Probability-density-function description of mesoscopic normal tunnel junctions

Masahito Ueda
NTT Basic Research Laboratories, 3-9-11 Midori-cho, Musashino-shi, Tokyo 180, Japan
(Received 8 February 1990)

Conventional treatments of the dynamics of mesoscopic normal tunnel junctions mainly reduce the problem to a stochastic master equation and finally resort to computer simulation. This paper presents a new methodology based on probability-density functions to solve the problem in a fully analytic manner. Analytic expressions of the charge distribution across the junction, current-voltage characteristics, and the degree of randomness of the single-electron-tunneling oscillations are obtained under an arbitrary bias condition, where the degree of randomness is defined as the ratio of the standard deviation of dwell times to their mean value. In particular, the minimum degree of randomness achievable under the constant-current-bias condition is found to be $(R_T C I_{dc}/e)^{1/2}$, where e is the electronic charge, R_T is the tunnel resistance, C is the junction capacitance, and I_{dc} is the bias current.

I. INTRODUCTION

In a mesoscopic system, which is large on the atomic scale, but small compared to the electron coherence length, both wave and corpuscular properties of a single electron are simultaneously important. Such a system opens up the possibility of regulating random phenomena in electron transport and tunneling by properly manipulating the wave-particle duality of a single electron. In microjunction physics, the effect of regularization manifests itself as suppression and enhancement of the dc tunneling rate when the electrostatic energy of a single electron is comparable to, or larger than, the masking thermal energy.[1-5]

Another characteristic feature of the mesoscopic system is its nonlinear response to the externally connected macroscopic system. The dynamics of a mesoscopic normal tunnel junction depends strongly on the manner in which it is connected to the external macroscopic system. When the source resistance is much larger than the tunnel resistance, an electron that has tunneled cannot be removed immediately; hence, a voltage drop e/C is generated across the junction, where e is the electronic charge and C is the junction capacitance. This voltage drop suppresses the probability of subsequent tunneling events until the accumulated charge exceeds $e/2$, and thereafter the voltage drop enhances the probability. Therefore, tunneling events are expected to occur regularly. In contrast, when the source resistance is much smaller than the tunnel resistance, an electron that has tunneled will be removed immediately; hence, the voltage across the junction is almost always pinned at a fixed value. Therefore tunneling events are expected to occur at random.

Single-electron-tunneling (SET) oscillations in ultrasmall normal-metal junctions are an excellent combination of these two effects: That is, *discrete* transfer of a single electron through the tunnel barrier and *continuous* injection of electric charge from outside the tunnel junction. There are a number of papers which discuss the dynamics of SET oscillations.[6-9] Most of them reduce the problem to a stochastic master equation and finally resort to computer simulation. This method correctly incorporates the effects of dissipation (shunt resistance) and fluctuations (thermal noise, etc.) on tunneling characteristics and gives results in excellent agreement with experiments.[10-13] However, some important problems concerning mesoscopic normal tunnel junctions still remain unsolved, namely, analytic expressions of the charge distribution across the junction, current-voltage (*I-V*) characteristics when there is a finite shunt resistance, and the degree of randomness of SET oscillations. In particular, what determines the minimum degree of randomness of SET events that can be achieved under the *ideal* constant-current-bias condition?

In a previous paper[14] we proposed a new analytic method for investigating the dynamics of mesoscopic normal tunnel junctions, and showed *in the time domain* the crossover from random to Coulomb-regulated single-electron tunneling as the external source changes from a voltage source into a current source. The present paper further develops this method to answer the above-mentioned problems. Extensive use is made of two kinds of probability-density functions which we call charge- and time-interval distributions. Using them, we derive analytic expressions of the charge distribution across the junction, *I-V* characteristics, and the mean and variance of dwell times for tunneling under an arbitrary bias condition. (By dwell time for tunneling, we mean the time interval that elapses between consecutive tunneling events.) In order to quantitatively evaluate how regularly SET events occur, a new quantity is introduced which we call the *degree of randomness of SET events.*[15] It is defined as the ratio of the standard deviation of dwell times to their mean value. In particular, the minimum degree of randomness of SET events achievable under the constant-current-bias condition is found to be $(R_T C I_{dc}/e)^{1/2}$, where R_T is the tunnel resistance and I_{dc}

the bias current. This limit may be referred to as the *standard quantum limit in tunneling current* because this limitation is imposed not by either the thermal noise or the external current noise but by the uncertainty inherent in quantum-mechanical tunneling.

This paper is organized as follows. Section II describes the semiclassical model adopted in the present paper, and discusses several assumptions that are implicit in this model. Two different sources of thermal noise are identified, and ways of suppressing them are discussed. Section III defines the charge- and time-interval distributions of consecutive tunneling events and discusses their fundamental properties and significance in the description of mesoscopic normal tunnel junctions. Section IV evaluates the mean and variance of dwell times and discusses, in the general case, the dependence of the degree of randomness of SET events on junction and source parameters. In particular, the minimum degree of randomness achievable under the constant-current-bias condition is found. Section V obtains analytic expressions of the charge distribution across the junction and *I-V* characteristics of tunneling current under an arbitrary bias condition. Section VI discusses the relationship between the present probability-density-function approach and the conventional master-equation approach, and summarizes the main results of this paper.

II. MODEL AND ASSUMPTIONS

This section describes the semiclassical model and assumptions used in the present paper. Most of the previous works[6–9] adopted a semiclassical model in which a constant current is injected into a normal-metal tunnel junction with capacitance C. A shunt resistance is connected in parallel with the junction. However, for our purposes, an equivalent, but slightly different, configuration is more appropriate. We adopt a semiclassical model in which a normal-metal tunnel junction with capacitance C is driven by a voltage source (voltage V). A source resistance R_S is connected in series to the junction. Here we observe that the source can be modeled exactly by a voltage plus a resistor.

Let us identify the assumptions made in the present paper. Both the tunnel resistance R_T and source resistance R_S are assumed to be much larger than the quantum unit of resistance $R_Q=(h/4e^2)\simeq 6.45k\Omega$ in order that the quantum-mechanical energy uncertainties,[16] which arise complementarily from the dwell time (tunneling lifetime) and circuit time constant, can be neglected compared to the electrostatic energy of a single electron. In particular, the condition $R_T \gg R_Q$ implies that an electron is almost always localized on one side or the other side of the barrier. Closely related to this condition is the assumption that the traversal time for tunneling[17] is negligible compared to the dwell time for tunneling. The thermal equilibration time inside the electrodes is also assumed to be negligible compared to the dwell time and the circuit time constant. This is necessary in order to assume the equilibrium Fermi distribution. We do not include any other elements such as stray capacitance or inductance. Although the consideration of these effects is essentially important for experimental observation of SET oscillations, we omit them, nevertheless, to present the theory in its simplest possible form.

Now we start with a discussion of the single-electron tunneling rate in an ultrasmall normal-metal tunnel junction. Suppose that the Fermi energy of the right electrode before tunneling is E_R and that of the left electrode after tunneling is E_L. Then the tunneling rate from the right to the left is, in general, given by[18]

$$R(E_L,E_R)=\int_{-\infty}^{\infty}\tau^{-1}(E)D_L(E)D_R(E)f_F(E-E_R)\times[1-f_F(E-E_L)]dE\ , \qquad (2.1)$$

where $\tau^{-1}(E)$ is the elastic tunneling rate in a state of energy E, $D_L(E)$ [or $D_R(E)$] is the density of states in the left (or right) electrode, and $f_F(E)$ is the equilibrium Fermi distribution which is defined by

$$f_F(E)=\frac{1}{1+\exp\left[\dfrac{E}{k_BT}\right]}\ . \qquad (2.2)$$

The factor $f_F(E-E_R)[1-f_F(E-E_L)]$ contributes significantly to the integral only within the energy range $|E_L-E_R|$, which is typically of the order of a few millielectronvolts, while the Fermi energies E_L and E_R themselves are of the order of a few electronvolts. Therefore, the elastic tunneling rate and the densities of states may be well approximated by their values at the Fermi energy. Thus we have

$$R(E_L,E_R)=\tau^{-1}(E_0)D_L(E_L)D_R(E_R)\times\frac{E_R-E_L}{1-\exp\left[-\dfrac{E_R-E_L}{k_BT}\right]}\ , \qquad (2.3)$$

where E_0 is some typical energy around E_R. If we assume that the Fermi energies are equal for both electrodes when no voltage is applied across the junction, then E_R-E_L equals the difference in the electrostatic energy of the junction before and after tunneling:

$$E_R-E_L=\frac{Q^2}{2C}-\frac{(Q-e)^2}{2C}=\frac{e}{C}\left[Q-\frac{e}{2}\right]\ , \qquad (2.4)$$

where $e>0$. We observe that the discrete nature of the electron tunneling appears as a dc offset in the electrostatic energy difference by an amount of $e^2/2C$. This term is precisely equal to the electrostatic energy of a single electron, and it suppresses or enhances the dc tunneling rate according to whether $Q<e/2$ or $Q>e/2$, respectively, provided that it is comparable to or larger than the masking-thermal energy. Substituting Eq. (2.4) into the right-hand side (rhs) of Eq. (2.3), we have a general formula for a forward transition rate:[6–8]

$$r(Q)=\frac{1}{eR_TC}\frac{Q-\dfrac{e}{2}}{1-\exp\left[-\dfrac{e}{Ck_BT}\left[Q-\dfrac{e}{2}\right]\right]}\ , \qquad (2.5)$$

where Q denotes the accumulated charge on the electrode before tunneling and

$$R_T = \frac{1}{e^2 \tau^{-1}(E_F) D_L(E_F) D_R(E_F)} \tag{2.6}$$

is called the tunnel resistance because it gives the resistance of the tunnel junction when a constant voltage is applied across the junction. When $k_B T \ll (e^2/2C)$, Eq. (2.5) reduces to

$$r(Q) = \begin{cases} 0 \quad \text{if } Q \le \frac{e}{2} , \\ \frac{Q-(e/2)}{e R_T C} \quad \text{otherwise} . \end{cases} \tag{2.7}$$

That is, the forward tunneling events are inhibited until the accumulated charge exceeds $(e/2)$. Similarly, the backward tunneling events are inhibited for $Q > -(e/2)$. This is the principle of Coulomb blockade.[6] The crucial observation here is that the tunneling rate depends only on the charge just before the tunneling event occurs and that it does not depend on any information concerning the earlier tunneling events. This reflects the fact that tunneling events obey a Markovian random-point process due to the wave property of an electron. The effect of quantization of charge (corpuscular property of an electron) is to suppress the dc tunneling rate until the charge reaches $(e/2)$ and to enhance it thereafter, as seen from Eq. (2.5). Considerable correlation in electron tunneling events can be achieved when these suppression and enhancement effects are properly manipulated by the external macroscopic system.

For the present model, the accumulated charge $Q(t)$ on the electrode obeys the following differential equation:

$$\frac{d}{dt} Q(t) = \frac{CV - Q(t)}{CR_s} + i_s(t) - e \sum_j \delta(t - t_j) , \tag{2.8}$$

where the second term on the rhs represents the current noise generated in the source resistance and the third term represents the change in charge due to tunneling where t_j denotes the times of tunneling. In the absence of current noise and tunneling events, Eq. (2.8) can be immediately solved to give

$$Q(t) = Q(t_i) \exp\left[-\frac{t - t_i}{CR_S} \right] + CV \left[1 - \exp\left[-\frac{t - t_i}{CR_S} \right] \right] . \tag{2.9}$$

It is opportune at this stage to discuss the effects of thermal noise on electron tunneling. The degree of regularity of electron tunneling is deteriorated by thermal noise generated in two different sources. The source resistance generates thermal noise, which randomly disturbs the prescribed current injection. The charge must be delivered to the junction continuously to make the fluctuations in the accumulated charge much smaller than the electronic charge e. This can be achieved by high-impedance suppression of the current noise. The other source of thermal noise is the tunnel junction and this noise blurs the effect of the Coulomb blockade via phonon-assisted tunneling, as seen from Eq. (2.5). This effect can be suppressed by either fabricating the junction capacitance very small or by lowering the temperature. Here we observe that the effect of the latter noise is already incorporated in the formula for the tunneling rate [Eq. (2.5)].

III. TIME AND CHARGE-INTERVAL DISTRIBUTIONS BETWEEN CONSECUTIVE TUNNELING EVENTS

In the previous section, we indicated that the tunneling process is Markovian; it does not depend on any information concerning earlier tunneling events. In general, such a process is completely specified by the two-time correlation function. Suppose that single-electron tunneling events occur at times t_j $(j = 1, 2, \ldots)$ [see Fig. 1(a)]. Then the tunneling characteristics can be best described with the probability distribution of time intervals between consecutive tunneling events: $\tau_j \equiv t_{j+1} - t_j$ $(j = 1, 2, \ldots)$. We denote this probability distribution as $P_{s11}(\tau)$.[19] That is, this quantity gives the probability density that the first subsequent tunneling event occurs τ seconds after the earlier one [Fig. 1(b)]. For example, for a completely random-tunneling process which obeys the Poisson random-point process, time intervals are exponentially distributed:

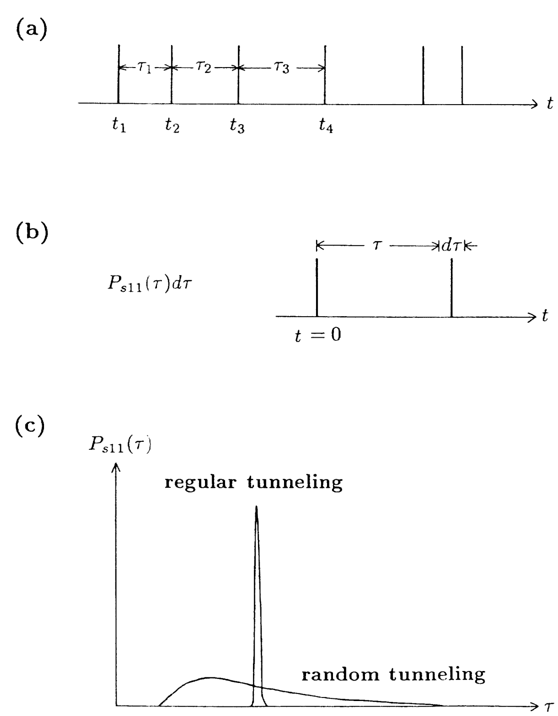

FIG. 1. (a) Time intervals between consecutive tunneling events. (b) Definition of time-interval distribution. (c) Time-interval distributions for regular and random-tunneling events.

$$P_{s11}(\tau)=\frac{1}{\bar{\tau}}\exp\left[-\frac{\tau}{\bar{\tau}}\right], \tag{3.1}$$

where $\bar{\tau}$ is the average time interval between consecutive tunneling events. We have already encountered this type of tunneling phenomenon in the macroscopic, constant-voltage-biased junction. Equation (3.1) is a microscopic version of the so-called "shot noise" whose current spectrum is characterized by $S_I(\omega)=(1/2\pi)e\bar{I}$, where $\bar{I}$ is the average current. In fact, this white spectrum can be derived from Eq. (3.1), provided that the process is Markovian, where $\bar{I}$ equals $e/\bar{\tau}$. For a completely regular tunneling process in which tunneling events occur at equal time intervals $\bar{\tau}$, we obtain

$$P_{s11}(\tau)=\delta(\tau-\bar{\tau}) . \tag{3.2}$$

In general, the more regularly tunneling events occur, the more sharply the time-interval distribution tends to peak around the average time interval, as is schematically illustrated in Fig. 1(c). With this probability distribution we have demonstrated the crossover from random to Coulomb-regulated single-electron tunneling as the external source changes from a voltage source into a current source.[14]

Another important distribution is the charge-interval distribution. Let $P_{s11}(Q_i,Q_f)$ be the probability density per (unit charge)^2 that the first subsequent tunneling event occurs at charge Q_f when the initial charge was Q_i, which is the charge immediately after a tunneling event has occurred. Since the tunneling probability does not depend on the past events, $P_{s11}(Q_i,Q_f)$ is equal to the product of (i) the initial-charge distribution $P^{\text{initial}}(Q_i)$, which gives the probability distribution of charges immediately after tunneling events occurred, and (ii) the probability density $P(Q_i,Q_f)$ that the first tunneling event occurs at charge Q_f, given that the initial charge was Q_i. Since the probability density $P(Q_i,Q_f)$ is equal to the probability density of a tunneling event occurring at charge Q_f multiplied by the probability of no tunneling events occurring until then, it is given by[14,20]

$$P(Q_i,Q_f)=\frac{r(Q_f)}{i(Q_f)}\exp\left[-\int_{\max(Q_i,e/2)}^{Q_f}\frac{r(Q)}{i(Q)}dQ\right], \tag{3.3}$$

where the lower bound of integration $\max(Q_i,e/2)$ appears since under condition $k_BT \ll e^2/2C$ the Coulomb blockade completely inhibits tunneling until the accumulated charge on the electrode exceeds $e/2$. Here, $i(Q)$ is the external current when the accumulated charge is Q. Equation (2.8) without current noise and tunneling currents gives

$$i(Q)=\frac{CV-Q}{CR_S} . \tag{3.4}$$

Substituting Eqs. (2.7) and (3.4) into the rhs of Eq. (3.3) yields[14]

$$P(Q_i,Q_f)=\frac{1}{e}\frac{R_S}{R_T}\frac{Q_f-e/2}{CV-Q_f}\times\left[\frac{CV-Q_f}{CV-A}\right]^{(1/e)(R_S/R_T)[CV-(e/2)]}\times\exp\left[\frac{1}{e}\frac{R_S}{R_T}(Q_f-A)\right], \tag{3.5}$$

where $A\equiv\max(Q_i,e/2)$. Equation (3.5) gives the desired probability distribution as a function of the ratio of source to tunnel resistances R_S/R_T, junction capacitance C, and source voltage V.

Before proceeding further, let us examine some fundamental properties of this function. The normalization condition is given by

$$\int_A^{CV}P(Q_i,Q_f)dQ_f=1 . \tag{3.6}$$

This condition is readily verified by substituting Eq. (3.5) into the left-hand side (lhs) of Eq. (3.6). The lower bound of integration, A, simply means that a tunneling event is prohibited until the accumulated charge Q exceeds $e/2$. A constant-current-bias limit can be achieved if we take the limits $V\to\infty$ and $R_S\to\infty$ with the ratio $I_{\text{dc}}\equiv V/R_S$ held constant. Then Eq. (3.5) reduces to

$$P(Q_i,Q_f)=\frac{Q_f-e/2}{eR_TCI_{\text{dc}}}\exp\left[-\frac{(Q_f-A)(Q_f+A-e)}{2eR_TCI_{\text{dc}}}\right]. \tag{3.7}$$

This gives a general expression for $P(Q_i,Q_f)$ under a constant-current-bias condition. For a special case of $Q_i=e/2$, Eq. (3.7) reduces to the result obtained in Ref. 7. A constant-voltage-bias limit can be achieved if we take the limit $\epsilon\equiv R_S/R_T\to 0$ with the bias voltage V held constant. Using ϵ, Eq. (3.5) can be rewritten as

$$P(Q_i,Q_f)=\frac{1}{e}\frac{Q_f-e/2}{CV-Q_f}\exp\left[\ln\epsilon+\frac{\epsilon}{e}\left[CV-\frac{e}{2}\right]\times\ln\frac{CV-Q_f}{CV-A}+\frac{\epsilon}{e}(Q_f-A)\right]. \tag{3.8}$$

For $\epsilon \ll 1$, it is sufficient to keep the first term in the exponent. Thus we have

$$P(Q_i,Q_f)=\begin{cases}\dfrac{\epsilon}{e}\dfrac{Q_f-e/2}{CV-Q_f}\xrightarrow{\epsilon\to 0}0 & (Q_f<CV) ,\\ \infty & (Q_f=CV) .\end{cases} \tag{3.9}$$

Since $P(Q_i,Q_f)$ must satisfy the normalization condition Eq. (3.6), we obtain

$$P(Q_i,Q_f)=2\delta(CV-Q_f) . \tag{3.10}$$

This result shows that the voltage across the junction is pinned at the applied voltage above $e/2C$, where the Coulomb blockade fails to work.

Now, to obtain the charge-interval distribution we have only to obtain the initial-charge distribution. Within the approximation in which the traversal time for tunneling is negligible compared to the dwell time and the circuit-time constant, the initial-charge distribution coincides with the final-charge distribution displaced by $-e$:

$$P^{\text{initial}}(Q)=P^{\text{final}}(Q+e)\ , \tag{3.11}$$

where the final-charge distribution $P^{\text{final}}(Q)$ gives the probability density that the tunneling event occurs at charge Q. On the other hand the final-charge distribution is related to the initial-charge distribution via $P(Q_i,Q_f)$:

$$P^{\text{final}}(Q_f)=\int_{-e/2}^{Q_f} P^{\text{initial}}(Q_i)P(Q_i,Q_f)dQ_i\ . \tag{3.12}$$

Dividing the range of interaction at $D\equiv\min(e/2,CV-e)$, we have

$$P^{\text{final}}(Q_f)=P\left[\frac{e}{2},Q_f\right]\int_{-e/2}^{D} P^{\text{initial}}(Q_i)dQ_i + \int_{D}^{Q_f} P^{\text{initial}}(Q_i)P(Q_i,Q_f)dQ_i\ , \tag{3.13}$$

where $P(Q_i,Q_f)$ in the first integrand is factored out because from Eq. (3.5) we have $P(Q_i,Q_f)=P(e/2,Q_f)$ for $Q_i<e/2$. Equation (3.13), together with Eq. (3.11), determines the desired initial-charge distribution. In the following discussion, however, we will concentrate on the case in which the source voltage V is smaller than $3e/2C$ because under this condition we are able to solve these equations exactly and hence we can obtain exact expressions of charge- and time-interval distributions. For $CV<\frac{3}{2}e$, the first integral in the rhs of Eq. (3.13) gives unity because of the normalization condition, while the second integral vanishes because $P^{\text{initial}}(Q_i)=0$ for $Q_i>CV-e$. Thus we obtain

$$P^{\text{final}}(Q_f)=P\left[\frac{e}{2},Q_f\right] \quad \text{for } CV<\tfrac{3}{2}\ , \tag{3.14}$$

and from Eq. (3.11), we obtain

$$P^{\text{initial}}(Q_i)=P\left[\frac{e}{2},Q_i+e\right] \quad \text{for } CV<\tfrac{3}{2}e\ . \tag{3.15}$$

The initial-charge distribution satisfies the following normalization condition:

$$\int_{-e/2}^{CV-e} P^{\text{initial}}(Q_i)dQ_i=1\ . \tag{3.16}$$

Figure 2 illustrates the initial- and final-charge distributions for several values of the ratio R_S/R_T with $CV=e$. All the final-charge distributions (e)–(h) start from $e/2$ because tunneling is inhibited until the accumulated charge exceeds $e/2$. In curve (e) the distribution is sharply localized just above $e/2$. The physical reason for this can be explained as follows. For a large value of the ratio R_S/R_T, the average dwell time for tunneling is much shorter than the circuit time constant CR_S; consequently, a tunneling event occurs immediately after the Coulomb blockade is lifted which happens when the charge exceeds $e/2$. However, once an electron has tunneled, the next tunneling event will be inhibited for a long-time interval ($\sim CR_S$) until the accumulated charge again exceeds $e/2$. Thus the tunneling events are expected to occur very regularly. As the ratio R_S/R_T decreases, the average dwell time for tunneling becomes relatively larger and, therefore, the distribution becomes broader and the regularity becomes worse. However, from this figure only we cannot say how regularly tunneling events occur for each case. For this purpose we will later obtain the time-interval distribution. Since the charge-interval distribution is the product of $P^{\text{initial}}(Q_i)$ and $P(Q_iQ_f)$, we finally obtain

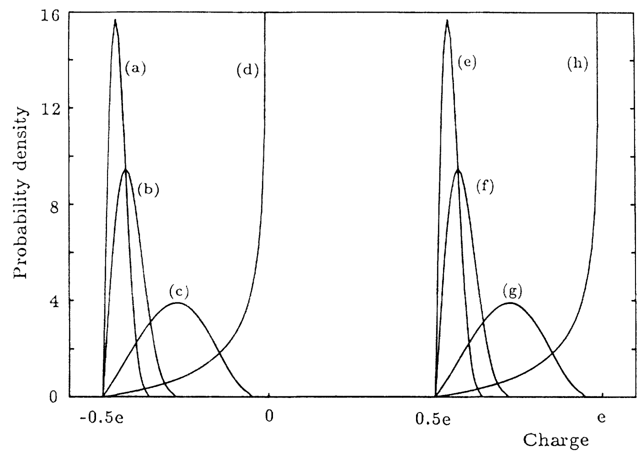

FIG. 2. Initial-charge distributions (a)–(d) and final-charge distributions (e)–(h) for (a) and (e) $R_S/R_T=300$, (b) and (f) $R_S/R_T=100$, (c) and (g) $R_S/R_T=10$, and (d) and (h) $R_S/R_T=1$ with $CV=e$.

$$P_{s11}(Q_i,Q_f)=P\left[\frac{e}{2},Q_i+e\right]P\left[\frac{e}{2},Q_f\right] \quad \text{for } CV<\tfrac{3}{2}e\ , \tag{3.17}$$

where $P(Q_i,Q_f)=P(e/2,Q_f)$ since $Q_i<e/2$ for $CV<\frac{3}{2}e$.

Now, let us obtain the time-interval distribution $P_{s11}(\tau)$ using this result. The time interval τ between two consecutive tunneling events is determined if we specify the corresponding initial charge Q_i and final charge Q_f. From Eq. (2.9) we have

$$\tau=-CR_S\ln\frac{CV-Q_f}{CV-Q_i}\ . \tag{3.18}$$

Each combination of Q_i and Q_f that gives the same time interval τ through Eq. (3.18) contributes to $P_{s11}(\tau)$ with weight function $P_{s11}(Q_i,Q_f)$. Thus we have

$$P_{s11}(\tau)=\int_{-e/2}^{CV-e}dQ_i\int_{e/2}^{CV}dQ_f P_{s11}(Q_i,Q_f) \times\delta\left[\tau+CR_S\ln\frac{CV-Q_f}{CV-Q_i}\right]\ , \tag{3.19}$$

where the δ function ensures that the time interval between the initial state with charge Q_i and the final state with charge Q_f is equal to τ. By noting that

$$\delta\left[\tau+CR_S\ln\frac{CV-Q_f}{CV-Q_i}\right]=\frac{CV-Q_f}{CR_S}\delta(Q_f-CV(1-e^{-\tau/CR_S})-Q_ie^{-\tau/CR_S}) , \tag{3.20}$$

Eq. (3.19) can be integrated with respect to Q_f to give

$$P_{s11}(\tau)=\begin{cases}0 \text{ if } B>CV-e \\ \int_B^{CV-e}dQ_iP_{s11}[Q_i,CV(1-e^{-\tau/CR_S})+Q_ie^{-\tau/CR_S}]\frac{e^{-\tau/CR_S}(CV-Q_i)}{CR_S} \text{ otherwise} ,\end{cases} \tag{3.21}$$

where $B\equiv\max[-e/2,CV-e^{\tau/CR_S}(CV-e/2)]$. The lower bound of integration B is obtained from the following consideration. If we specify τ and Q_i, then Q_f no longer takes arbitrary values but is uniquely determined from Eq. (3.18). On the other hand, Q_f cannot take values lower than $e/2$ because of the Coulomb blockade. To meet this condition, therefore, the integration range for Q_i must have a lower bound which yields B. In other words, the tunnel junction cannot be charged up to $e/2$ within a fixed time τ, when the initial charge is below B. Equation (3.21) gives a general expression of the time-interval distribution in terms of the charge-interval distribution. For $CV<\frac{3}{2}e$, we can obtain the analytic expression of $P_{s11}(\tau)$ with the help of Eqs. (3.5) and (3.17). Figure 3 shows time-interval distributions for various values of the ratio R_S/R_T with fixed bias voltage $V=e/C$, where the time axis is normalized by CR_S. Curve (a) corresponds to the ratio $R_S/R_T=300$ and is sharply localized around $1.2CR_S$. This curve clearly demonstrates SET oscillations in the time domain. We observe that this curve has a finite width. It will be discussed in Sec. IV that this width is due to the quantum-mechanical uncertainty with respect to the time when an electron starts to tunnel, and it *never* reaches zero even in the limit of the ideal constant-current-bias limit. As the ratio R_S/R_T decreases, the distribution becomes less and less localized and the regularity of SET events becomes worse. Thus we have demonstrated the crossover from random to Coulomb-regulated single-electron-tunneling oscillations in the *time domain* as the external source changes from a voltage source into a current source. A similar crossover can be seen as the bias voltage decreases towards an optimum value above $e/2C$.[14]

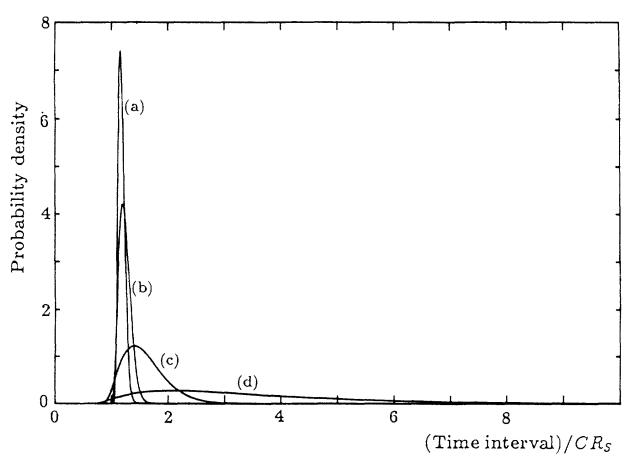

FIG. 3. Time-interval distributions for (a) $R_S/R_T=300$, (b) $R_S/R_T=100$, (c) $R_S/R_T=10$, and (d) $R_S/R_T=1$ with $CV=e$, where the time axis is normalized by CR_S.

Finally, let us examine the normalization conditions for $P_{s11}(Q_i,Q_f)$ and $P_{11}(\tau)$. The normalization condition for $P_{s11}(Q_i,Q_f)$ is given by

$$\int_{-e/2}^{CV-e}dQ_i\int_A^{CV}dQ_fP_{s11}(Q_i,Q_f)=1 . \tag{3.22}$$

This equation is readily verified if we substitute $P_{s11}(Q_i,Q_f)=P^{\text{initial}}(Q_i)P(Q_i,Q_f)$ into the lhs of Eq. (3.21):

$$\begin{aligned}\text{lhs}&=\int_{-e/2}^{CV-e}dQ_iP^{\text{initial}}(Q_i)\int_A^{CV}dQ_fP(Q_iQ_f) ,\\ &=\int_{-e/2}^{CV-e}dQ_iP^{\text{initial}}(Q_i)=1 ,\end{aligned} \tag{3.23}$$

where Eqs. (3.6) and (3.16) are used. The normalization condition for $P_{s11}(\tau)$ is given by

$$\int_0^\infty P_{s11}(\tau)d\tau=1 . \tag{3.24}$$

This equation can be verified if we substitute Eq. (3.19) into the lhs of Eq. (3.24) and use Eq. (3.22).

IV. DEGREE OF RANDOMNESS OF SET EVENTS AND STANDARD QUANTUM LIMIT OF TUNNELING CURRENT

A. Average dwell time

In general, the dynamics of mesoscopic normal tunnel junctions is characterized by the power spectrum of voltage across the junction. However, since in our case at most one electron tunnels at one time and the tunneling process is Markovian, tunneling characteristics can be more directly described by the probability distribution of time intervals between consecutive tunneling events, i.e., the probability distribution of dwell times. Suppose that a tunneling event occurred at time t_i and that the first subsequent tunneling event occurred at time t_f. Then the dwell time can be defined as

$$\tau(Q_i,Q_f)\equiv t_f-t_i , \tag{4.1}$$

where Q_i and Q_f represent the charge just after the first tunneling event and the charge just before the second tunneling event, respectively. From Eq. (2.9) we have

$$\tau(Q_i,Q_f)=CR_S\ln\frac{CV-Q_i}{CV-Q_f} . \tag{4.2}$$

The average dwell time $\bar{\tau}$ is given by

$$\bar{\tau}=\int_{-e/2}^{CV-e}dQ_i\int_{e/2}^{CV}dQ_f\tau(Q_i,Q_f)P_{s11}(Q_i,Q_f)\ . \quad (4.3)$$

For $CV<\frac{3}{2}e$, substituting Eq. (3.17) into the rhs of Eq. (4.3) yields

$$\bar{\tau}=\int_{e/2}^{CV}dQ\ CR_S\ln\frac{CV+e-Q}{CV-Q}P(e/2,Q)\ . \quad (4.4)$$

This is an exact expression of the average dwell time as a function of the junction capacitance C, source voltage V, tunnel resistance R_T, and source resistance R_S, where an explicit representation of the distribution $P(e/2,Q)$ is given from Eq. (3.5). To proceed further with the calculation, we expand the logarithmic term in the integrand to the second order in quantities $(Q-e)/CV$ and Q/CV (whose absolute magnitudes are less than unity), obtaining

$$\bar{\tau}=CR_S\frac{e}{(CV)^2}\left[\left(CV-\frac{e}{2}\right)+\int_{e/2}^{CV}QP\left(\frac{e}{2},Q\right)dQ\right]\ . \quad (4.5)$$

Substituting Eq. (B4) in Appendix B into this equation finally yields

$$\bar{\tau}=\frac{eR_S}{V}\left[1+\frac{e}{CV}\frac{R_T}{R_S}\frac{\exp(d)}{d^d}\gamma(d+1,d)\right]\ , \quad (4.6)$$

where $\gamma(a,x)$ is the incomplete γ function defined by

$$\gamma(a,x)\equiv\int_0^x t^{a-1}\exp(-t)dt,\quad \mathrm{Re}a>0\ , \quad (4.7)$$

and $d\equiv 1/e(R_S/R_T)[CV-(e/2)]$. In the limit of the constant-current-bias condition where $R_S/R_T\gg 1$ and $V/R_S=I_{dc}$, Eq. (4.6) reduces, using Eq. (A9) in Appendix A, to

$$\bar{\tau}=\frac{e}{I_{dc}}\left\{1+\left[\frac{\pi}{2}\frac{R_T}{R_S}\frac{e}{CV}\left(1-\frac{e}{2CV}\right)\right]^{1/2}\right\}\ . \quad (4.8)$$

The first term in the rhs of this equation gives the celebrated relation

$$\bar{\tau}=\frac{1}{f}\ , \quad (4.9)$$

where $f=I_{dc}/e$ is characteristic frequency of SET oscillations. The second term represents the first-order correction to it due to a finite dissipation, represented by R_T, in the junction.

B. Degree of randomness of SET events

It is natural to define the degree of randomness of SET events as the standard deviation of dwell times divided by their mean value:

$$\sigma\equiv\frac{[\overline{(\Delta\tau)^2}]^{1/2}}{\bar{\tau}}\ , \quad (4.10)$$

where $\overline{(\Delta\tau)^2}\equiv\overline{\tau^2}-\bar{\tau}^2$. A Poisson random-point process whose time-interval distribution is given by Eq. (3.1) yields $\sigma=1$. A completely regular-point process whose time-interval distribution is given by Eq. (3.2) yields $\sigma=0$. In general, the smaller the value of σ is, the more regularly the tunneling events occur. The mean square of dwell times is given by

$$\overline{\tau^2}=\int_{-e/2}^{CV-e}dQ_i\int_{e/2}^{CV}dQ_f\tau^2(Q_i,Q_f)P_{s11}(Q_i,Q_f)\ . \quad (4.11)$$

We form $\overline{\tau^2}-\bar{\tau}^2$ and expand the logarithmic terms in quantities $(Q-e)/CV$ and Q/CV. Then the lowest-order remaining terms give

$$\begin{aligned}\overline{(\Delta\tau)^2}&=\left(\frac{R_S}{V}\right)^2\left[1+\frac{1}{4}\left(\frac{e}{CV}\right)^2\right]\int_{e/2}^{CV}dQ\int_{e/2}^{CV}dQ'(Q-Q')^2P\left(\frac{e}{2},Q\right)P\left(\frac{e}{2},Q\right)\ ,\\ &=2\left(\frac{R_S}{V}\right)^2\left[1+\frac{1}{4}\left(\frac{e}{CV}\right)^2\right]\overline{(\Delta Q_f)^2}\ ,\end{aligned} \quad (4.12)$$

where

$$\overline{(\Delta Q_f)^2}=\int_{e/2}^{CV}Q^2P\left(\frac{e}{2},Q\right)dQ-\left[\int_{e/2}^{CV}QP\left(\frac{e}{2},Q\right)dQ\right]^2\ . \quad (4.13)$$

Here the subscript f refers to the final charge at which the tunneling event occurs. Substituting (B9) in Appendix B into Eq. (4.12), we obtain

$$\overline{(\Delta\tau)^2}=2\left(\frac{eR_T}{V}\right)^2\left[1+\frac{1}{4}\left(\frac{e}{CV}\right)^2\right]\left[2d-\left(\frac{\exp(d)}{d^d}\gamma(d+1,d)\right)^2-2\frac{\exp(d)}{d^d}\gamma(d+1,d)\right]\ . \quad (4.14)$$

Thus the degree of randomness of SET events is therefore given, from Eqs. (4.6) and (4.14), by

$$\sigma=\frac{R_T}{R_S}\frac{\left\{2\left[1+\frac{1}{4}\left(\frac{e}{CV}\right)^2\right]\left[2d-\left(\frac{\exp(d)}{d^d}\gamma(d+1,d)\right)^2-2\frac{\exp(d)}{d^d}\gamma(d+1,d)\right]\right\}^{1/2}}{1+\frac{e}{CV}\frac{R_T}{R_S}\frac{\exp(d)}{d^d}\gamma(d+1,d)}\ . \quad (4.15)$$

This equation gives a general expression of the degree of randomness of SET events under an arbitrary bias condition. In the limit of the constant-current-bias condition, Eq. (4.15) reduces to

$$\sigma=\left\{(4-\pi)\left[1+\frac{1}{4}\left[\frac{e}{CV}\right]^2\right]\frac{R_T CI_{\rm dc}}{e}\right\}^{1/2}$$
$$\simeq\left[\frac{R_T CI_{\rm dc}}{e}\right]^{1/2} . \quad (4.16)$$

Thus we find the minimum degree of randomness that can be achieved under the constant-current-bias condition; it is proportional to the square root of the tunnel resistance, junction capacitance, and bias current.

C. Standard quantum limit of mesoscopic normal tunneling current

We note that the limit given by Eq. (4.16) is not caused by either current or thermal fluctuations. The limit is imposed by the quantum-mechanical uncertainty with respect to the time when an electron starts to tunnel. Thanks to a finite (rather than infinite) tunnel resistivity, single-electron tunneling is made possible. This finiteness, however, permits delocalization of the electron wave function over both electrodes, and thus blurs the time of tunneling through the time-energy uncertainty relation[16] to the extent of

$$\overline{(\Delta\tau)^2}\simeq\frac{eR_T C}{I_{\rm dc}} . \quad (4.17)$$

Thus we find that the limit given by Eq. (4.16) has a purely quantum-mechanical origin and therefore by analogy with quantum optics it may be referred to as the *standard quantum limit of mesoscopic normal tunneling current.*

Finally, let us consider how much the tunneling events are regularized by the Coulomb blockade compared to the Poisson random-point process which gives $\sigma=1$. Substituting $I_{\rm dc}=V/R_S$ into Eq. (4.16), we find

$$\sigma\simeq\left[\frac{CV}{e}\frac{R_T}{R_S}\right]^{1/2} \quad (4.18)$$

which is a factor of $\sqrt{R_T/R_s}$ smaller than unity under the constant-current-bias condition ($R_S\gg R_T$).

V. CURRENT-VOLTAGE CHARACTERISTICS OF MESOSCOPIC NORMAL TUNNEL JUNCTIONS

In the previous section we obtained the average dwell time whose general expression is given in Eq. (4.3). In a similar manner, the average time $\overline{\tau}(Q)dQ$, during which the charge on the electrode is between Q and $Q+dQ$, is given by

$$\overline{\tau}(Q)dQ=\int_{-e/2}^{CV-e}dQ_i\int_{e/2}^{CV}dQ_f\tau(Q,Q+dQ)\times P_{s11}(Q_i,Q_f)\Theta(Q-Q_i)\times\Theta(Q_f-Q) , \quad (5.1)$$

where $\tau(Q,Q+dQ)$ is given from Eq. (4.2) and $\Theta(Q)$ is the Heaviside unit-step function defined by

$$\Theta(Q)=\begin{cases}1 & (Q>0) ,\\ 0 & (Q<0) .\end{cases} \quad (5.2)$$

Since $\tau(Q,Q+dQ)$ does not include any of the variables of integration, it can be factored out to give

$$\overline{\tau}(Q)=\frac{CR_S}{CV-Q}\int_{-e/2}^{CV-e}dQ_i\int_{e/2}^{CV}dQ_fP_{s11}(Q_i,Q_f)\times\Theta(Q-Q_i)\Theta(Q_f-Q) . \quad (5.3)$$

As long as the traversal time for tunneling may be neglected, the probability distribution of charge across the junction, $P(Q)$, is given by the ratio of $\overline{\tau}(Q)$ to the average dwell time $\overline{\tau}$:

$$P(Q)=\frac{\overline{\tau}(Q)}{\overline{\tau}} . \quad (5.4)$$

Substituting Eqs. (4.3) and (5.3) into Eq. (5.4) yields

$$P(Q)=\frac{\frac{CR_S}{CV-Q}\int_{-e/2}^{CV-e}dQ_i\int_{e/2}^{CV}dQ_fP_{s11}(Q_i,Q_f)\Theta(Q-Q_i)\Theta(Q_f-Q)}{\int_{-e/2}^{CV-e}dQ_i\int_{e/2}^{CV}dQ_f\tau(Q_i,Q_f)P_{s11}(Q_i,Q_f)} . \quad (5.5)$$

It can be readily verified that $P(Q)$ in Eq. (5.5) satisfies the normalization condition

$$\int_{-e/2}^{CV}P(Q)dQ=1 . \quad (5.6)$$

For $CV<\frac{3}{2}e$, the expression in Eq. (5.5) can be greatly simplified. Substituting Eq. (3.17) into the numerator of Eq. (5.5) and integrating it by parts yields

$$\int_{-e/2}^{CV-e} dQ_i \int_{e/2}^{CV} dQ_f P\left[\frac{e}{2},Q_i+e\right] P\left[\frac{e}{2},Q_f\right] \Theta(Q-Q_i)\Theta(Q_f-Q)$$
$$= \int_{-e/2}^{CV-e} dQ_i \Theta(Q-Q_i)\left[-\frac{d}{dQ_i}\exp\left[-\int_{e/2}^{Q_i+e}\frac{r(q)}{i(q)}dq\right]\right]$$
$$\times \int_{e/2}^{CV} dQ_f \Theta(Q_f-Q)\left[-\frac{d}{dQ_f}\exp\left[-\int_{e/2}^{Q_f}\frac{r(q)}{i(q)}dq\right]\right]$$
$$= \left[\Theta\left[Q+\frac{e}{2}\right]-\exp\left[-\int_{e/2}^{Q+e}\frac{r(q)}{i(q)}dq\right]\Theta(CV-e-Q)\right]$$
$$\times\left[\Theta\left[\frac{e}{2}-Q\right]+\exp\left[-\int_{e/2}^{Q}\frac{r(q)}{i(q)}dq\right]\Theta\left[Q-\frac{e}{2}\right]\right]. \quad (5.7)$$

Substituting Eq. (5.7) into Eq. (5.5), we obtain

$$P(Q)=\begin{cases}\dfrac{1}{\bar{\tau}}\dfrac{CR_S}{CV-Q}\left[1-\exp\left[-\displaystyle\int_{e/2}^{Q+e}\frac{r(q)}{i(q)}dq\right]\right] & \text{for } -e/2<Q<CV-e\ ,\\ \dfrac{1}{\bar{\tau}}\dfrac{CR_S}{CV-Q} \quad \text{for } CV-e<Q<e/2\ , \\ \dfrac{1}{\bar{\tau}}\dfrac{CR_s}{CV-Q}\exp\left[-\displaystyle\int_{e/2}^{Q}\frac{r(q)}{i(q)}dq\right] & \text{for } e/2<Q<CV\ .\end{cases} \quad (5.8)$$

Here $\bar{\tau}$ is given by Eq. (4.4). Equation (5.8) gives an exact expression of the charge distribution across the junction under an arbitrary bias condition. The voltage distribution across the junction $P(V)$ is uniquely related to the charge distribution $P(Q)$ by $P(V)=P(Q)(dQ/dV)=CP(Q)$. Figure 4 illustrates the charge distributions for several values of the ratio R_S/R_T with fixed bias voltage $V=e/C$. Curve (a) corresponds to the ratio $R_S/R_T=300$. We have already seen that for this case both initial- and final-charge distributions are sharply localized [see Figs. 2(a) and 2(e)] and that SET events occur very regularly [see Fig. 3(a)]. Correspondingly, the charge distribution rapidly increases at values above $Q=-e/2$ and rapidly decreases above $Q=e/2$. As the ratio decreases, both rises and falls become slower, and finally the distribution diverges at the charge corresponding to the applied voltage [Fig. 4(d)]. At this point the bias condition changes into a constant-voltage-bias operation. For a constant-current-bias condition where $\bar{\tau}=e/I_{dc}$ and

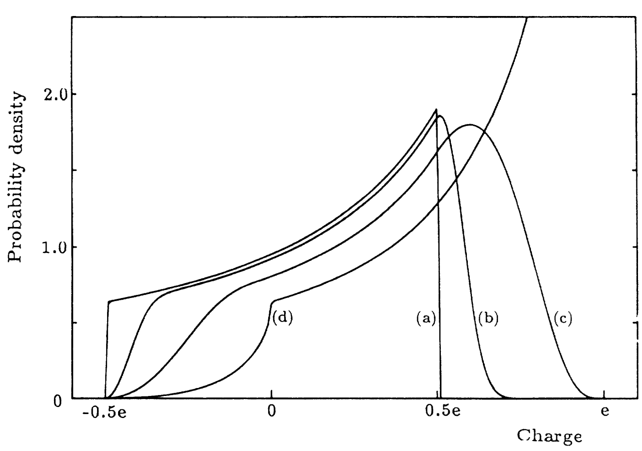

FIG. 4. Charge distribution for (a) $R_S/R_T=300$, (b) $R_S/R_T=100$, (c) $R_S/R_T=10$, and (d) $R_S/R_T=1$ with $CV=e$.

$$\exp\left[-\int_{e/2}^{Q}\frac{r(q)}{i(q)}dq\right]=\exp\left[-\frac{(Q-e/2)^2}{2eR_TCI_{dc}}\right], \quad (5.9)$$

Eq. (5.8) reduces to

$$P(Q)=\begin{cases}\dfrac{1}{e}\left[1-\exp\left[-\dfrac{(Q+e/2)^2}{2eR_TCI_{dc}}\right]\right] \\ \quad\text{for } -e/2<Q<CV-e\ ,\\ \dfrac{1}{e} \quad \text{for } CV-e<Q<e/2\ ,\\ \dfrac{1}{e}\exp\left[-\dfrac{(Q-e/2)^2}{2eR_TCI_{dc}}\right] \quad \text{for } e/2<Q<CV\ .\end{cases} \quad (5.10)$$

For the special case of $CV=\frac{3}{2}e$, Eq. (5.10) reduces to the result obtained in Ref. 9.

The expected value of the charge on the electrode is given by

$$\bar{Q}=\int_{-e/2}^{CV} QP(Q)dQ \ . \tag{5.11}$$

Substituting Eq. (5.8) into the rhs of Eq. (5.11), we obtain

$$\bar{Q}=CV-CR_S\frac{e}{\bar{\tau}} \ , \tag{5.12}$$

and hence

$$\bar{V}=V-R_S\frac{e}{\bar{\tau}} \ . \tag{5.13}$$

This equation has a simple physical interpretation. The quantity

$$\bar{I}\equiv\frac{e}{\bar{\tau}} \tag{5.14}$$

is the electronic charge multiplied by the average number of tunneling events per unit time, that is the average current through the barrier. On average, the same amount of current should flow in the external circuit, causing a voltage drop of $R_S\bar{I}$ in the source resistance. Therefore Eq. (5.13) simply means that the average voltage across the junction is equal to the source voltage V minus the voltage drop in the source resistance $R_S\bar{I}$ (Kirchhoff's second law).

Finally, let us discuss the current-voltage characteristic of mesoscopic normal tunnel junctions under an arbitrary bias condition. The average voltage across the junction is obtained by substituting Eq. (4.6) into Eq. (5.13).

$$\bar{V}=\frac{\dfrac{e}{C}\dfrac{R_T}{R_S}\dfrac{\exp(d)}{d^d}\gamma(d+1,d)}{1+\dfrac{e}{CV}\dfrac{R_T}{R_S}\dfrac{\exp(d)}{d^d}\gamma(d+1,d)} \ . \tag{5.15}$$

The average current through the junction is obtained by substituting Eq. (4.6) into Eq. (5.14):

$$\bar{I}=\frac{\dfrac{V}{R_S}}{1+\dfrac{e}{CV}\dfrac{R_T}{R_S}\dfrac{\exp(d)}{d^d}\gamma(d+1,d)} \ . \tag{5.16}$$

Forming the ratio, we obtain

$$\frac{\bar{V}}{\bar{I}}=\frac{eR_T}{CV}\frac{\exp(d)}{d^d}\gamma(d+1,d) \ . \tag{5.17}$$

This relation gives the current-voltage characteristics under an arbitrary bias condition. In the constant-current-bias limit, Eq. (5.17) reduces to

$$\bar{V}=\left[\frac{\pi e R_T I_{\mathrm{dc}}}{2C}\right]^{1/2}\left[1-\frac{2}{3}\left[\frac{2}{\pi d}\right]^{1/2}\right] \ . \tag{5.18}$$

The first term on the rhs is identical to the result obtained for the constant-current-bias condition,[6,7] while the second term gives the first-order correction due to a finite dissipation in the junction represented by R_T. In the constant-voltage-bias limit, Eq. (5.17) reduces to

$$\frac{\bar{V}}{\bar{I}}=\frac{R_S}{V}\left[V-\frac{e}{2C}\right]\left[1-\frac{d}{2}\right] , \tag{5.19}$$

where in the first parentheses we see a dc offset in the voltage caused by the Coulomb blockade, while the term $d/2$ in the second parentheses gives the first-order correction due to a finite dissipation in the source resistance R_S.

VI. DISCUSSION AND CONCLUSIONS

In this paper we have presented a new methodology based on the probability-density functions and have solved some problems concerning mesoscopic normal tunnel junctions in a fully analytic manner. The crucial observation which leads us to this method is that a tunneling process is Markovian regardless of how it is regulated by the external macroscopic system. Such a process can be completely described with a second-order correlation function. This is why we introduced charge- and time-interval distributions. In a previous paper,[14] the time-interval distribution was used to describe the crossover from random to Coulomb-regulated SET oscillations in the *time domain* as the external source changes from a voltage source into a current source, and as the bias voltage decreases towards an optimum value above $e/2C$. In the present paper, several techniques have been developed to obtain observable quantities such as the charge distribution across the junction and I-V characteristics from the charge-interval distribution. To quantitatively evaluate how regularly tunneling events occur, the degree of randomness of SET events is defined as the ratio of the standard deviation of dwell times to their mean value. A general expression of this quantity is obtained under an arbitrary bias condition. In particular, the minimum degree of randomness that can be achieved under the constant-current-bias condition is found to be $(R_TCI_{\mathrm{dc}}/e)^{1/2}$. The physical origin of this residual uncertainty is identified as the time-energy uncertainty relation due to delocalization of the electron wave function over both electrodes. For this reason and by analogy with quantum optics, we refer to this limit as the *standard quantum limit of mesoscopic normal tunneling current*.

Let us consider the relationship of the present theory to the conventional master-equation approach.[6-9,21] The master-equation approach deals with the probability distribution $P(Q,t)$ of accumulated charge Q at time t which is assumed to obey the following stochastic master equation:

$$\begin{aligned}\frac{\partial P(Q,t)}{\partial t}=&-I_{\mathrm{dc}}\frac{\partial P(Q,t)}{\partial Q}+r(Q+e)P(Q+e,t)\\&+l(Q-e)P(Q-e,t)-[r(Q)+l(Q)]P(Q,t)\\&+\frac{1}{CR_S}\frac{\partial}{\partial Q}[QP(Q,t)]+\frac{k_BT}{R_S}\frac{\partial^2}{\partial Q^2}P(Q,t) \ ,\end{aligned} \tag{6.1}$$

where the last two terms represent the CR_S relaxation and the thermal noise generated in the shunt resistance. The present theory neglects the backward transition rate

$l(Q)$ by assuming that $eV \gg k_BT$ and also neglects the noise term. Otherwise, the present theory includes all effects in Eq. (6.1) and should, therefore, give equivalent results. In particular, Eq. (5.8) gives a stationary solution of Eq. (6.1) under the same assumptions; the distribution rapidly grows up at first, and then exhibits a rather flat plateau, and finally rapidly falls, in good agreement with results obtained by computer simulation.[6,9] Although the present theory neglects the noise term, it can be incorporated into the theory by simply assuming an appropriate filter function because the noise term contributes additively to Eq. (6.1).

Finally, it should be emphasized that the formalism developed in the present paper is not limited to normal-metal junctions but applies equally to Josephson junctions because within the semiclassical approximation the (forward) transition rate of the Josephson junction reduces to a formula similar to the one in Eq. (2.5), although the physics of Bloch-wave oscillations[23] and SET oscillations differ substantially. This subject will be discussed elsewhere.

ACKNOWLEDGMENTS

The author would like to thank Dr. Y. Yamamoto, Dr. S. Kurihara, and Dr. T. Takayanagi of NTT basic research laboratories for stimulating discussions.

APPENDIX A

Let us evaluate the quantity

$$f\left[CV,\frac{R_T}{R_S}\right] \equiv \int_{e/2}^{CV} \exp\left[-\int_{e/2}^{Q}\frac{r(q)}{i(q)}\right] dQ \qquad \text{(A1)}$$

when $k_BT \ll e^2/2C$ and no noise current is generated in the source resistance. Then substituting the expressions

$$r(q)=\frac{q-e/2}{eR_TC}\Theta\left[q-\frac{e}{2}\right] \qquad \text{(A2)}$$

and

$$i(q)=\frac{CV-q}{CR_S} \qquad \text{(A3)}$$

into Eq. (A1) yields

$$f\left[CV,\frac{R_T}{R_S}\right]=\int_{e/2}^{CV}\left[\frac{CV-Q}{CV-\frac{e}{2}}\right]^{(1/e)(R_S/R_T)[CV-(e/2)]}\exp\left[\frac{1}{e}\frac{R_S}{R_T}\left[Q-\frac{e}{2}\right]\right]dQ\ . \qquad \text{(A4)}$$

If we change the integration variable to $t=(1/e)(R_S/R_T)(CV-Q)$ and set $d\equiv(1/e)(R_S/R_T)[CV-(e/2)]$, we obtain

$$f\left[CV,\frac{R_T}{R_S}\right]=\frac{eR_T}{R_S}\frac{\exp(d)}{d^d}\gamma(d+1,d)\ , \qquad \text{(A5)}$$

where $\gamma(a,x)$ is the incomplete γ function defined by

$$\gamma(a,x)\equiv\int_0^x t^{a-1}\exp(-t)dt,\quad \mathrm{Re}\,a>0\ . \qquad \text{(A6)}$$

This function has an asymptotic expansion[22]

$$\gamma(a,x)=x^a\exp(-x)\sum_{n=0}^{\infty}\frac{(a-1)!}{(a+n)!}x^n\ (a\ll 1)\ . \qquad \text{(A7)}$$

Substituting Eq. (A7) into Eq. (A5) yields

$$f\left[CV,\frac{R_T}{R_S}\right]=\left[CV-\frac{e}{2}\right]\left[1-\frac{d}{2}+O(d^2)\right]\quad (d\ll 1)\ . \qquad \text{(A8)}$$

For $d\gg 1$, $\gamma(d+1,d)$ has an asymptotic expansion[22]

$$\gamma(d+1,d)=\left[\frac{\pi}{2}\right]^{1/2}d^{d+(1/2)}\exp(-d)\times\left[1-\frac{2}{3}\left[\frac{2}{\pi d}\right]^{1/2}+\frac{1}{12d}+\cdots\right],\quad \left[d\gg 1,|\arg d|<\frac{\pi}{2}\right]\ . \qquad \text{(A9)}$$

Substituting Eq. (A9) into Eq. (A5) yields

$$f\left[CV,\frac{R_T}{R_S}\right]=\left[\frac{\pi}{2}\frac{eR_T}{R_S}\left[CV-\frac{e}{2}\right]\right]^{1/2}-\frac{2eR_T}{3R_S}\quad (d\gg 1)\ . \qquad \text{(A10)}$$

APPENDIX B

Let us evaluate two integrations used in text:

$$\overline{Q_f}\equiv\int_{e/2}^{CV}QP\left[\frac{e}{2},Q\right]dQ\ , \qquad \text{(B1)}$$

$$\overline{Q_f^2}\equiv\int_{e/2}^{CV}Q^2P\left[\frac{e}{2},Q\right]dQ\ , \qquad \text{(B2)}$$

where the subscript f refers to the final charge at which the tunneling event occurs. With the observation that

$$P\left[\frac{e}{2},Q\right]=-\frac{d}{dQ}\exp\left[-\int_{e/2}^{Q}\frac{r(q)}{i(q)}dq\right]\ , \qquad \text{(B3)}$$

we integrate the rhs of Eq. (B1) by parts, obtaining

$$\begin{aligned}\overline{Q_f}&=\frac{e}{2}+\int_{e/2}^{CV}\exp\left[-\int_{e/2}^{Q}\frac{r(q)}{i(q)}dq\right]dQ\ ,\\&=\frac{e}{2}+\frac{eR_T}{R_S}\frac{\exp(d)}{d^d}\gamma(d+1,d)\ ,\end{aligned} \qquad \text{(B4)}$$

where the last line follows from Eq. (A5). Similarly we integrate the rhs of Eq. (B2) by parts, obtaining

$$\overline{Q_f^2} = \left[\frac{e}{2}\right]^2 + 2\int_{e/2}^{CV} Q \exp\left[-\int_{e/2}^{Q}\frac{r(q)}{i(q)}dq\right]dQ\ ,$$
$$= \left[\frac{e}{2}\right]^2 + 2CV\int_{e/2}^{CV}\exp\left[-\int_{e/2}^{Q}\frac{r(q)}{i(q)}dq\right]dQ$$
$$-2\left[CV-\frac{e}{2}\right]\int_{e/2}^{CV}\left[\frac{CV-Q}{CV-\frac{e}{2}}\right]^{(1/e)(R_S/R_T)[CV-(e/2)]+1}\exp\left[\frac{R_S}{eR_T}\left[Q-\frac{e}{2}\right]\right]dQ\ , \tag{B5}$$

where the first integration in the rhs again given by Eq. (A5) and the second integration, when substituted with $t\equiv(1/e)(R_S/R_T)(CV-Q)$ and $d\equiv(1/e)(R_S/R_T)[CV-(e/2)]$, can be rewritten as

$$\frac{eR_T}{R_S}\frac{\exp(d)}{d^{d+1}}\int_0^d t^{d+1}e^{-t}dt = \frac{eR_T}{R_S}\frac{\exp(d)}{d^{d+1}}\gamma(d+2,d)$$
$$= \frac{eR_T}{R_S}\frac{\exp(d)}{d^{d+1}}(d+1)\gamma(d+1,d) - \frac{eR_T}{R_S}\ , \tag{B6}$$

where the last line follows from the identity[22]

$$\gamma(a+1,x) = a\gamma(a,x) - x^a e^{-x}\ . \tag{B7}$$

Substituting Eq. (B6) into (B5), we obtain

$$\int_{e/2}^{CV} Q^2 P\left[\frac{e}{2},Q\right]dQ = \left[\frac{e}{2}\right]^2 + 2CV\frac{eR_T}{R_S}\frac{\exp(d)}{d^d}\gamma(d+1,d) - 2\left[CV-\frac{e}{2}\right]\left[\frac{eR_T}{R_S}\frac{\exp(d)}{d^{d+1}}(d+1)\gamma(d+1,d) - \frac{eR_T}{R_S}\right]$$
$$= \left[\frac{e}{2}\right]^2 + \frac{2eR_T}{R_S}\left[CV-\frac{e}{2}\right] + \frac{2eR_T}{R_S}\left[\frac{e}{2}-\frac{eR_T}{R_S}\right]\frac{\exp(d)}{d^d}\gamma(d+1,d)\ . \tag{B8}$$

The variance of the final charge is obtained from Eqs. (B4) and (B8) as

$$\overline{(\Delta Q_f)^2} = \overline{Q_f^2} - \overline{Q_f}^2\ ,$$
$$= 2\frac{eR_T}{R_S}\left[CV-\frac{e}{2}\right] - 2\left[\frac{eR_T}{R_S}\right]^2\frac{\exp(d)}{d^d}\gamma(d+1,d) - \left[\frac{eR_T}{R_S}\frac{\exp(d)}{d^d}\gamma(d+1,d)\right]^2\ ,$$
$$= \left[\frac{eR_T}{R_S}\right]^2\left[2d - 2\frac{\exp(d)}{d^d}\gamma(d+1,d) - \left[\frac{\exp(d)}{d^d}\gamma(d+1,d)\right]^2\right]\ . \tag{B9}$$

In the limit $d \gg 1$, we can use the asymptotic expansion from Eq. (A9) as

$$\gamma(d+1,d) \simeq \left[\frac{\pi d}{2}\right]^{1/2}\frac{d^d}{\exp(d)}\ , \tag{B10}$$

and therefore Eq. (B9) reduces to

$$\overline{(\Delta Q_f)^2} = \left[2-\frac{\pi}{2}\right]\left[\frac{eR_T}{R_S}\right]^2 d\ , \tag{B11}$$

where we have neglected the second term in the rhs of Eq. (B9) since $d \gg \sqrt{d}$.

[1] I. Giaever and H. R. Zeller, Phys. Rev. Lett. **20**, 1504 (1968); Phys. Rev. **181**, 789 (1969).

[2] J. Lambe and R. C. Jacklevic, Phys. Rev. Lett. **22**, 1371 (1969).

[3] R. E. Cavicchi and R. H. Silsbee, Phys. Rev. Lett. **52**, 1453 (1984).

[4] T. A. Fulton and G. J. Dolan, Phys. Rev. Lett. **59**, 109 (1987).

[5] J. B. Barner and S. T. Rugiero, Phys. Rev. Lett. **59**, 807 (1987).

[6] D. V. Averin and K. K. Likharev, J. Low Temp. Phys. **62**, 345 (1986).

[7] E. Ben-Jacob, Y. Gefen, K. Mullen, and Z. Schuss, Phys. Rev. B **37**, 7400 (1988).

[8] K. Mullen, Y. Gefen, and E. Ben-Jacob, Physica B+C **152B**, 172 (1988).

[9] U. Gegenmüller and G. Schön, Physica B+C **152B**, 186 (1988).

[10] P. J. M. van Bentum, H. van Kempen, L. E. C. van de Leemput, and P. A. A. Teunissen, Phys. Rev. Lett. **60**, 369 (1988).

[11] L. S. Kuzmin, P. Delsing, T. Claeson, and K. K. Likharev, Phys. Rev. Lett. **62**, 2539 (1989).
[12] R. Wilkins, E. Ben-Jacob, and R. C. Jaklevic, Phys. Rev. Lett. **63**, 801 (1989).
[13] P. Delsing, K. K. Likharev, L. S. Kuzmin, and T. Claeson, Phys. Rev. Lett. **63**, 1861 (1989).
[14] M. Ueda and Y. Yamamoto, Phys. Rev. B **41**, 3082 (1990).
[15] In Ref. 14, we refer to this quantity as the degree of *regularity* of SET events. In the present paper, we rename it as the degree of *randomness* of SET events because this quantity is proportional to the standard deviation of dwell times.
[16] Y. Aharonov and D. Bohm, Phys. Rev. **122**, 1649 (1961).
[17] M. Büttiker and R. Landauer, Phys. Rev. Lett. **49**, 1739 (1982); IBM J. Res. Develop. **30**, 451 (1986).
[18] C. B. Duke, *Tunneling Phenomena in Solids,* edited by E. Burnstein and S. Lundqvist (Plenum, New York, 1969).
[19] M. Ueda, Phys. Rev. A **38**, 2937 (1988); **40**, 1096 (1989).
[20] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1985).
[21] E. Ben-Jacob, D. J. Bergman, B. J. Matkowsky, and Z. Schuss, Phys. Rev. B **34**, 1572 (1986).
[22] *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* edited by M. Abramowitz and I. A. Stegun, Nat. Bur. of Stand. (U.S.) Circ. No. 55 (Dover, New York, 1972).
[23] K. K. Likharev and A. B. Zorin, J. Low Temp. Phys. **59**, 347 (1985).