap edge $\varepsilon=|\Delta(\hat{\mathbf{p}},u)|$, and therefore the ballistic solutions will themselves end up satisfying the Andreev boundary condition Eq. (46).

Since the QP's with $\varepsilon < \Delta_{\max}(\hat{\mathbf{p}})$ are abruptly scattered at $u_0(\varepsilon)$, $\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon=|\Delta(\hat{\mathbf{p}},u)|,u)$ need not be continuous at $u=u_0(\varepsilon)$, and this implies that $C(\hat{\mathbf{p}},\varepsilon)$ may be nonzero [see Eq. (44)]. Now we need to fix $C(\hat{\mathbf{p}},\varepsilon)$.

Recall that a particlelike excitation (particle for short, +branch) with $\hat{p}_x \gtrless 0$ has group velocity $v_{\mathbf{p}}^x \gtrless 0$, while a holelike excitation (hole for short, −branch) with $\hat{p}_x \gtrless 0$ has a group velocity $v_{\mathbf{p}}^x \lesseqgtr 0$. To fix $C(\hat{\mathbf{p}},\varepsilon)$ we use the fact that $u_0(\varepsilon) \to -\infty$ as $\varepsilon \to \Delta_{\max}(\hat{\mathbf{p}})^-$ to argue that $\delta f_+(\hat{p}_x<0;u)$ and $\delta f_-(\hat{p}_x>0;u)$ [the *incoming* excitations from the $\Delta_{\min}(\hat{\mathbf{p}})$ phase] with $\varepsilon < \Delta_{\max}(\hat{\mathbf{p}})$ should coincide for $\varepsilon \to \Delta_{\max}(\hat{\mathbf{p}})^-$ with the solutions (42) and (43) obtained for $\delta f_{\pm}$ with $\varepsilon \to \Delta_{\max}(\hat{\mathbf{p}})^+$. Using Eqs. (42)–(44) and (47), applying the above continuity argument to $\delta f_+(\hat{p}_x<0)$ and $\delta f_-(\hat{p}_x>0)$ for $\varepsilon \to \Delta_{\max}(\hat{\mathbf{p}})$, and making use of the constraints discussed above Eq. (40) we find that

$$C(\hat{\mathbf{p}},\varepsilon) = -\frac{v}{v_F}\frac{f_0'}{|\hat{p}_x|}[\varepsilon^2-|\Delta_{\min}(\hat{\mathbf{p}})|^2]^{1/2} . \quad (48)$$

[Even though $C(\hat{\mathbf{p}},\varepsilon)$ is fixed by boundary conditions on the distribution function in the regions where the ballistic solutions are valid, it is also the *exact* value of the distribution function right at the gap edge, which is one of the regions where the ballistic solutions formally break down.] Using Eq. (44) we then arrive at the complete solutions for $\Delta_{\min}(\hat{\mathbf{p}}) \le \varepsilon \le \Delta_{\max}(\hat{\mathbf{p}})$ and $l \gg u \ge u_0(\varepsilon)$:

$$\delta f_+(\hat{\mathbf{p}},\varepsilon;u) = \begin{cases} -\dfrac{v}{v_F}\dfrac{f_0'}{\hat{p}_x}\{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}+[\varepsilon^2-|\Delta_{\min}(\hat{\mathbf{p}})|^2]^{1/2}\} & \text{for } \hat{p}_x>0 \\ -\dfrac{v}{v_F}\dfrac{f_0'}{\hat{p}_x}\{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}-[\varepsilon^2-|\Delta_{\min}(\hat{\mathbf{p}})|^2]^{1/2}\} & \text{for } \hat{p}_x<0 \end{cases} \quad (49)$$

and

$$\delta f_-(\hat{\mathbf{p}},\varepsilon;u) = \begin{cases} -\dfrac{v}{v_F}\dfrac{f_0'}{\hat{p}_x}\{[\varepsilon^2-|\Delta_{\min}(\hat{\mathbf{p}})|^2]^{1/2}-[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}\} & \text{for } \hat{p}_x>0 \\ +\dfrac{v}{v_F}\dfrac{f_0'}{\hat{p}_x}\{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}+[\varepsilon^2-|\Delta_{\min}(\hat{\mathbf{p}})|^2]^{1/2}\} & \text{for } \hat{p}_x<0 . \end{cases} \quad (50)$$

From Fig. 3 we see how the incoming, say, particles are pushed out of equilibrium by the moving interface and then are Andreev reflected at $u_0(\varepsilon)$; after the incoming particles undergo branch conversion at $u_0(\varepsilon)$, they reemerge as holes and then propagate deep into the $\Delta_{\min}(\hat{\mathbf{p}})$ phase where they are damped by QP-QP and QP-wall collisions.

The above solutions satisfy all the boundary conditions (1)–(3), as can easily be checked. One can also easily check that $\delta\nu_{\mathbf{p}}(u)$ constructed from the above solutions for $\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u)$ [Eqs. (42), (43), (49), and (50)] and Eq. (32) satisfies the excitation KE [Eq. (25)] to first order in v/v_F. [Although these solutions for $\delta\nu_{\mathbf{p}}(u)$ are rather simple they appear to be difficult to obtain by directly solving Eq. (25).]

Before proceeding to the calculation of the Andreev friction coefficient, we first discuss the breakdown of the ballistic solution near the local gap edge [see discussion below Eq. (41)]. Even though the ballistic solutions formally break down near the local gap edge, they coincide with the exact solution to the full KE right at the gap edge, and they are perfectly finite over the whole region near the gap edge; this behavior should be contrasted with the divergence that occurs for the grazing angle ($\hat{p}_x \to 0$) QP's. Furthermore, the interval in energy near the local gap edge where the ballistic solutions break down is very small, as will be the error made by extrapolating the ballistic solutions right down to the gap edge. For example, in Sec. V we will see that for T not too low $d/l \gg v_{AB}/v_F$, and we then find that the QP states near the gap edge can be treated hydrodynamically even in the EBL. In this case we estimate that the relative error incurred by extrapolating the ballistic solution right up to the gap edge is $\sim (v_{AB}/v_F)^2(l/d) \ll 1$. (A similar sort of argument should hold in the lower T region where $d/l \lesssim v_{AB}/v_F$.) Thus from now on (in the EBL) we will use the ballistic solutions over the whole $(\varepsilon,\hat{\mathbf{p}})$ phase space except in the region of grazing angles,

$\hat{p}_x \lesssim \max\{d/l, v_{AB}/v_F\}$.

To get the irreversible energy absorbed by the QP fluid due to the motion of the interface, we simply insert the solutions Eqs. (42) and (43) for $\varepsilon > \Delta_{\max}(\hat{\mathbf{p}})$ and the solutions Eqs. (49) and (50) for $\Delta_{\min}(\hat{\mathbf{p}}) \le \varepsilon \le \Delta_{\max}(\hat{\mathbf{p}})$ into expression (40) for $\dot{\mathcal{E}}_{\rm irr}^{\rm QP}$ and evaluate the elementary integrals obtained after using the following interchange in the order of integration:

$$\int_{-\infty}^{+\infty} du \int_{|\Delta(\hat{\mathbf{p}},u)|}^{+\infty} d\varepsilon = \int_{\Delta_{\min}(\hat{\mathbf{p}})}^{\Delta_{\max}(\hat{\mathbf{p}})} d\varepsilon \int_{u_0(\varepsilon)}^{+\infty} du + \int_{\Delta_{\max}(\hat{\mathbf{p}})}^{+\infty} d\varepsilon \int_{-\infty}^{+\infty} du \ , \tag{51}$$

which is valid provided that $\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u) \equiv 0$ for $\varepsilon < \Delta_{\min}(\hat{\mathbf{p}})$, as we have assumed.

Using the ballistic solutions for $\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u)$ we find that to an excellent approximation, the relevant integrals can be shown to be

$$\int_{-\infty}^{+\infty} du \frac{|\Delta(\hat{\mathbf{p}},u)|[\partial_u|\Delta(\hat{\mathbf{p}},u)|]}{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}}[\delta f_+(\hat{\mathbf{p}},\varepsilon;u)+\delta f_-(\hat{\mathbf{p}},\varepsilon;u)] = \frac{v}{v_F}\frac{f_0'}{|\hat{p}_x|}\{[\varepsilon^2-|\Delta_{\min}(\hat{\mathbf{p}})|^2]^{1/2}-[\varepsilon^2-|\Delta_{\max}(\hat{\mathbf{p}})|^2]^{1/2}\}^2 \tag{52}$$

for $\varepsilon > \Delta_{\max}(\hat{\mathbf{p}})$, and

$$\int_{u_0(\varepsilon)}^{+\infty} du \frac{|\Delta(\hat{\mathbf{p}},u)|[\partial_u|\Delta(\hat{\mathbf{p}},u)|]}{[\varepsilon^2-|\Delta(\hat{\mathbf{p}},u)|^2]^{1/2}}[\delta f_+(\hat{\mathbf{p}},\varepsilon;u)+\delta f_-(\hat{\mathbf{p}},\varepsilon;u)] = 2\frac{v}{v_F}\frac{f_0'}{|\hat{p}_x|}[\varepsilon^2-|\Delta_{\min}(\hat{\mathbf{p}})|^2] \tag{53}$$

for $\Delta_{\min}(\hat{\mathbf{p}}) \le \varepsilon < \Delta_{\max}(\hat{\mathbf{p}})$, where we have used property that the only place where $[\partial_u|\Delta(\hat{\mathbf{p}},u)|]$ (and therefore the integrand in the above integrals) is nonzero is in the core region of the interface, and for most QP states in this region $\delta f_{\pm}(\hat{\mathbf{p}},\varepsilon;u)$ is given to an excellent approximation by the ballistic solutions.

After inserting the above results in Eq. (40), we arrive at an expression for $\Gamma_{\rm and} = \dot{\mathcal{E}}_{\rm irr}^{\rm QP}/v^2$ which is in complete agreement with Kopnin's quasiclassical result [see Eqs. (8) and (9)]. We can see right away that in order of magnitude $\Gamma_{\rm and} \sim v_F^{-1} N(0) \Delta_0^2$. In the limit $T \to T_c$, we can attempt to expand $\Gamma_{\rm and}$ in powers of the small quantity $\Delta_B(T)/T_c$, and although it turns out that $\Gamma_{\rm and}$ is nonanalytic in this expansion parameter a rough estimate gives (up to nonanalytic terms)[13]

$$\Gamma_{\rm and} \sim v_F^{-1} N(0) \frac{[\Delta_B(T)]^3}{T} \ln(1/\mu_c) \sim (1-T/T_c)^{3/2} \ , \tag{54}$$

which is in agreement with Kopnin[11] (μ_c is the QP momentum cutoff, see Appendix C).

In summary, we have solved the nonlinear KE to leading order in v/v_F in the EBL to find the nonequilibrium QP distribution induced by the motion of the *A-B* interface. We then directly calculated the nonadiabatic energy transfer from the moving interface to the QP fluid and finally extracted the friction coefficient $\Gamma_{\rm and}$. We have also suggested how a dynamical equation for the order parameter, which depends self-consistently on the QP distribution function (via the dependence of the QP KE on the gap itself), can in principle be used to obtain the modification of the moving phase boundary from its static form, although we have also demonstrated that to get the interface friction coefficient to lowest order in v/v_F there is no need to actually obtain an explicit expression for this modification.

The numerical evaluation of the Andreev friction coefficient $\Gamma_{\rm and}$ requires a discussion of the appropriate cutoff in the divergent integral over the QP momentum direction, and we postpone this until Sec. V after we have discussed the hydrodynamic and gapless regimes. We will find that our theoretical estimates based on the Andreev friction mechanism alone overestimate v_{AB} by roughly a factor of 2. As we will discuss in detail below, this leads us to suspect that the grazing angle QP states, not taken into account in $\Gamma_{\rm and}$, may play an important role even in the EBL. For not too low T we find that these states may be treated hydrodynamically, and a rough estimate seems to show that they do indeed give rise to an important contribution. Before doing all this, we first generalize our calculation of $\Gamma_{\rm and}$ to a special class of nonunitary interface configurations and show that result is the same. This generalization will be important when we discuss (in Sec. V) the grazing angle contribution to the friction coefficient in the EBL and the friction coefficient in the extreme hydrodynamic limit (EHL) in Sec. IV D.

Generalization to nonunitary interface configurations. In the previous sections we showed how to use the superfluid KE's to study the dynamics of a unitary phase boundary. Here we will generalize slightly to allow for a special class of nonunitary interface configurations. This is the first step in attempting to deal with the general nonunitary problem. This generalization is also needed to study the dynamics of the *A-B* phase front in large magnetic fields of the size ($H \sim 2$ kG) currently being used in the Los Alamos (LANL) experiments, where the front is propagating through an already magnetized liquid. The bizarre and complicated results (for details see Refs. 10, 35, and 36) coming out of these experiments at low temperatures seem to indicate that the simple signal originally identified at higher T as the magnetization front connected with the *A-B* phase boundary has split into two objects. The data are not yet in a form where it is at all easy to sort things out, but it is clear that one of the objects is moving at ~ 800 cm/sec, which is close to

the condensate spin-wave velocity. These complicated experimental results are very likely connected with the magnetized, nonequilibrium QP distribution induced by the moving interface *plus* the local conversion of QP spin into the spin-wave collective modes in the condensate. Since at low T there are essentially no QP's in the B phase and all the QP"s in the A phase are reflected back into the A phase, the interface should shovel QP magnetization in front of it like a snowplow, and local conversion of QP→condensate spin should take place rather sharply near the interface region. Since there is no time for local relaxation of the condensate spin, there should be spin currents propagating into the bulk phases. (This spin conversion is analogous to the normal current→supercurrent conversion that takes place at a normal-superconductor interface due to Andreev scattering processes). Once the actual A-B front (which may be distinct from the magnetization front) surpasses the spin-wave velocity it also appears as if a Cherenkov spin shock wave should appear in the system (cf. Ref. 37), and this may provide an explanation for at least some of the phenomena observed in the experiments (see also Ref. 36). The above discussion is speculative, and we have not even begun to work out many of the details.

With this motivation we now consider nonunitary phase boundaries. The generalized Kaul-Kleinert (KK)[30] variational ansatz for the gap matrix of the static interface can be written in the form

$$\hat{\Delta}^{\mathrm{KK}}_{\alpha\beta}(\hat{\mathbf{p}},x)=\lambda(x)\hat{\Delta}^{A}_{\alpha\beta}(\hat{\mathbf{p}})+\kappa(x)\hat{\Delta}^{B}_{\alpha\beta}(\hat{\mathbf{p}}) , \tag{55}$$

where

$$\lambda(x)=\begin{cases}0 & \text{for } x\to-\infty\\ 1 & \text{for } x\to+\infty\end{cases} \tag{56}$$

and

$$\kappa(x)=\begin{cases}1 & \text{for } x\to-\infty\\ 0 & \text{for } x\to+\infty\end{cases} . \tag{57}$$

For the static interface, λ and κ are determined by minimizing the static free energy. The simple KK ansatz takes $\lambda=1-\kappa$, and the A-B surface energy determined using this form for the order parameter is not too far off the more accurate numerical calculations in the GL regime[23] (which agree surprisingly well with the experiments performed at melting pressure[38].) The orientations of the bulk A and B phase order parameters are fixed by appropriately choosing the unit vectors $\hat{\mathbf{w}}_1$, $\hat{\mathbf{w}}_2$, and $\hat{\mathbf{d}}$ for $\hat{\Delta}^{A}_{\alpha\beta}$ and the rotation matrix R_{ai} for $\hat{\Delta}^{B}_{\alpha\beta}$. We will now show that if we choose $\hat{\Delta}^{(0)}_{\alpha\beta}(\hat{\mathbf{p}},u)=\hat{\Delta}^{\mathrm{KK}}_{\alpha\beta}(\hat{\mathbf{p}},u)$ for the zero-order moving gap configuration, then we arrive at the same expression for Γ_{and} that we obtained earlier with the unitary assumption.

We begin by observing that in the KE (25) the direction of the momentum $\hat{\mathbf{p}}$ enters simply as a parameter (this result is valid to within small terms $\sim\Delta_0/\varepsilon_F$, which we are consistently ignoring.) It can also be shown[8,22] that *the KK gap matrix [Eq. (55)] can be diagonalized provided that the QP spin quantization axis, $\hat{\mathbf{z}}_s(\hat{\mathbf{p}})$, is chosen appropriately for each $\hat{\mathbf{p}}$ separately.* The correct choice is $\hat{\mathbf{z}}_s(\hat{\mathbf{p}})=\hat{\mathbf{d}}\times\hat{\mathbf{p}}^{R}$, where $(\hat{\mathbf{p}}^{R})_a=R_{ai}\hat{\mathbf{p}}_i$, and with this choice of spin axis

$$\hat{\Delta}^{\mathrm{KK}}(\hat{\mathbf{p}},u)=\begin{bmatrix}\Delta_+(\hat{\mathbf{p}},u) & 0\\ 0 & \Delta_-(\hat{\mathbf{p}},u)\end{bmatrix} ; \tag{58}$$

here

$$\Delta_\pm(\hat{\mathbf{p}},u)=i\lambda(u)\Delta_A(\hat{\mathbf{p}}\cdot\hat{\mathbf{w}}_1+i\hat{\mathbf{p}}\cdot\hat{\mathbf{w}}_2) +\kappa(u)\Delta_B(\pm|\hat{\mathbf{d}}\times\hat{\mathbf{p}}^{R}|+i\hat{\mathbf{p}}^{R}\cdot\hat{\mathbf{d}}) \tag{59}$$

with $|\Delta_\pm(\hat{\mathbf{p}},u)|$ going over asymptotically to the A phase gap as $x\to+\infty$ and the B phase gap as $x\to-\infty$.

With the simple KK ansatz ($\lambda+\kappa=1$), it is possible to check explicitly[8] that the gap $|\Delta_\pm(\hat{\mathbf{p}},u)|$ never has a maximum in the region of the interface; although we have not checked it, we will assume that the same conclusion holds for the generalized KK form. We now generalize the KE (25) by affixing a spin index, $\sigma(\hat{\mathbf{p}})$ to both the QP distribution function and the gap to get $\nu_{\mathrm{p}\sigma}$ and $|\Delta_\sigma(\hat{\mathbf{p}},u)|$, with $\sigma(\hat{\mathbf{p}})=\pm$. The KE then decouples into two independent KE's one for the up-spin distribution function and the other for the down spin, and the QP energy $E_{\mathrm{p}\sigma}=[\varepsilon_p^2+|\Delta_\sigma(\hat{\mathbf{p}},u)|^2]^{1/2}$ becomes spin dependent. The KE's can be solved exactly as before and the solutions inserted into the expression for the irreversible energy transfer [Eq. (24)] with the momentum-dependent spin sum performed before the angular integral:

$$\dot{\mathcal{E}}^{\mathrm{QP}}_{\mathrm{irr}}=N(0)\int\frac{d\hat{\Omega}}{4\pi}\sum_{\sigma(\mathbf{p})}\int dx\int d\varepsilon_p(\delta\nu_{\mathrm{p}\sigma}\partial_t E_{\mathrm{p}\sigma}) . \tag{60}$$

In the EBL the contribution of each spin species to $\dot{\mathcal{E}}^{\mathrm{QP}}_{\mathrm{irr}}$ depends only on the asymptotic forms of the gaps $|\Delta_\sigma(\hat{\mathbf{p}},u)|$ [see discussion after Eq. (51)], and since these forms are the same for both species, both spin species contribute equal amounts to Γ_{and}, and the sum of the two contributions equals our previous result. On the other hand, in situations where $\dot{\mathcal{E}}^{\mathrm{QP}}_{\mathrm{irr}}$ depends explicitly on the structure of the gap in the interface region, as in the hydrodynamic regime discussed in the next section, both spin species will not contribute equal amounts to the friction coefficient.

D. Hydrodynamic (local approximation) regime

In this section we consider the extreme hydrodynamic limit (EHL), which is the opposite limit from the EBL considered in the previous section (cf. Ref. 39). The essential requirements are that the liquid is hypercooled, while at the same time $\Delta_0\tau\gg1$, and the QP MFP satisfies the inequalities $\xi_0\ll l(T)\ll d(T)\sim\xi(T)$. Further details outlining the conditions for the EHL are given in Appendix A, where we also show that there exists a small portion of the P,T phase diagram near the PCP, given roughly by $10^{-4}\lesssim[1-T/T_c(P)]\lesssim10^{-2}$, where the system is in the hydrodynamic regime *and* the A-B transition is still strongly hypercooled. Within this window, we expect the dynamics of the interface to be governed by the hydrodynamic theory described here. In analogy with the hydrodynamic theory of orbital relaxation in the A phase, the hydrodynamic regime can be called the nor-

mal locking regime, since the motion of the interface is strongly coupled to the QP fluid thorough the dependence of the QP energy on the instantaneous gap and the efficient relaxation of the nonequilibrium QP distribution through inelastic collisions. For temperatures very close to T_c, $\Delta_0\tau \ll 1$, and provided the liquid is hypercooled the dynamics of the interface should be described by the gapless theory discussed in the next section.

In both the hydrodynamic and gapless regimes, the QP's can be eliminated from the problem (*integrated out*) and the superfluid dynamics can be reduced to an effective dynamics for the condensate order parameter alone. In both these regimes the dynamical evolution equation for the gap has the following schematic diffusive TDGL form:[32]

$$\gamma[\Delta]\partial_t\Delta = -\frac{\delta\mathcal{F}_0}{\delta\Delta^\dagger} = N(0)(\Lambda\nabla^2\Delta + \alpha\Delta - \beta|\Delta|^2\Delta) , \quad (61)$$

where $\mathcal{F}_0[\Delta,\Delta^\dagger]$ is the static GL free energy,

$$\mathcal{F}_0 = N(0)\left[\Lambda|\nabla\Delta|^2 - \alpha|\Delta|^2 + \frac{\beta}{2}|\Delta|^4\right] , \quad (62)$$

$\alpha \equiv 1 - T/T_c$ is the reduced temperature, $\beta \sim (\xi_0/v_F)^2 \sim |\Delta(0)|^{-2}$, and $\Lambda \sim \xi_0^2$. In the EHL the effective diffusion coefficient $\gamma \sim N(0)|\Delta|\tau/T$ can be obtained from the generalized gap equation by taking into account the deviation of the QP distribution from local equilibrium.

Once we have an expression for the coefficient γ we can use irreversible thermodynamic arguments[40] to calculate the energy dissipated per unit area during the propagation of a moving 1D kink in the gap, $\Delta(x,t) = \Delta(x - vt)$:

$$\dot{\mathcal{E}}_{\rm irr} = \int dx \left[\frac{\delta\mathcal{F}_0}{\delta\Delta}\partial_t\Delta + \text{H.c.}\right] = -2\int d\,\gamma[\Delta]|\partial_t\Delta|^2 . \quad (63)$$

We can then identify the friction coefficient from $\dot{\mathcal{E}}_{\rm irr} = -\Gamma v^2$; i.e.,

$$\Gamma_{\rm hyd} = 2\int dx\,\gamma[\Delta]|\partial_x\Delta|^2 \quad (64)$$

Thus in the EHL the friction coefficient and therefore the terminal velocity $v_{AB} = \Delta G_{AB}/\Gamma$ depend on the detailed structure of the moving kink and on the QP collision time (this is in contrast to the EBL). Although the above discussion contains the essential physics (and is entirely correct for a hypothetical *s*-wave neutral Fermi superfluid), it is only schematic for superfluid ^{3}He where the presence of the tensor order parameter makes it necessary to keep track of spin and orbital indices, etc.

We will confine ourselves to the KK type of *A-B* interface configurations, for at this stage we are only interested in obtaining a reasonable estimate of the hydrodynamic friction coefficient (for which the KK form should be entirely adequate), and we do not want to deal with the complication of general nonunitary interface configurations. Even with the simplification of the KK order parameter it is not completely trivial to derive the hydrodynamic TDGL equation for the moving interface, and we will not attempt it here. We will instead compute the power dissipated directly in terms of the QP distribution function, as we did in the EBL calculation of the previous section.

The starting point is the semiclassical KE, Eq. (25), in the particle representation, but now that we are in the EHL ($l \ll d$), instead of neglecting the collision term (as we did in the EBL), we neglect the streaming and force terms which are smaller than the collision term by at least a factor of l/d (again using $v/v_F \ll 1$). The simple RTA that we use here for the collision integral has been shown[32] to be asymptotically exact in the limit $T \to T_c$, and since this is the only place where the local hydrodynamic approximation applies anyway, the RTA should be quantitatively correct and not just a crude approximation.

The resulting KE is of the usual form for calculating transport coefficients in the hydrodynamic limit,

$$v[\partial_v|\Delta_\sigma(\hat{\mathbf{p}},u)|]\frac{|\Delta_\sigma(\hat{\mathbf{p}},u)|}{E_{p\sigma}(u)}\left[\frac{\partial\nu_{p\sigma}^{\rm LE}(u)}{\partial E}\right] = \frac{\delta\nu_{p\sigma}}{\tau_p} , \quad (65)$$

where the LHS is the driving force and the RHS is the response of the QP fluid. As discussed in the previous section, the spin index $\sigma(\hat{\mathbf{p}}) = \pm$ is related to a choice of spin quantization axis that depends on $\hat{\mathbf{p}}$. Equation (65) also shows that in the EHL the QP distribution function only deviates from local equilibrium in the region where the local gap is changing (see Fig. 4); this should be contrasted with what happens in the ballistic regime (see Figs. 2 and 3) where the nonequilibrium QP distributions only relax far from the interface core region.

Using Eq. (65) in the expression for the irreversible energy dissipated yields

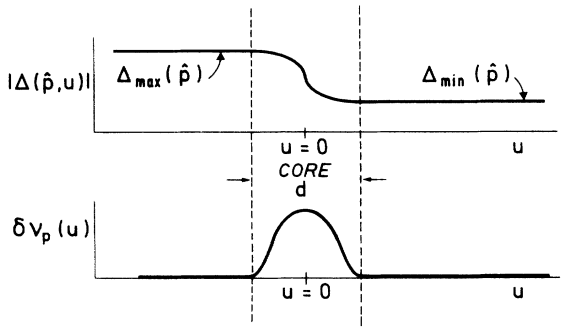

FIG. 4. Schematic diagram of the nonequilibrium QP distribution induced by the moving *A-B* interface in the hydrodynamic regime.

$$\dot{\mathcal{E}}_{\rm irr}^{\rm QP}=\int dx \sum_{p\sigma(\hat{\mathbf{p}})} \delta\nu_{p\sigma}(u)\partial_t E_{p\sigma}(u) ,$$

$$\equiv v^2 \int dx \sum_{p\sigma(\hat{\mathbf{p}})} \tau_p[\partial_u |\Delta_\sigma(\hat{\mathbf{p}},u)|]^2 \left[\frac{|\Delta_\sigma(\hat{\mathbf{p}},u)|}{E_{p\sigma}(u)}\right]^2 \left[\frac{\partial\nu_{p\sigma}^{\rm LE}(u)}{\partial E}\right] . \quad (66)$$

This expression should be compared with Eq. (63). (We have also checked that the rate of energy dissipation, $\dot{\mathcal{E}}_{\rm irr}^{\rm QP}=-T\int dx\,\partial_t S$, calculated from the rate of change of the entropy density, where $S=-\sum_{p\sigma(\hat{\mathbf{p}})}[\nu_{p\sigma}\ln\nu_{p\sigma}+(1-\nu_{p\sigma})\ln(1-\nu_{p\sigma})]$ agrees with Eq. (66).)

A normalized form of the gap for the KK order parameter is

$$g_\pm(\hat{\mathbf{p}},u)\equiv\frac{|\Delta_\pm(\hat{\mathbf{p}},u)|}{\Delta_B}=\{[-\lambda(u)r\hat{\mathbf{p}}\cdot\hat{\mathbf{w}}_2\pm\kappa(u)|\hat{\mathbf{d}}\times\hat{\mathbf{p}}^R|]^2+[\lambda(u)r\hat{\mathbf{p}}\cdot\hat{\mathbf{w}}_1+\kappa(u)\hat{\mathbf{p}}^R\cdot\hat{\mathbf{d}}]^2\}^{1/2} , \quad (67)$$

where in the GL regime $r\equiv\Delta_A/\Delta_B$ is a number ~ 1.

After some manipulations using the above results, we find that the hydrodynamic friction coefficient $\Gamma_{\rm hyd}=\dot{\mathcal{E}}_{\rm irr}^{\rm QP}/v^2$ has the form

$$\Gamma_{\rm hyd}=\frac{\pi}{4T}\frac{\tau}{\xi(T)}N(0)\Delta_B^3 I_{AB}\sim\left[1-\frac{T}{T_c}\right]^2 , \quad (68)$$

where

$$I_{AB}\equiv\int_{-\infty}^{+\infty}d\zeta\int\frac{d\hat{\Omega}}{4\pi}\sum_{\sigma(\hat{\mathbf{p}})=\pm} g_\sigma(\hat{\mathbf{p}},\zeta)[\partial_\zeta g_\sigma(\hat{\mathbf{p}},\zeta)]^2 , \quad (69)$$

is a temperature-independent dimensionless integral ~ 1 that depends on the structure of the interface and the textural boundary condition between the A and B phases; the general form [Eq. (68)] for $\Gamma_{\rm hyd}$, including the temperature dependence, should be generally valid, independent of the KK ansatz. In deriving Eq. (68) we have also used

$$\int_{-\infty}^{+\infty}d\varepsilon_p\tau_p\left[\frac{|\Delta_\sigma(\hat{\mathbf{p}},u)|}{E_{p\sigma}(u)}\right]^2\left[\frac{\partial\nu_{p\sigma}^{\rm LE}(u)}{\partial E}\right] \approx -\frac{\pi|\Delta_\sigma(\hat{\mathbf{p}},u)|\tau}{4T} , \quad (70)$$

which holds for $T\to T_c$ and the property that in the GL regime the gap can only depend on u in the dimensionless combinations $\zeta=u/\xi(T)$ (here τ is to be interpreted as the QP MFT in the normal state evaluated on the Fermi surface at $T=T_c$).

We conclude from Eq. (68) that near T_c the temperature dependence of $\Gamma_{\rm hyd}\sim(1-T/T_c)^2$, which differs from the $T\to T_c$ limit of $\Gamma_{\rm and}$, and the difference in exponent should be experimentally observable in measurements of v_{AB}.

E. Gapless regime

For temperatures very close to T_c (but still outside the temperature range where critical fluctuations destroy the validity of mean-field theory) the magnitude of the gap is smaller than the energy width $\sim 1/\tau$ (due to inelastic collisions) of the QP states making up the Cooper pairs. This so-called gapless regime ($\Delta_0\tau\ll 1$) covers reduced temperatures $10^{-6}\lesssim 1-T/T_c\lesssim t_{\rm gapl}\sim 10^{-3}$ or 10^{-4}. Since it may be very difficult to perform A-B interface velocity experiments that close to T_c, this section may be of only theoretical interest, and we will be very brief. (This conclusion may be too pessimistic: The gapless dynamics of the A phase $\hat{\mathbf{l}}$ vector is discussed in Appendix B, where *it is suggested that the crossover from local hydrodynamic to gapless behavior may have already been seen in experiments investigating orbital relaxation.*)

The gapless dynamics is a purely quantum-mechanical many-body effect, and therefore recourse must be made to a suitable microscopic theory. Many derivations of the gapless dynamical equation have been given for a hypothetical s-wave Fermi superfluid,[41–43] with the result that the gap obeys a simple TDGL equation. By generalizing this work to the p-wave (^{3}He) case we find that the evolution equation for the ^{3}He tensor order parameter, Δ_{ai}, is

$$\gamma\partial_t\Delta_{ai}=-\frac{\delta\mathcal{F}_0}{\delta\Delta_{ai}^\dagger} , \quad (71)$$

where $\mathcal{F}_0[\Delta_{ai},\Delta_{ai}^\dagger]$ is the static GL free energy (including bending terms) and the diffusion coefficient is

$$\gamma=\frac{\pi N(0)}{24T} , \quad (72)$$

which is independent of Δ, as was expected from the s-wave calculations where $\gamma=[\pi N(0)/8T]$.

The derivation of the gapless dynamics is valid only if

$$\left|\frac{1}{\Delta}\frac{\partial\Delta}{\partial t}\right|\ll\tau^{-1},\frac{v_F}{\xi_0},\quad \left|\frac{1}{\Delta}\frac{\partial\Delta}{\partial x}\right|\ll\xi^{-1},\quad \Delta_0\ll\tau^{-1} ,$$

where $v_F/\xi_0\sim\Delta_0(0)\sim T_c$ and the characteristic time and length associated with the dynamics of the gap are $\bar{d}\sim|(1/\Delta)\partial\Delta/\partial x|^{-1}$ and $\bar{t}\sim|(1/\Delta)\partial\Delta/\partial t|^{-1}$. For a moving kink in the order parameter the above inequalities translate into

$$\tau, T_c^{-1}\ll\bar{t}\sim d/v,\quad \xi_0\ll\bar{d}\sim d .$$

The approach to equilibrium of the nonequilibrium gap in the gapless regime (also valid for fluctuation effects above T_c) occurs through the Cooper-pair formation and pair-breaking caused directly by the inelastic QP collisions. For $\Delta_0\tau\ll 1$ these processes are important be-

cause the QP energy gap is smeared out by collisions since the width $\sim 1/\tau$ of the QP energy states making up the Cooper pairs is bigger than the pair binding energy $\sim\Delta_0$. This is in contrast to the hydrodynamic regime, where such direct processes are not important, and gap relaxation takes place almost entirely through the inelastic-scattering processes that relax the nonequilibrium QP distribution with the gap adjusting essentially adiabatically through the instantaneous gap equation (see Sec. IV D).

Using the gapless diffusion coefficient and the irreversible thermodynamic arguments, Eq. (63), for the power dissipated, we find that in the gapless regime the friction coefficient for a moving planar interface A-B interface has the form

$$\Gamma_{\rm gapl}=\frac{\pi N(0)}{12T}\int du\,|\partial_u\Delta_{ai}|^2\ . \tag{73}$$

We can now substitute a form for the zero-order moving kink $\Delta_{ai}^{(0)}(u)$ [for example, the KK ansatz Eq. (55)] to get

$$\Gamma_{\rm gapl}=\frac{\pi N(0)\Delta_B^2}{12T\xi(T)}J_{AB}\sim\left[1-\frac{T}{T_c}\right]^{3/2}, \tag{74}$$

where

$$J_{AB}=\int d\zeta\frac{|\partial_\zeta\Delta_{ai}^{(0)}(\zeta)|^2}{\Delta_B^2} \tag{75}$$

is a temperature-independent dimensionless integral ~ 1 that depends on the structure of the interface and the textural boundary conditions between the A and B phases (again $\zeta=u/\xi$). Thus we see from Eq. (74) that the exponent of the temperature dependence for Γ is $\frac{3}{2}$ in the gapless regime, which differs from the hydrodynamic result, 2.

V. TERMINAL VELOCITY OF THE A-B INTERFACE: THEORY VERSUS EXPERIMENT

In Sec. IV we attempted to derive expressions for the friction coefficient (inverse of the mobility) for the A-B interface moving under hypercooled conditions for three dynamical regimes: the gapless, hydrodynamic, and extreme ballistic. The terminal velocity data published so far[10] lie in the extreme ballistic regime, for at all pressures at which the published data were taken the ratio $d/l\ll 1$ already at T_{AB} and is decreasing quickly with temperature, and the pair-breaking contribution to the dissipation is expected to be negligible. In the EBL our first guess might be that since the vast majority of the QP states participate in the Andreev processes, the experimentally measured interface mobility should be given to a good approximation by the Andreev contribution alone (see Sec. IV C). This turns out not to be the case, and below we attempt to account for this discrepancy. A rough estimate will show that the grazing incidence QP states not accounted for in $\Gamma_{\rm and}$ may be as important as the Andreev contribution, although this is not the only possibility.

Since the A phase energy gap is strongly anisotropic around the Fermi surface with nodes appearing at $\pm\hat{l}$, inspection of the expression for $\Gamma_{\rm and}$ [Eq. (8)] suggests that the Andreev friction coefficient ought to depend strongly on the angle that $\hat{l}$ makes with $\hat{\mathbf{n}}$, the normal to the interface (by convention, pointing towards the B phase). As shown in Refs. 8 and 9, for all but the slowest interface speeds the A phase $\hat{l}$ vector is effectively frozen in the configuration it had before the B phase nucleated. If we define $\mu=\hat{l}\cdot\hat{\mathbf{n}}$, then, due to the influence of the cell walls and the applied magnetic field on the $\hat{l}$ vector, $\mu=0$ should hold in the LANL experiments[10] (this value for μ turns out to correspond to the minimum-energy configuration for the static interface[23]).

As we discussed earlier, to get an approximate numerical estimate for $\Gamma_{\rm and}$ the formal logarithmic divergence in the integral over the momentum component $\hat{p}_x$ should be cut off at the largest value of $\hat{p}_x$ for which the approximations inherent in the calculation breakdown. The two semiclassical cutoffs, d/l and v_{AB}/v_F, come directly out of our analysis of the kinetic equation (see Sec. IV), and as discussed in Appendix C, the quantum cutoff $\sim(T/\varepsilon_F)^{1/2}$ comes from a breakdown in the conditions for Andreev reflection. Thus the appropriate choice of cutoff should be

$$\mu_c\sim\max\{v_{AB}/v_F,d/l,(T/\varepsilon_F)^{1/2}\}\ . \tag{76}$$

Since each of these quantities depends on pressure and temperature, the choice of cutoff will depend on the position in the P,T phase diagram. In Fig. 5 we plot the possible cutoffs as a function of T/T_{AB} at $P=34$ bars (we use the temperature dependence for the MFP l estimated

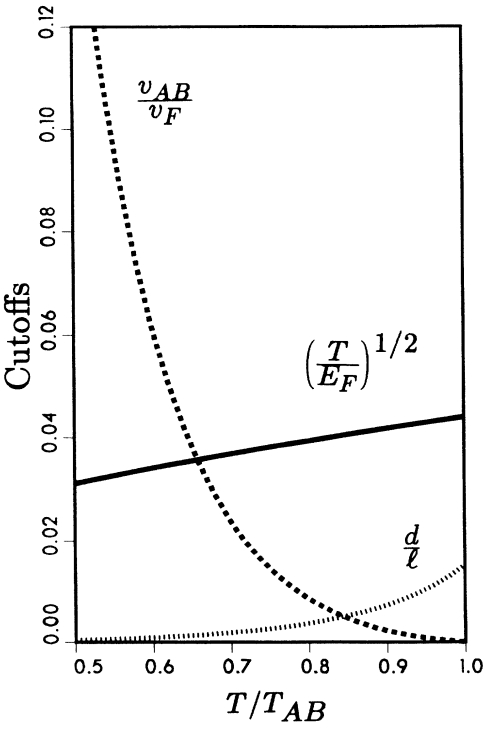

FIG. 5. The three possible QP momentum direction cutoffs for $\Gamma_{\rm and}$ as a function of T/T_{AB} at $P=34$ bars.

in Ref. 25). In the temperature range for which published interface velocity data exist ($T/T_{AB} \gtrsim 0.65$) the clear choice of cutoff is $\mu_c \sim (T/\varepsilon_F)^{1/2}$, while for lower T the appropriate cutoff should be v_{AB}/v_F. Although the expression for $\Gamma_{\rm and}$ takes on fairly simple forms as $T \to 0$,[11] the data lie in a region where the full integral expression must be evaluated numerically to get quantitatively accurate results. In Fig. 6, using $v_{AB} = \Delta G_{AB}/\Gamma_{\rm and}$, we plot our theoretical estimates for v_{AB} versus T/T_{AB} for two pressures, $P = 24.5$ and 34 bars; also plotted are the published LANL data. In evaluating $\Gamma_{\rm and}$ we used experimental data from Ref. 44 to get T_{AB}, $N(0)$, v_F, etc. and used the quantum cutoff in the integral over $\hat{p}_x$. To estimate the temperature dependence of the A and B phase energy gaps we used standard interpolation formulas for the gap maxima, which account for strong-coupling renormalizations of the gaps near T_c and reduce to the weak-coupling BCS values for $T = 0$. In evaluating v_{AB} at $P = 34$ bars we used an expression for ΔG_{AB} in ergs/cm^3 derived in Ref. 8 from experimental entropy data:[45]

$$\Delta G_{AB}(P,T) = T_c\{1.01[(T/T_c)^2 - (T_{AB}/T_c)^2] - 1.88(T/T_c - T_{AB}/T_c)\}\ , \quad (77)$$

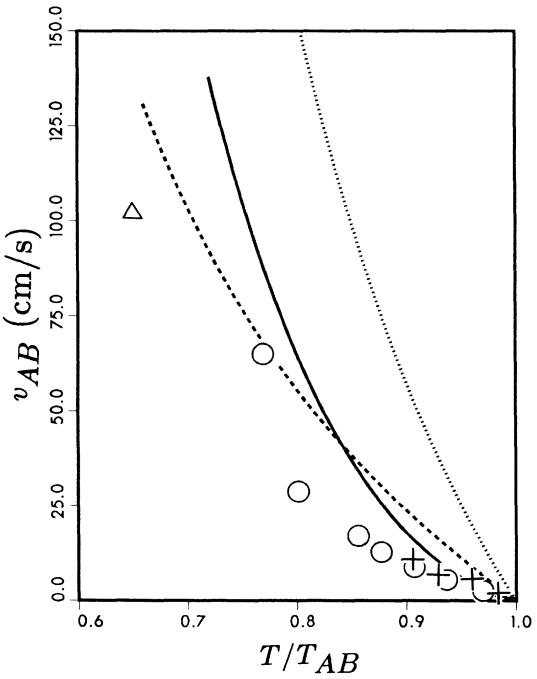

FIG. 6. Terminal velocity v_{AB} of the planar phase boundary as a function of $T/T_{AB}(P)$. Circles (crosses) are from Ref. 10 for $P = 33.6$ (24.5) bars and magnetic fields $H \approx 100$ (200) G. Triangular data point is from Ref. 35 for $P \approx 30$ bars. Dotted and dashed curves are the theoretical results for $\mu = \hat{l}\cdot\hat{\mathbf{n}} = 1$ and 0, respectively, at $P = 34$ bars and $H = 0$. Solid line is the theoretical result for $\mu = 0$ at $P = 24.5$ bars and $H = 0$.

where T_c is in mK. For $P = 24.5$ bars we used

$$\Delta G_{AB}(P,T) \approx 10\frac{T_{AB}}{T_c}\left[1 - \frac{T}{T_c}\right]^2\left[1 - \frac{T}{T_{AB}}\right], \quad (78)$$

in ergs/cm^3, which was obtained in Ref. 46 from dynamic magnetic susceptibility data. [ΔG_{AB} estimated from static susceptibility data (cf. Ref. 8) tends to overestimate ΔG_{AB} (as checked, for example, by latent heat measurements) by about a factor of 2. This discrepancy arises from the puzzling discrepancy between the static and dynamic B phase susceptibilities, a problem that to this day has not been cleared up.].

Comparing the dashed curve ($\hat{l}$ in the plane of the interface, appropriate for the LANL experiments, and $P = 34$ bars) in Fig. 6 with the data shows that, except for the lowest T data, the theoretical predictions overestimate v_{AB} by a factor ~ 2. There are not really enough data points at $P = 24.5$ kbars to allow us to draw any firm conclusions on how well theory and experiment compare at this pressure, but again theory seems to overestimate v_{AB}. Since there are no adjustable parameters in the theory, we cannot try to fit the velocity data. Since $\Gamma_{\rm and}$ diverges only logarithmically as the cutoff goes to zero (and therefore depends only weakly on the exact value of the cutoff), it is difficult to believe that the fault lies in our estimate of the quantum cutoff. [As shown in Appendix C, the true quantum cutoff for a QP state with momentum $\mathbf{p}$ is actually $\sim(\varepsilon_p/\varepsilon_F)^{1/2}$, which we have approximated as $(T/\varepsilon_F)^{1/2}$. This replacement is not likely to give rise to a factor of 2 in $\Gamma_{\rm and}$.] Furthermore, since we have previously shown[12,13] that the moving planar A-B interface is linearly stable, we do not believe that the above discrepancy can be explained away by invoking a nonplanar interface. At this point we can only offer some possible explanations.

(1) The LANL experiment actually measures the speed of the magnetization profile that accompanies the moving A-B interface. The v_{AB} that we have computed is the predicted speed of the kink in the superfluid order parameter. As pointed out by LY (Ref. 9) it is conceivable, but at first sight perhaps implausible, that the magnetic signal and the kink are traveling at different speeds. In fact the latest data coming from LANL (Ref. 36) reveal that the propagation of the phase boundary shows a much richer behavior than was previously imagined, with two distinct signals traveling at different speeds. Still it is difficult to believe that in the simpler high-temperature regime such an effect could give rise to the abovementioned factor of 2 discrepancy.

(2) The nontrivial transmission coefficients that LY claim should appear in the expression for $\Gamma_{\rm and}$ will tend to increase the friction coefficient (over the barrier quantum Andreev reflection leads to a greater momentum transfer to the QP fluid) and therefore decrease v_{AB}. Since the effects of the nontrivial transmission coefficients should disappear at lower T (the only important QP states at low T are the ones near the nodes in the A phase and they are all reflected), the semiclassical theoretical

prediction for v_{AB} should come into agreement with the data at lower T, which very roughly seems to be the case, at least for $P=34$ bars (see Fig. 6). There is no quantitative prediction for how much realistic nontrivial transmission coefficients would modify the value of $\Gamma_{\rm and}$, and in any case Kopnin's quasiclassical GF calculation implies that the nontrivial coefficients do not even appear.

(3) We now come to the most plausible explanation for the discrepancy. As we already mentioned, it could be that even in the EBL the relatively small number of grazing angle QP states gives rise to a contribution to the friction coefficient comparable with $\Gamma_{\rm and}$. This can occur if their small number is compensated by a large energy transfer per state. It will also be important, especially for low temperatures, that for $\hat{l}$ in, or close to, the plane of the interface (the probable LANL experimental configuration), the low-energy QP states (with momenta near the nodes in the A phase gap) will be among the grazing incidence states. The QP states with momenta within the quantum cutoff region cannot undergo Andreev reflection, and there are then three possibilities for such states: (1) they may behave hydrodynamically by undergoing inelastic collisions within the boundary region, (2) they may remain frozen and respond under a sudden approximation, or (3) they may be scattered in the ordinary (non Andreev) way, and which process actually occurs depends on the relative sizes of the various cutoffs and the state $(\varepsilon,\hat{\mathbf{p}})$ in question [see discussion above Eq. (38)]. To study normal QP scattering process in the presence of the A-B interface requires going back to the microscopic Bogoliubov–de Gennes equations or the microscopic GF's, since the terms that account for these processes have been systematically dropped in the quasiclassical and semiclassical theories. However, an estimate of the normal scattering contribution to the interface mobility can probably be made along the lines of the method LY used for the Andreev processes. Unfortunately, we have not yet been able to sort out the details of how to consistently and quantitatively evaluate the contribution to the friction coefficient from the QP states neglected in the derivation of $\Gamma_{\rm and}$. We can, however, use the hydrodynamic results discussed in Sec. IV D to estimate the contribution to the friction coefficient from *some* of the grazing angle QP states and see how these states can give rise to an important contribution. In the not too low temperature region of the phase diagram where the quantum cutoff is the correct one and $d/l \gg v_{AB}/v_F$ (see Fig. 5), we expect the QP states with $d/l < |\hat{p}_x| < (T/\varepsilon_F)^{1/2}$ to be scattered in the ordinary way (no branch conversion) since they are still behaving ballistically and Andreev reflection is forbidden, while for $|\hat{p}_x| < d/l$ we expect the QP states to behave hydrodynamically (i.e., relax due to collisons in the region of the interface). Using the hydrodynamic solutions obtained in Sec. IV D, we can attempt to estimate the contribution of these hydrodynamic states to the interface mobility in the EBL. Even though the simple RTA to the collision integral in the QP KE is probably not quantitatively reliable in the EBL, the QP collision time ends up dropping out of the final expression for the hydrodynamic contribution, $\Gamma_{\rm hyd}$, lending some credence to our final estimate. We find that the hydrodynamic solution in Eq. (65) is indeed accurate in the EBL for those states with $|\hat{p}_x| < d/l$ provided that $d/l \gg v_{AB}/v_F$. For these hydrodynamic states Eq. (68) for $\Gamma_{\rm hyd}$ should hold, but with the angular integration restricted to angles for which $|\hat{p}_x| < d/l$. It is then easy to see that the small factor d/l arising from the angular integration is compensated by the large factor l/d appearing in the expression for $\Gamma_{\rm hyd}$, resulting in $\Gamma_{\rm hyd} \sim v_F^{-1} N(0) \Delta_0^2$, which is independent of τ and *apparently of the same order of magnitude as* $\Gamma_{\rm and}$. Whether or not the contribution to the interface mobility from the grazing angle states can bring the theoretical prediction for v_{AB} into line with the experimental data requires a careful numerical estimate in the EBL of both $\Gamma_{\rm hyd}$ and the contribution due to the QP states that undergo normal scattering, and this had not yet been carried out. Since the mean-field theories for superfluid dynamics are expected to produce quantitatively accurate results, this program seems worth carrying out, especially considering that in one setting there is the opportunity to study such a rich variety of kinetic phenomena.

VI. SUMMARY AND CONCLUSIONS

We have studied the motion of the A-B interface using superfluid KE's and have attempted to calculate the mobility of the interface in three different dynamical regimes: the extreme ballistic, the extreme hydrodynamic, and the gapless regimes. For the first two regimes we were able to use semiclassical theories, while a microscopic quantum treatment was needed to get the dynamics in the gapless regime. We are in the process of attempting to use the superfluid KE's to study the low-temperature ($T/T_c \lesssim 0.5$) regime where pairbreaking processes dominate the dynamics. [LY (Ref. 9) have done some calculations in the pair-breaking regime, and they argue that the friction coefficient should be roughly temperature independent there.] The difficulty here is that since the pair-breaking regime is (like the gapless regime) outside the scope of a semiclassical treatment, it is necessary to use the quantum (4×4) matrix KE's or the quasiclassical GF equations of motion (see Sec. II). We also believe that both the quasiclassical GF and matrix kinetic equation formalisms can be used to shed further light on whether or not nontrivial transmission and reflection coefficients appear in $\Gamma_{\rm and}$ and to obtain a quantitative estimate of the effective mass of the phase boundary, which plays an important role in the theory of the oscillations of the pinned interface.

We have found that the semiclassical KE result for the Andreev friction coefficient $\Gamma_{\rm and}$ agrees with the quasiclassical GF calculation of Kopnin, but not with the semiphenomenological calculation of LY as reported in Ref. 7. The probable reason for this discrepancy was discussed in Sec. III, where we explained that it now appears likely that the original LY calculation missed a term by using an incorrect ballistic approximation for the QP transmission coefficient when the interface is in motion.[27]

In the limit $T \to T_c$, we can summarize the results for

the friction coefficient Γ by writing the temperature dependence as

$$\Gamma \sim \left[1-\frac{T}{T_c}\right]^{\eta} \tag{79}$$

and displaying the various results for the exponent η:

$$\eta=\begin{cases}\frac{3}{2}, & \text{Kopnin and KE's, Ballistic regime, } \alpha > 10^{-2}\ , \\ 2, & \text{hydrodynamic regime, } 10^{-3} < \alpha < 10^{-2} \\ \frac{3}{2}, & \text{gapless regime, } \alpha < 10^{-3}\ ,\end{cases} \tag{80}$$

where $\alpha=(1-T/T_c)$ is the reduced temperature and we have glossed over the crossover regimes where the temperature dependence will deviate from a simple power law. Very close to T_{AB} the diffusion limited dynamics should come into play, but we have not discussed this regime at all here. We have also compared the theoretical predictions for the terminal velocity of the interface (determined from the Andreev friction coefficient in the ballistic regime) with experiment and find that the theory appears to overestimate the LANL data[10] by about a factor of 2, a discrepancy that we have tried to attribute at least partly to the neglect of the grazing angle QP's.

The moving *A-B* interface shows a rich array of dynamical behavior and therefore is an excellent setting in which to study superfluid dynamics. So far only the ballistic regime has been studied experimentally, and we hope that the investigations reported here will motivate some experimental work at high temperatures near T_c to check the limiting form of the Andreev friction coefficient and to probe the hydrodynamic and maybe even the gapless regimes (experimental work studying the low-T regime is already in progress at LANL).

Note added in proof. We have assumed throughout this work that it is possible to expand the relevant quantities in powers of v/v_F. We have no proof, however, that the relevant quantities are analytic in this parameter; thus we must keep in mind the (perhaps unlikely) possibility that by expanding in v/v_F we have missed important nonanalytic terms, and this may be another (or additional) reason for the discrepancy between theory and experiment. I thank Nils Schopohl for discussions on this point.

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APPENDIX A: HYDRODYNAMIC AND GAPLESS WINDOW FOR THE MOTION OF THE *A-B* INTERFACE

In this appendix we show that there exists a portion of the phase diagram in the vicinity of the PCP where the motion of the interface should be governed by the hydrodynamic and gapless dynamics developed in Secs. IV D and IV E. In order for the local hydrodynamic theory to apply, the following inequalities must be satisfied:

$$\xi_0 \ll l(T) \ll d(T)\ , \tag{A1}$$

$$\Delta_0^{-1} \ll \tau(T)\ , \tag{A2}$$

$$v_{AB} \ll v_F\ . \tag{A3}$$

For the gapless theory to apply we need $\Delta_0\tau \ll 1$ instead of Eq. (A2) and this only occurs very close to T_c. As emphasized by LY the first of these inequalities is *not* satisfied over most of the phase diagram, and therefore the extreme ballistic and pair-breaking regimes discussed in Secs. II and IV apply for almost all P and T. Here we are interested in the small region near the PCP where the above inequalities do apply.

Using $d(T) \sim c\xi(T)$ $(c \sim 5-10)$, $l \gtrsim 10^4$ Å above the PCP pressure, and $\xi_0 \sim 100$ Å we find that $l(T) < d(T)$ for $t < t_{\rm hyd} \sim 10^{-2}$, where in this appendix we write the reduced temperature as $t = 1 - T/T_c(P)$ and introduce a reduced temperature $t_{\rm hyd}$ that marks the crossover from the ballistic to hydrodynamic regime. Conveniently, the other two inequalities, Eqs. (A1) and (A2) are extremely well satisfied for $t < t_{\rm hyd}$.

We now must make sure that the transition is still hypercooled in the hydrodynamic window, for otherwise latent heat effects govern the dynamics and the system is in the well-studied diffusion-limited regime. The system will be hypercooled if the undercooling parameter $\Delta_u = H(T)/L(T) \gg 1$, where $L(T)$ is the latent heat and $H(T)$ is the heat necessary to warm the B phase up from T to T_{AB}. If the specific heat C_p is roughly constant between T and T_{AB}, then $H(T) \approx C_p(T - T_{AB})$ and

$$\Delta_u(P,T,H) \approx C_p(T - T_{AB})/L\ .$$

The LANL experiment can be run at very small ambient magnetic fields $\sim 5-10$ G. In such small fields the splitting of the A transition into the A_1 and A_2 transitions will be very tiny $(1 - T_{c2}/T_c \sim 10^{-5})$, and $T_{AB}(P,H)$ will be shifted only slightly down from its $H=0$ value. For purposes of demonstration, we will work at exactly the PCP pressure ~ 21 bars, but any pressure in the vicinity will do.

Near the PCP and in zero field, $\Delta G_{AB}(P,T) \approx 10t^2 x_0$ ergs/cm^3, where $x_0(P) \equiv 1 - T/T_{AB}(P,H=0)$. In a finite field ΔG_{AB} becomes (in ergs/cm^3) (Refs. 38 and 46)

$$\Delta G_{AB}(P,T,H) \approx 10t^2 x_0 - \chi_N \alpha t H^2\ ,$$

where $\chi_N \approx 9 \times 10^{-8}$ is the normal-state susceptibility and $\alpha \approx 2.3$. At the PCP pressure in zero field $T_{AB}(P) = T_c(P)$ and $x_0 = t$. The reduced temperature of the A-B transition in a field is then $t_{AB}(H) \approx \alpha \chi_N H^2/10$ or $t_{AB} \approx 7 \times 10^{-4} < t_{\rm hyd}$ in a 5 G field, so there is no problem there. Near the PCP superheating of the B phase has also been observed, so that velocity measurements might be possible both above and below $T_{AB}(H)$.

From the definition of the latent heat,[47] we find that L as a function of the reduced temperature t has the form,

$$L(P,t,H) = -\frac{T}{T_c}\left[\frac{\partial \Delta G_{AB}}{\partial t}\right] + \Delta G_{AB}(P,t,H)\ .$$

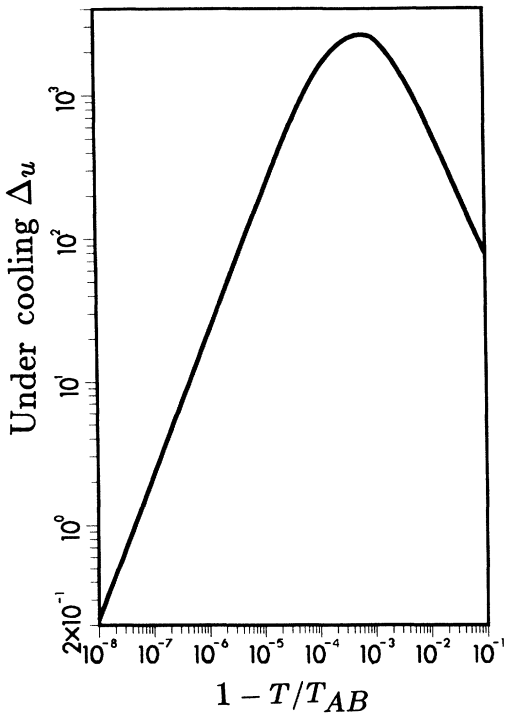

FIG. 7. Undercooling parameter Δ_u as a function of the reduced temperature $x=1-T/T_{AB}$ at the PCP pressure in a 5 G magnetic field.

The specific heat of the superfluid phases near T_c is $C\sim 90$ (ergs/cm^3 mK) and it is roughly constant over the small temperature range of interest here. Then using the above expressions and estimates for L, ΔG_{AB}, and C_p, we can plot $\Delta_u(x)$, where $x\equiv 1-T/T_{AB}(P,H)$. We find (see Fig. 7) that unless x is very tiny, $<10^{-8}$ [and T is therefore extremely close to $T_{AB}(P,H)$], $\Delta_u(x)\gg 1$. Since $t_{\rm hyd}\sim 10^{-2}$ and the crossover to the gapless regime takes place over $10^{-3}\lesssim t\lesssim 10^{-4}$, we see that in the vicinity of the PCP there is for small fields a hydrodynamic and gapless window in the phase diagram where the A-B transition is still strongly hypercooled. Within this window the dynamics of the interface should be governed by the local hydrodynamic and gapless theories discussed in Secs. IV D and IV E.

APPENDIX B: ORBITAL DYNAMICS OF THE A PHASE $\hat{l}$ VECTOR

In this appendix we calculate the orbital viscosity of the A phase $\hat{l}$ vector in the gapless regime using the TDGL equation [Eq. (71)] and use this result to suggest that the crossover from the low-T hydrodynamic regime to the high-T gapless regime may have already been seen in orbital viscosity experiments. This problem is simpler than the interface problem because the system remains at all times in the unitary A phase, although the symmetry-breaking $\hat{l}$ vector is moving.

Near T_c in the hydrodynamic and gapless regimes the $\hat{l}$ vector obeys a dissipative equation of motion of the form

$$\hat{l}\times\frac{\delta\mathcal{F}_0}{\delta\hat{l}}=\mu(\hat{l}\times\partial_t\hat{l})\ , \tag{B1}$$

where the LHS is the torque on $\hat{l}$ due to bending energies, walls, dipole forces, etc., and the RHS is a viscous torque that gives rise to energy dissipation with μ the orbital viscosity.

The energy per unit area dissipated during the motion of a planar $\hat{l}$ soliton (cf. moving interface problem) would be $\mathcal{E}=\mu\int dx\,|\partial_t\hat{l}(x,t)|^2$ (see, e.g., Ref. 11). The hydrodynamic regime[14] ($\Delta_0\tau\gg 1$, and $\omega\tau, ql\ll 1$, etc.) is sometimes also known as the normal locking regime, because the QP energy depends on the instantaneous position of the $\hat{l}$ vector, and the motion of $\hat{l}$ is hindered by collisional damping of the QP's as they try to follow the motion of $\hat{l}$. In this regime the well-known result for μ in the limit $T\to T_c$ is[48]

$$\mu_{\rm hyd}\approx\frac{\pi^2\Delta_0^3N(0)\tau}{64T}\sim\left[1-\frac{T}{T_c}\right]^{3/2}\ . \tag{B2}$$

To obtain μ in the gapless regime ($\Delta_0\tau\ll 1$) we make some simplifying assumptions. We assume that the system is homogeneous and take the equilibrium A phase order parameter to be

$$\hat{\Delta}_{\alpha\beta}(\hat{\mathbf{p}})=(i\hat{\sigma}_a\hat{\sigma}_2)_{\alpha\beta}\hat{d}_a\tilde{\Delta}\cdot\hat{\mathbf{p}}\propto\delta_{\alpha\beta}$$

with the (fixed) spin $\hat{\mathbf{d}}$ vector chosen along the y axis and the equilibrium orbital configuration $\tilde{\Delta}_0=\Delta_0(\hat{\mathbf{w}}_1^{(0)}+i\hat{\mathbf{w}}_2^{(0)})$ (thus in equilibrium $\hat{l}=\hat{l}_0=\hat{\mathbf{w}}_1^{(0)}\times\hat{\mathbf{w}}_2^{(0)}=\hat{\mathbf{y}}$). If for simplicity we consider only a restricted set of rotations

$$\delta\hat{\Omega}=\hat{\mathbf{x}}(\delta\hat{\Omega}_x)+\hat{\mathbf{z}}(\delta\hat{\Omega}_z)\ ,$$

of $\hat{l}$ so that

$$\delta\hat{l}=\delta\hat{\Omega}\times\hat{l}_0=\hat{\mathbf{x}}(\delta\hat{l}_x)+\hat{\mathbf{z}}(\delta\hat{l}_z)\ ,$$

then

$$\delta\tilde{\Delta}=\Delta_0(\delta\hat{\mathbf{w}}_1+i\delta\hat{\mathbf{w}}_2)=\Delta_0(-\delta\hat{l}_z-i\delta\hat{l}_x)\hat{\mathbf{y}}'\ . \tag{B3}$$

Next we use the simple TDGL equation with $\gamma=[\pi N(0)/24T]$, and insert the separable A phase order parameter $\Delta_{ai}=\hat{d}_a\tilde{\Delta}_i$. Assuming that the spin ($\hat{\mathbf{d}}$) vector is fixed, the TDGL equation becomes

$$\gamma\partial_t\tilde{\Delta}=\frac{\delta\mathcal{F}_0}{\delta\tilde{\Delta}^*}$$

We now use Eq. (B3) to get

$$-\gamma\Delta_0(\partial_t\hat{l}_z+i\partial_t\hat{l}_x)\hat{\mathbf{y}}=\frac{\delta\mathcal{F}_0}{\delta\tilde{\Delta}^*}\ , \tag{B4}$$

and then by applying the vector operator $(\hat{l}_0\times\tilde{\Delta}_0^*)\hat{l}_0\cdot$ to Eq. (B4), taking the real part, and making use of

$$\hat{l}\times\frac{\delta\mathcal{F}_0}{\delta\hat{l}}=(\hat{l}\times\tilde{\Delta}^*)\left[\frac{\delta\mathcal{F}_0}{\delta\tilde{\Delta}^*}\cdot\hat{l}\right]+(\hat{l}\times\tilde{\Delta})\left[\frac{\delta\mathcal{F}_0}{\delta\tilde{\Delta}}\cdot\hat{l}\right]$$

and

$$\mathrm{Re}[\partial_t\hat{l}_z+i\partial_t\hat{l}_x)(\hat{l}_0\times\tilde{\Delta}_0^*)]=\Delta_0(\hat{l}_0\times\partial_t\hat{l})\ ,$$

we find an equation of the form of Eq. (B1), but now with μ given by

$$\mu_{\rm gapl}=\frac{\pi N(0)\Delta_0^2}{12T}\sim\left[1-\frac{T}{T_c}\right],$$

and therefore the temperature dependence of the orbital viscosity changes in the gapless regime from its hydrodynamic form [cf. Eq. (B2)]. Since the exact expression for μ (valid even in the crossover region $\Delta_0\tau\sim 1$) has already been worked out in detail using other nonequilibrium techniques,[49,50] we will settle for calculating the asymptotic result obtained above. In Fig. 8 we plot the exact expression μ (obtained from Ref. 49) as a function of reduced temperature, and compare it with the limiting forms, $\mu_{\rm hyd}$ and $\mu_{\rm gapl}$; clearly the hydrodynamic result fails very close to T_c as the dynamics becomes gapless at a crossover temperature given roughly by $(1-T/T_c)\sim 10^{-3}$. The crossover from the low-T hydrodynamic exponent $\frac{3}{2}$ to the high-T gapless exponent 1 is clearly visible in the log-log plot.

A deviation from the hydrodynamic $\frac{3}{2}$ power-law temperature dependence for μ seems to have been observed in Ref. 51 in what we would expect to be the crossover temperature region; the trend of the deviation is just what one would expect for the crossover to the gapless behavior, although more data closer to T_c are really needed to tell for sure if the gapless theory works quantitatively. This possible crossover to gapless dynamical relaxation may be the first time such behavior has been observed in pure Fermi superfluids (the gapless regime in pure superconductors is way beyond experimental resolution), although gapless relaxation has been observed for heavily doped superconductors with a high concentration of magnetic impurities, which are efficient Cooper pair breakers, and can therefore lead to $\Delta_0\tau_{\rm imp}<1$.[52]

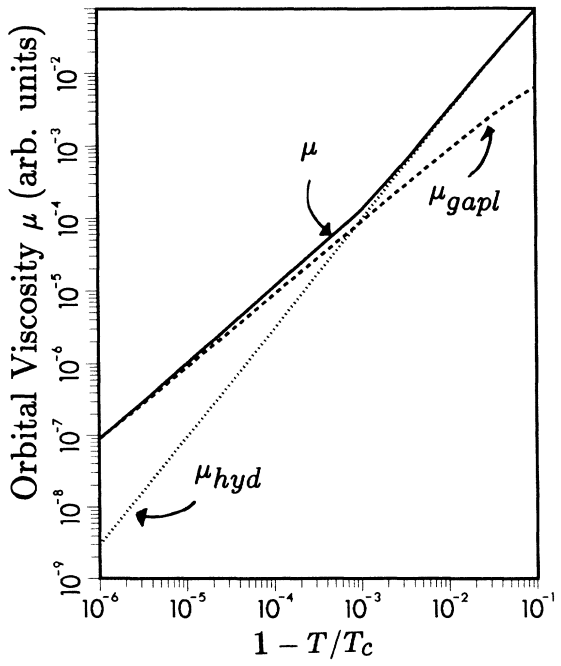

FIG. 8. A plot of the A phase orbital viscosity μ as a function of the reduced temperature $1-T/T_c$ showing the crossover from the Cross-Anderson (hydrodynamic) form at low T to the high-T gapless form. Also shown are the limiting low-T (hydrodynamic) and high-T (gapless) forms for μ.

The gapless form of the orbital viscosity also has implications for the mutual friction between vortices and the normal component in superfluid ^{3}He. The current theoretical predictions[21] for the mutual friction coefficient for continuous vortices in the A phase, which were obtained using the hydrodynamic (Cross-Anderson) form for μ, do not agree with experiments performed very close to T_c, and it is conceivable that the experiments[53] are seeing the crossover to the gapless behavior. More work is necessary to sort this out.

APPENDIX C: QUASIPARTICLE MOMENTUM CUTOFF

In this appendix we address the question of the proper cutoff for the logarithmically divergent integral over QP momentum direction in the Andreev friction coefficient, $\Gamma_{\rm and}$. The Andreev reflection of a QP incident on the A-B boundary (here with normal $\hat{\mathbf{n}}=\hat{\mathbf{z}}$) involves a branch conversion *particle* $(\varepsilon_p>0)\leftrightarrow$*hole* $(\varepsilon_p<0)$ with p_z changed but $\hat{\mathbf{p}}_\perp$ conserved. For simplicity we consider the reflection of a QP off of a static A-B boundary, since this should be sufficient for estimating the order of magnitude of the cutoff. In this case the total QP energy $E=[\varepsilon_p^2+\Delta_A(\hat{\mathbf{p}})^2]^{1/2}$ is conserved in the scattering process $p_z\rightarrow p_z'$. Here $\varepsilon_p\approx p_z^2/2m^*+p_\perp^2/2m^*-\varepsilon_F$ with $\varepsilon_F=p_F^2/2m^*$. The Andreev reflection occurs when a QP scatters from outside to inside the Fermi surface (or vice versa) with $\varepsilon_p=\varepsilon_{p'}$, $\hat{\mathbf{p}}$ and the sign of p_z conserved, and only a very small change in the magnitude of p_z ($\Delta p_z\ll p_F$); this is in contrast to normal scattering processes where the QP undergoes specular reflection with

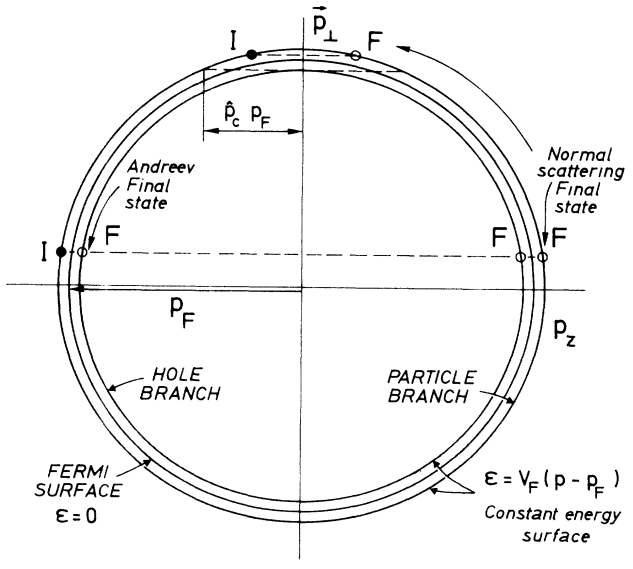

FIG. 9. Schematic diagram of the Andreev and normal scattering processes that take place in the presence of the inhomogeneous order-parameter field associated with the A-B interface. I = initial state and F = final state. $\hat{p}_c$ is the quantum cutoff where the Andreev scattering mechanism breaks down (not to scale).

$p_z' = -p_z$ and retains its original branch character. Figure 9 clearly shows that Andreev processes can only take place if $p_z' \leq 0$ when, for example, $p_z < 0$. A simple bit of algebra shows that for $|\hat{p}_z| \geq \hat{p}_c$, where

$$\hat{p}_c^{\,2} \equiv 2\sqrt{2}\frac{[E^2 - \Delta_A(\hat{\mathbf{p}})^2]^{1/2}}{\varepsilon_F}\ , \tag{C1}$$

there are three possible final states for an incident QP, but only the Andreev process is important for $p_z \sim p_F$. In this case the other processes require a momentum change of $\sim 2p_F$, and since the kink in the gap has a width $d \sim \xi \gg p_F^{-1}$, the scattering potential does not have the appropriate Fourier components to induce such large transfers of momentum [in other words, the matrix elements for such processes are suppressed by factors $\exp(-\xi p_F)$]. On the other hand, for $\hat{p}_z < \hat{p}_c$ only normal scattering processes take place (see Fig. 9); now the momentum change $2p_z \ll 2p_F$ is small enough, so that normal processes are not exponentially suppressed.

The upshot of the above discussion is that the divergent integral in the expression for Γ_{and} should be cut off at $\hat{p}_z \geq \hat{p}_c \sim \sqrt{\varepsilon_p/\varepsilon_F}$, where at the intermediate temperatures of experimental interest in the EBL ($T/T_{AB} > 0.7$ near melting pressure) a typical QP kinetic energy is $\varepsilon \sim T \sim \Delta_0(T)$. There are, however, other possibilities for the cutoff, which come from other approximations made in the derivation of Γ_{and}. Roughly speaking, the true cutoff should be taken to be $\mu_c \sim \max\{\sqrt{\varepsilon_p/\varepsilon_F}, v_{AB}/v_F, d/l\}$, where the last two candidates for the cutoff enter for obvious physical reasons that are discussed in further detail in Sec. IV C. Over most of the temperature range of current experimental interest the appropriate cutoff is the Andreev cutoff defined in Eq. (C1) (see Fig. 5). We can now estimate the angular cutoff $\mu_c \sim (T/\varepsilon_F)^{1/2}$ and proceed with the evaluation of Γ_{and}.

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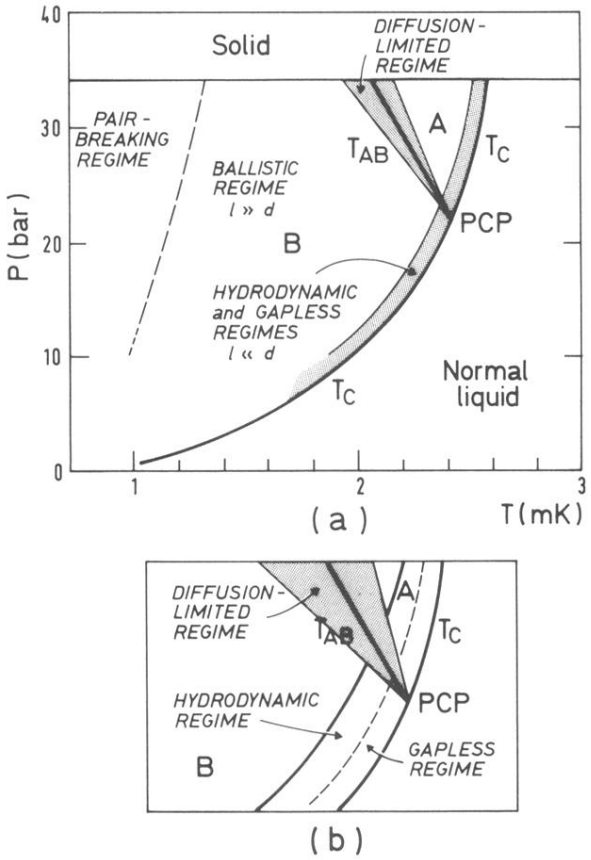

FIG. 1. A schematic diagram of the various dynamical regimes for the motion of the *A-B* interface in the *P-T* phase diagram (not to scale).