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Regular fractal models of snowflakes and critical dynamics of the kinetic Ising model on fractals

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A group of regular fractal models for real snowflakes is proposed on a triangular lattice. The scaling form of the dynamics exponent of the kinetic Ising model on this group of fractal models, $Z=D_f+2/\nu$, is found by the exact time-dependent renormalization-group method. The dynamics exponent $Z=D_f+3/\nu$ is suggested for real snowflakes. Critical-dynamics behavior of the kinetic Ising model on fractals with $R_{\min}=2$ is investigated. The universal form of the dynamics exponent $Z=D_f+R/(2\nu)$ is suggested for fractals with $R_{\min}=2$.

Recently, more and more attention has been given to the diffusion-limited-aggregation (DLA) model.[1] Many phenomena are described by the DLA model.[2] One of the most interesting of these phenomena is the snowflake.[3] It has been fascinating human beings for many centuries due to its delicate features. The DLA model has deepened our understanding of it, but this is far from satisfactory. Here, we study the critical dynamics of the kinetic Ising model on snowflakes.

We introduce a group of regular fractal models as an approximation to real snowflakes. This is an extension of the Christon and Stinchcombe work[4] with a triangular lattice. This group of regular fractals must characterize the most important feature of the snowflakes' sixfold symmetry. Obviously, they must be dendritic and have suitable dimension. First, we define that a fractal with a point as the smallest unit is the site model of the fractal. Similarly, a fractal with a bond as the smallest unit is the bond model of the fractal. Koch curves are examples. The fractals proposed in this paper are represented by the site model. A seed has N generations. Such a fractal has $D_f=\ln(N)/\ln(b)$, where b is a rescaling factor. The simplest regular fractal is shown in Fig. 1. The fractal of Fig. 2 has $D_f=\ln 391/\ln 35$, which is very near the Monte Carlo result of 1.678 for snowflakes.

An Ising spin is set on every point of the fractal. Here, the bonds connecting points of the fractals stand for interaction between the spins. We study the kinetic Ising model[5] on fractals by the time-dependent renormalization-group (TDRG) method. We will not repeat statements about the TDRG method here, and employ all TDRG equations and notes from Ref. 6 in the following study. We define decimation as the multiple δ function $\prod_{i=1}^{N}\delta(\mu_i-\sigma_j)$ (see Fig. 1), and only study magnetic perturbation.

Up to now, critical dynamics of the kinetic Ising model has only dealt with a few kinds of fractals.[6-8] Also, a set of geometrical parameters has been introduced to classify the universality classes.[9] The most important one is the order of ramification R.[9] Gefen, Mandelbrot, and Aharony[9] showed that an Ising system located on a fractal with $R_{\min}<\infty$ has no finite-temperature transitions ($T_c=0$ only). One of the aims of this paper is to try to understand the relation of R to critical dynamics.

First, we perform TDRG on the simplest fractal model (see Fig. 1). The fractal in any stage ($N\geq 2$) can be made up of the three kinds of basic structures, which have a four-spin linear cell with 0, 4, and 5 hanging spins. The field parameters h_1, h_2, and h_6 relate the points with the nearest interacting numbers 1, 2, and 6, respectively. Following the decimation of Fig. 1, we start from Eq. (7) of Ref. 6. After taking the trace, the first term on the left-hand side is transformed into $\prod_i A_{s(i)}e^{k'\mu_i\mu_j}$ with $A_{s(i)}=\{[(2\cosh k)^{s(i)}4\cosh^3 k]/\cosh k'\}$ $k'=\tanh^{-1}(\tanh^3 k)$,

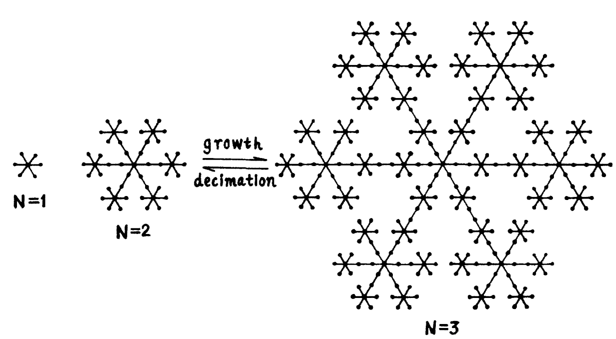

FIG. 1. The simplest regular model for snowflakes. The rescaling factor of length is $b=3$. The fractal dimension $D_f=\ln 7/\ln 3$.

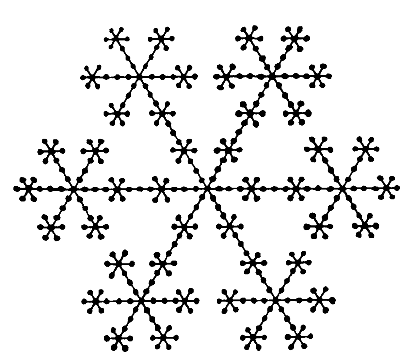

FIG. 2. The generator of the generalized regular model of snowflakes. $b=35$, $D_f=\ln 391/\ln 35$.

and $s(i)=0,4,5$ indicates the number of hanging bonds at point i. There is only an unstable fixed point at $K^*=\infty$. The transformation of the second term is described by the following recursion relationship of the field parameters:

$$h_1'=5b+a+h_6,\quad h_2'=4b+2a+h_6,\quad h_6'=6a+h_6. \tag{1}$$

In Eq. (1), $b=\tanh k h_1$ comes from a hanging spin while $a=\{[\tanh k(1+\tanh k)]/(1+\tanh^3 k)\}h_2$ is from a linear cell. The matrix Λ has the eigenvalues 7, 1, and 0 at $k^*=\infty$ ($\tanh k^*=1$). The transformation of the field parameters is $h_q'=b^{D_f}h_q$ at the critical point. The magnetic perturbation contributes D_f to the dynamics exponent Z.

The right-hand side of Eq. (7) of Ref. 5 becomes the terms $[(4\cosh^2 k)^6/A_0^6]P_e'W'(\mu)h_6\mu$, $[(4\cosh^2 k)/A_5]\times P_e'W'(\mu)h_6\mu$, $[(4\cosh^2 k)^2/(A_4A_0)]P_e'W'(\mu)h_6\mu$, which result in h_6', h_1', h_2', respectively. The matrix $\Omega=\{a_{ij}\}$ ($i,j=1-3$) only has three nonzero elements, which are $a_{13}=\cosh k'/(32\cosh^6 k)$, $a_{23}=\cosh^2 k'/(16\cosh^6 k)$, $a_{33}=\cosh^6 k'/\cosh^6 k$. So $W_{\max}$, the largest eigenvalue of the matrix Ω, is equal to $\cosh^6 k'/\cosh^6 k$. From $k'=\tanh^{-1}(\tanh^3 k)$, $W_{\max}$ can be rewritten as $b^{-3/\nu}$ with $\nu=1$ at $k^*=\infty$. So, the right-hand side of Eq. (7) of Ref. 6 contributes $3/\nu$ to Z. The scaling law of the dynamics exponent $Z=D_f+3/\nu$ is found. In the following study, we will show that it is universal to a group of regular fractal models for snowflakes.

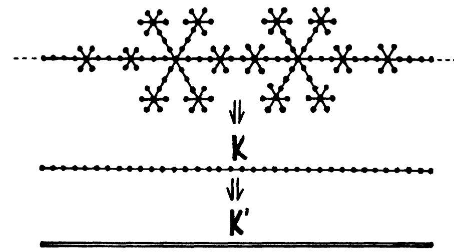

FIG. 3. One of the basic structures of the fractal of Fig. 2 and the renormalization procedure.

So, we perform TDRG on the generalized fractal model for snowflakes, which is generated by Fig. 2. The three kinds of basic structures of the fractal are the basic cell of Fig. 3 with 0, 4, and 5 hanging parts, which is a $\frac{1}{6}$ part of Fig. 2 except its seed point, respectively. The field parameters are still h_1, h_2, and h_6 as before. After Fig. 3's decimation, we obtain $k=\tanh^{-1}(\tanh^{35}k)$ and $A_{s(i)}=\{[(2\cosh k)^{s(i)}(2\cosh k)^{48}2^{34}\cosh^{35}k]/\cosh k'\}$ with $s(i)=0$, 240, 300, which is the number of hanging spins at point i.

As we have done in Ref. 6, it is easy to show that a hanging spin contributes the factor $h_1\tanh k$ to h_q' and the basic cell shown in Fig. 3 denotes the factor

$$\begin{aligned}a=[&h_2\tanh k+h_2\tanh^2 k+h_2\tanh^3 k+(4h_1\tanh k+4h_6)\tanh^4 k+\cdots\\&+h_2\tanh^{11}k+(20h_1\tanh^5 k+4h_2\tanh^4 k+4h_2\tanh^3 k+4h_2\tanh^2 k+4h_6\tanh^4 k+h_6)\\&\times\tanh^{12}k+\cdots+h_2\tanh^{34}k]/(1+\tanh^{35}k)\end{aligned} \tag{2}$$

to h_q'. So the recursion relations of the field parameters are

$$h_1'=b+a+h_6,\quad h_2'=c+2a+h_6,\quad h_6'=6a+h_6, \tag{3}$$

with

$$\begin{aligned}b=&(125\tanh^{17}k+20\tanh^9 k+20\tanh^5 k)h_1\\&+(25\tanh^{15}k+25\tanh^{14}k+25\tanh^{13}k+5\tanh^{11}k+5\tanh^{10}k+5\tanh^9 k+5\tanh^7 k+5\tanh^6 k+5\tanh^5 k\\&\quad+5\tanh^3 k+5\tanh^2 k+5\tanh k)h_2+(25\tanh^{16}k+5\tanh^{12}k+5\tanh^8 k+5\tanh^4 k)h_6,\end{aligned} \tag{4}$$

$$\begin{aligned}c=&(100\tanh^{17}k+16\tanh^9 k+16\tanh^5 k)h_1\\&+(20\tanh^{15}k+20\tanh^{14}k+20\tanh^{13}k4\tanh^{11}k+4\tanh^{10}k+4\tanh^9 k+4\tanh^7 k+4\tanh^6 k+4\tanh^5 k\\&\quad+4\tanh^3 k+4\tanh^2 k+4\tanh k)h_2+(20\tanh^{16}k+4\tanh^{12}k+4\tanh^8 k+4\tanh^4 k)h_6.\end{aligned} \tag{5}$$

From Eqs. (2)–(5), the matrix Λ can be written out. At $k^*=\infty$ it has eigenvalues 391, 1, and 0. The largest one $\lambda_{\max}=391$ can be rewritten as b^{D_f} ($b=35$, $D_f=\ln 391/\ln 35$). The $h_q'=b^{D_f}h_q$ is obtained again at $k^*=\infty$. The magnetic perturbation contributes D_f to Z.

It is not difficult to show that $W_{\max}$ is equal to $\cosh^6 k'/\cosh^6 k$ with $k=\tanh^{-1}(\tanh^b k)$ on any one of this group of fractals and can be rewritten as $b^{-3/\nu}$ with $\nu=1$. So the right-hand side of Eq. (7) of Ref. 6 always contributes $3/\nu$ to Z. We get $Z=D_f+3/\nu$ again.

Up to now, we have shown that there is the scaling law of the dynamics exponent $Z=D_f+3/\nu$ on this group of fractal models for snowflakes and that ν is always equal to 1. Through the above investigation, we arrive at the following conclusions.

(i) We introduce a group of fractal models for snowflakes. By the exact TDRG method, we find that there is a universal scaling form of the dynamics exponent of the kinetic Ising model $Z=D_f+3/\nu$ on this group of fractals and that ν is constant on them. So this group of fractal models characterizes essential features of snowflakes and is a reasonable model for snowflakes. Therefore, we suggest the dynamics exponent of the kinetic Ising model $Z=D_f+3/\nu$ for snowflakes.

(ii) In the renormalization transformation, the parameters $\{h_q\}$ form a constant subspace in the large parameter space of the RG. The parameters $\{h_q\}$ involve all important information produced in the RG transformation. We find that the h_q' consist of the factors $\tanh k$, $\tanh^2 k, \ldots, \tanh^n k, \ldots$. Because the parameter k is positive (antiferromagnetic system), the contribution above the factors h_q' is additive as the pieces of small mass become large ones. When the system arrives at the critical point ($k^* = \infty$), all the factors become 1. At this time, each factor behaves as the unit mass of a fractal. The change of $\{h_q\}$ caused by these factors from the renormalization is just like that of the mass M of a fractal. So, the $h_q' = b^{D_f} h_q$ like the $M' = b^{D_f} M$ is the typical scaling form of $\{h_q\}$ in critical area. Because an Ising spin system located on a fractal with $R_{\min} < \infty$ only has a zero-temperature transition, $k^* = \infty$, we can conclude that the magnetic perturbation of the master equation contributes D_f to the dynamics exponent Z of the kinetic Ising model on fractals[6-8] with $R_{\min} < \infty$. Therefore, the dynamics exponent Z has the form $D_f + f(R, \nu)$ for all fractals with $R_{\min} < \infty$. What is the $f(R, \nu)$? It comes from the contribution of the Liouville operator, which is a response to a spin wave. The correlation between Ising spins is the key factor to $f(R, \nu)$. When $R_{\min} = 2$, ν is finite for Ising systems on fractals, however, when $R_{\min} > R_c = 3$, $\nu = \infty$ for Ising systems on fractals. So, Ising systems on fractals with $R_{\min} = 2$ and $R_{\min} \geq 3$ have essential differences. Having considered all the studies here and before,[6-8] we write

$$W_{\max} = (\cosh k' / \cosh k)^R = (dy'/dy)^{-R/2} = b^{-R/2\nu} \tag{6}$$

for fractals with $R_{\min} = 2$. In more detail, $W_{\max} = b^{-R_{\max}/2\nu}$ is for site models of fractals and $W_{\max} = b^{-R_{\min}/2\nu}$ is for bond models of fractals. So, we conclude that the dynamics exponent of the kinetic Ising model on fractals of $R_{\min} = 2$ is

$$Z = D_f + R/(2\nu) \tag{7}$$

with the condition that if the fractal is a site model, $R = R_{\max}$ in Eq. (7), if the fractal is a bond model, $R = R_{\min}$ in Eq. (7).

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