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Supersolitons in layered Josephson structures

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It is demonstrated that in a system of parallel-coupled long Josephson junctions forming a layered superconducting structure there are nonlinear excitations of coupled fluxon arrays in the form of dynamical "supersolitons" [A. V. Ustinov, Phys. Lett. A **136**, 155 (1989)]. The supersolitons in the system may be of two types, dynamical kinks and envelope solitons. The former ones are described by the elliptic-lattice equation which is transformed into the sine-lattice equation in the case of the dense fluxon arrays or the modified Boussinesq equation in the continuum limit. The latter solitons are oscillating ones and are described by the nonlinear Schrödinger equation in the discrete carrier case. These solitons may be important in transport properties of the flux flow in layered superconductors or high-T_c superconductors with twins under external magnetic fields. The stability of the nonlinear excitations is briefly discussed.

Nonlinear effects in long Josephson junctions (LJJ's) have been investigated intensively in recent years both theoretically and experimentally (see, e.g., Refs. 1 and 2). The LJJ may be simulated by the perturbed sine-Gordon (SG) equation,[1] its soliton (kink) solutions describe the motion of the Josephson vortices (often called "fluxons") in such a junction. Statics and dynamics of fluxons in a single homogeneous LJJ have been studied in detail so far (see, e.g., Refs. 1 and 2, and references therein).

In the last few years some interesting effects in LJJ's with arranged inhomogeneities (local regions where the Josephson critical current density is changing) were studied.[3-9] Of particular interest is the case of inhomogeneities periodically installed in the LJJ (the so-called lattice of inhomogeneities).[3,5-9] Recently, a feature for periodically modulated LJJ's has been found by numerical simulations,[7] and also verified experimentally[8] and explained analytically.[9] It was demonstrated that in LJJ with a lattice of pointlike inhomogeneities, besides the well-known flux-flow type of current-voltage (I-V) steps with numbers equal to the number of fluxons in the fluxon array, pronounced low-voltage steps arise,[7,8] and the effect may be explained due to a type of nonlinear excitations which was called "supersoliton."[7] These excitations may be described analytically in a spatially modulated SG system as local solitonlike variations in the density of the fluxon array.[9] The system is, as a matter of fact, a variant of the well-known Frenkel-Kontorova model, and the supersoliton may be identified with "a dislocation" in that model, primary fluxons being the effective interacting "particles." Unlike the Frenkel-Kontorova model, the supersoliton is a solution of the elliptic SG equation.[9]

In this paper we describe another model in which supersolitons are predicted analytically and may be observed experimentally. The system under consideration is a layered superconducting structure. The interest in such a type of superconducting systems has increased recently[10-12] because, on one hand, progress in technology made it possible to produce S-I-S-I-... (S represents superconductor, I represents isolator) systems of good quality and, on the other hand, some of the high-T_c superconductors, e.g., Bi-Sr-Ca-Cu-O, are layered ones indeed, or a layered Josephson structure may be formed by twinning plates in the superconductors (see, e.g., Ref. 13). In this paper we demonstrate that in the layered superconducting structures it is possible to have propagation of supersolitons, at least, of two types, dynamical kinks and envelope solitons. We briefly consider a stability of the solitons and also discuss a possibility of an experimental verification of supersolitons using the I-V characteristics of the layered Josephson systems.

We start from the model of the S-I-S-I-... system which is described by the system of dynamical equations[10]

$$\frac{\partial^2\phi_n}{\partial t^2}-\frac{\partial^2\phi_n}{\partial x^2}+\sin\phi_n=\sum_{m\neq n}\gamma^{|m-n|}\frac{\partial^2\phi_m}{\partial x^2}, \tag{1}$$

$$\gamma=\exp(-a/\lambda), \tag{2}$$

where $\phi_n(x,t)$ is the phase difference on the nth I layer, and we have used dimensionless variables: the coordinate x is measured in units of the Josephson length, $\lambda_J=(\Phi_0 c/16\pi^2 j_c\lambda)^{1/2}$ and the time is in units of $\omega_0^{-1}=\lambda_J/c_0$, where c_0 is the Swihart velocity, j_c is the Josephson critical current, and λ is the London penetration length. The coupling between the superconductors is determined by the dimensionless parameter (2), a being the thickness of the S layer (Fig. 1).

In the case $\gamma=0$ each Josephson layer of the system is described by the SG equation without a perturbation (coupling). An exact solution to the unperturbed SG equation describing the periodic fluxon array (the fluxons are of the equal polarities) is

$$\phi_n(x,t)=\pi-2\,\mathrm{am}\left(\frac{x-\zeta_n^{(t)}}{k},k\right), \tag{3}$$

where am is the Jacobi elliptic amplitude, k being the cor-

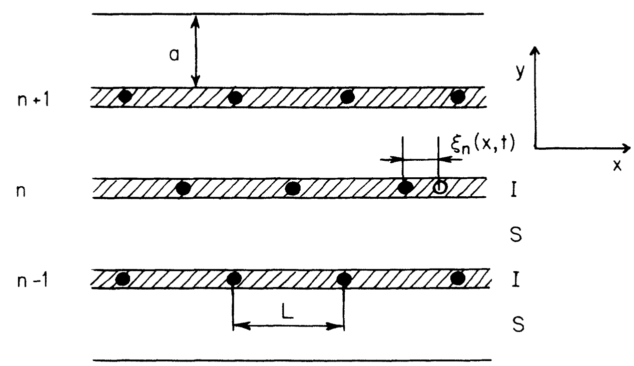

FIG. 1. Interacting arrays of fluxons in the layered Josephson structure: filled circles indicate stable fluxon positions; S represents superconductors, I represents isolators. The functions $\xi_n(x,t)$ describe the deviations of fluxons' coordinates from the equilibrium states.

responding modulus ($0<k<1$) generated by an external magnetic field, and $\zeta = Vt$, V being the arrays' velocity (we assume $V^2 \ll 1$). The spatial period of the array is $L = 2k\,K(k)$, $K(k)$ being the complete elliptic integral of the first kind. In the limit $k \to 1$ the array (3) tends to a single Josephson fluxon, and in the opposite case, when $k \ll 1$, the fluxons form a densely packed array.

The interaction between solitary fluxons belonging to different layers of the system is repulsive[11] and, as a result, the fluxons form the stable structure shown in Fig. 1. These interacting arrays may be considered as a "nonlinear medium" for propagation of perturbation waves. To study the waves analytically we will consider the coupling parameter γ as small using the Hamiltonian version of the perturbation theory for solitons.[14] In the framework of the approach, the interaction between the fluxon arrays is described by the perturbed Hamiltonian

$$H = \sum_n \left\{ H_n^{(0)} + \sum_{m \neq n} H_{nm}^{(\mathrm{int})} \right\}, \tag{4}$$

where$H_n^{(0)}$ is the proper well-known Hamiltonian of the unperturbed SG system for each nth array, and the averaged interaction Hamiltonian,

$$H_{nm}^{(\mathrm{int})} = \gamma^{|n-m|} \frac{1}{L} \int_0^L \frac{\partial \phi_n}{\partial x} \frac{\partial \phi_m}{\partial x} dx, \tag{5}$$

describes the fluxon arrays' coupling. Straightforward calculations of the interaction energy $W(\Delta_{nm}) = \frac{1}{2}(H_{nm} + H_{mn})$ with the unperturbed fluxon shapes (3) yield

$$W(z) = \frac{2\gamma^{|n-m|}}{k^2} \frac{\mathrm{dn} z}{\mathrm{sn}^2 z} \left[1 - \frac{\mathrm{cn}^2 z}{K(k)} \prod (k^2 \mathrm{sn}^2 z; k) \right], \tag{6}$$

where $z \equiv \Delta_{nm} \equiv (\zeta_n - \zeta_m)/k$. Here sn, cn, and dn are the standard elliptic Jacobi functions, and

$$\prod (x;k) = \int_0^K du [1 - x\,\mathrm{sn}^2(u,k)]^{-1}$$

is the complete elliptic integral of the third kind.

In the equilibrium (ground) state the fluxons of the system form a two-dimensional triangular lattice (filled circles in Fig. 1), the fluxons' positions are

$$\zeta_m^{(0)} = \zeta_n^{(0)} + \tfrac{1}{2}[1 - (-1)^{m-n}] k K(k).$$

We will describe oscillations of the fluxons near the equilibrium points, when the parameters $\xi_n \equiv \zeta_n - \zeta_n^{(0)}$ are dynamical variables of the fluxon lattice. The starting point of our analysis for the perturbation-induced system (1) is to impose a long-wave modulation on the array (3), replacing the parameters ξ_n by the functions $\xi_n(x,t)$ varying on a large scale. Inserting the corresponding wave form (3) into Eq. (4), it is possible to find the effective Hamiltonian written in the variables $\xi_n(x,t)$ (cf. Ref. 9),

$$H = \sum_n \int dx \left\{ \tfrac{1}{2}\rho \left[\left(\frac{\partial \xi_n}{\partial t} \right)^2 + \left(\frac{\partial \xi_n}{\partial x} \right)^2 \right] + \sum_m W(\xi_n - \xi_m) \right\}, \tag{7}$$

where $W(z)$ is defined in Eq. (6), and $\rho = 4E(k)/k^2 K(k)$, $E(k)$ being the complete elliptic integral of the second kind. The Hamiltonian (7) is the basis used to describe nonlinear excitations in the layered system.

The Hamiltonian (7) gives rise to the system of the dynamical equations,

$$\rho \left(\frac{\partial^2 \xi_n}{\partial t^2} - \frac{\partial^2 \xi_n}{\partial x^2} \right) + \sum_{m = \pm 2, \dots} W'(\xi_n - \xi_m) + \sum_{m = \pm 1, \dots} W'(\xi_n - \xi_{n+m} + kK) = 0, \tag{8}$$

where $W'(z) = dW/dz$. The system (8) may be naturally called a system of elliptic-lattice equations.

When the parameter γ is small, one can take into account only the nearest neighbors in Eq. (8), i.e., $m = \pm 1$. In such an approach, for the limit of a densely packed array, $k^2 \ll 1$, we obtain from Eq. (8) the so-called two-dimensional sine-lattice (SL) equation,

$$\frac{\partial^2 u_n}{\partial t^2} - \frac{\partial^2 u_n}{\partial x^2} - G[\sin(u_{n+1} - u_n) + \sin(u_{n-1} - u_n)] = 0, \tag{9}$$

where $u_n = 2\xi_n/k$, $G = \gamma/\rho \approx \gamma k^2/4$. Equation (9) in the limit of "hard" arrays, $\xi_n = \xi_n(t)$, has the form of the one-dimensional SL equation studied in Ref. 15. As is known, the SL equation has approximate, but well-defined, one- and multisoliton solutions in the form $u_n = A \tan^{-1}(\alpha_n/\beta_n)$ for an arbitrary constant $A > 0$, where the quantities α_n and β_n are simply discrete versions of the corresponding ones in the SG equation.[15] The simplest solution of such a type is a one-kink solution (superkink),

$$u_n = A \tan^{-1}[\exp(a\kappa n - \omega\sqrt{G}t)], \tag{10}$$

where the constants κ and ω are connected by the dispersion relation $\omega^2=4\sinh^2(\kappa a/2)$. As was demonstrated by Homma and Takeno,[15] although the SL equation does not appear to be exactly integrable, it shares the soliton generating properties with the Toda lattice equation. Numerical tests of the existence of approximate N-soliton solutions were done for $a\kappa=0.2$, 0.4, and 0.8, and the existence of the nearly integrable kinks appears to be guaranteed for $a\kappa$ at least up to $a\kappa=0.8$.[15]

Let us return to a general case defined in Eq. (8). Small-amplitude oscillations of the fluxon arrays are described by the equation

$$\rho\left(\frac{\partial^2\xi_n}{\partial t^2}-\frac{\partial^2\xi_n}{\partial x^2}\right)=\gamma A(k)(\xi_{n+1}+\xi_{n-1}-2\xi_n)+\gamma^2B(k)(\xi_{n+2}+\xi_{n-2}-2\xi_n)+\gamma C(k)[(\xi_{n+1}-\xi_n)^3-(\xi_n-\xi_{n-1})^3]\,, \tag{11}$$

where

$$A(k)=\frac{4k'}{k^4}\left[1-\frac{k^2}{2}-\frac{E(k)}{K(k)}\right],\quad B(k)=\frac{4}{3k^4}[k'^2-k^2E(k)/2K(k)]\,,$$

$$C(k)=\frac{2k'}{3k^6}\left[(k^2-2)\frac{E(k)}{K(k)}+2k'^2+\frac{k^4}{4}\right],$$

$k'=(1-k^2)^{1/2}$. In the linear limit Eq. (10) gives rise to the dispersion law of soundlike oscillations propagating across the layers,

$$\omega^2=q^2+\frac{4\gamma}{\rho}[A(k)\sin^2(\kappa a/2)+\gamma B(k)\sin^2(\kappa a)]\,. \tag{12}$$

In the continuum limit, when

$$\xi_{n\pm1}(x,t)\simeq\xi(x,y;t)\pm a\xi_y+\tfrac{1}{2}a^2\xi_{yy}\pm\tfrac{1}{6}a^3\xi_{yyy}+\tfrac{1}{24}a^4\xi_{yyyy}+\cdots\,,$$

Eqs. (10) and (11) may be transformed into the equation

$$\frac{\partial^2\xi}{\partial t^2}=\frac{\partial^2\xi}{\partial x^3}+C^2\frac{\partial^2\xi}{\partial y^2}+p\left[\frac{\partial\xi}{\partial y}\right]^2\frac{\partial^2\xi}{\partial y^2}+h\frac{\partial^4\xi}{\partial y^4}\,, \tag{13}$$

where

$$C^2=\frac{\gamma a^2}{\rho}[A(k)+4\gamma B(k)],\quad p=\frac{3a^4\gamma}{\rho}C(k),$$

$$h=\frac{\gamma a^4}{12\rho}[A(k)+16\gamma B(k)]\,,$$

which in the linear limit supports propagation of waves with the velocities (cf. Ref. 12), $V_x^2=1$ and $V_y^2=C^2$. If we set $u=\xi_y$, we obtain the two-dimensional modified Boussinesq (2MB) equation. Equation (13) has no kink solutions because for all k we have checked that $C(k)<0$ and the parameter p is negative (see, e.g., Ref. 16). In the continuum limit the superkinks exist when we will take account of the next nonlinear terms in Eq. (13).

One-dimensional excitations and superkinks propagating across the layers are stable against x-dependent perturbations. This may be easy to prove in the small-amplitude limit when the 2MB equation (13) is transformed into the modified Kadomtsev-Petviashvili equation (see, e.g., some proofs in Ref. 17).

Besides the kink soliton, the lattice equation (11) possesses breather solitons. These solutions may be obtained as envelope oscillating solitons in the "discrete carrier limit,"[16] when we look for solutions in the form

$$\xi_n(x,t)=F_n(x,t)\exp[i\theta_n(x,t)]+\text{c.c.},$$

where $\theta_n(x,t)=(qx+\kappa an+\omega t)$. In this approach, we can treat the phase $\theta_n(x,t)$ exactly and only use the continuum approximation for the envelope function $F(x,y;t)$. Thus, in Eq. (11) first we consider $\theta_n(x,t)$ and $F_n(x,t)$ as functions of the discrete variable n and after taking differences we go to the continuum limit only for $F_n(x,t)\approx F(x,y;t)$. As a result, we obtain the two-dimensional nonlinear Schrödinger (2NLS) equation,

$$-2i\omega\frac{\partial F}{\partial t}-\frac{\partial^2F}{\partial^2x}+g\frac{\partial^2F}{\partial^2z}+Q|F|^2F=0\,,$$

where

$$g=V_g^2-C^2\left[\cos(\kappa a)+4\gamma\frac{B(k)}{A(k)}\cos(2\kappa a)\right],$$

$V_g=\partial\omega/\partial\kappa$, and $z\equiv y-V_gt$, while Q is the function of κ and it is, as a matter of fact, a partial case of the general expression (5.14) given in Ref. 16. As was demonstrated in a number of papers, the 2NLS equation is unstable (see, e.g., Ref. 18), and it may give rise to a self-focusing wave from special classes of initial date.[17]

Since the dynamical superkink (10) propagates with a large velocity, it may be very important in the flux-flow transport properties of layered Josephson structures. To study the superkink motion in a real situation, one needs to add the effective bias current f and dissipative losses, $-\alpha\partial\phi_n/\partial t$ in the right-hand side of Eq. (1). Considering the terms as small, we lead to the modified Eq. (8) which includes the terms $-\alpha\rho\partial\xi/\partial t$ and $-\pi f/kK(k)$ (cf. Ref. 9). In the model of hard fluxon arrays, the energy-balance analysis yields the equilibrium velocity V_{eq} of the superkink providing equilibrium between dissipative losses ($\sim\alpha$) and energy input from the drive ($\sim f$).

The real dynamics of LJJ's is, as a matter of fact, more complicated and it must take account of two-dimensional flux motion along and across the layers. But, in any case, the motion of superkink will lead to peculiarities in the I-V curves at the voltage $\sim V_{\text{eq}}/l$, l being the length of the system in the y direction (see Fig. 1).

The modulus k of the elliptic functions may be determined by minimization of the thermodynamical potential

$\tilde{G}$ in the presence of the external magnetic field H_{ex}. In the case of densely packed array that yields $k^{-1} \approx H_{ex}/2$, and it defines all dependences as functions of the external magnetic field. For example, $C^2 \approx (\gamma a^2/2H_{ex}^2)(1+\gamma H_{ex}^4/3)$ (cf. Ref. 12).

In conclusion, we have predicted analytically supersolitons in a system of coupled Josephson junctions forming a layered superconducting structure. The supersolitons in the system are collective dynamical excitations of interacting fluxon arrays, and they may be described in some special cases by the two-dimensional elliptic-lattice or the modified Boussinesq equations (dynamical kinks) or by the nonlinear Schrödinger equation (envelope solitons). The predicted nonlinear excitations must give contributions to transport properties of the flux-flow in layered superconductors or systems with twinning planes, and they may be detected as peculiarities of the current-voltage characteristics of the fluxon-array dynamics in the presence of external magnetic fields.

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