Path-integral representation and critical properties of the quantum Potts model

L. De Cesare, K. Lukierska-Walasek,* I. Rabuffo, and K. Walasek*
Dipartimento di Fisica Teorica e Sue Metodologie per le Scienze Applicate, Università degli Studi di Salerno, I-84100 Salerno, Italy
(Received 27 October 1989)

A path-integral representation for the d-dimensional q-state quantum Potts model has been obtained starting from the microscopic Hamiltonian. The terms in the functional Hamiltonian, which are relevant in the renormalization-group sense, are given explicitly up to a quartic term in the order-parameter field. The critical properties of the model at zero temperature near five dimensions are analyzed within the field-theoretic renormalization-group approach with the help of the minimal subtraction method. In particular, some critical exponents to first order in $\epsilon=5-d$ are presented, and the occurrence of a $d\rightarrow d+1$ dimensional crossover is pointed out.

The q-state Potts model, as a possible generalization of the Ising model, has attracted a great deal of attention in condensed-matter physics.[1] It exhibits much richer critical properties but much less is known about it. In particular there have been several attempts to understand the nature of its phase transitions by varying the lattice dimension d and the number of states q regarded as continuous parameters.

A new investigation into the properties of the model has been made by looking at its quantum version.[2-7] The d-dimensional q-state quantum Potts model is worthy of study for its own sake both for possible applications in statistical physics and for its relevance in understanding the physics of elementary particles at very high temperature.[5]

The initial motivation for studying the quantum Potts model was the underlying belief that the d-dimensional classical models and their $(d-1)$-dimensional quantum counterparts have the same phase diagrams and lie in the same class of universality.[2] This is certainly true for the $d=2$ classical Potts model case where the exact Baxter results[8] can be carried over[9] to the one-dimensional quantum model and they are borne out in higher dimensions by approximate calculations.[3-7] However, no definitive conclusions have been drawn for any d and q. Up to now, mean-field[5-7] and real-space renormalization-group[3] (RG) calculations have been realized by working on the quantum lattice model. Unfortunately, the real-space RG transformations are not free from ambiguities and uncontrollable approximations which prevent reliable results for realistic values of d and q from being obtained, and may fail to predict correctly the order of possible phase transitions. So, it would be desirable to have RG transformations which have little arbitrariness in them and allow for systematic improvements.

In this Brief Report we present a path-integral representation of the q-state quantum Potts model in d dimensions which gives the possibility of conveniently studying its critical properties and the role of quantum fluctuations by varying q and d with the use of the reliable techniques, as the field-theoretic and Wilson RG treatments, well established for other quantum models.[10] As an evidence of the effectiveness of our functional representation, we apply the field-theoretic RG near five dimensions for arbitrary values of q. In particular the effects of quantum fluctuations are pointed out: a dimensional crossover $d\rightarrow d+1$ occurs as the temperature goes to zero.

An analogous study in the vicinity of three dimensions is not so immediate and a detailed analysis will be presented elsewhere.

The quantum model we shall consider is described by the Hamiltonian[3-7]

$$\hat{H}=-\frac{1}{2q}\sum_{i,j}\sum_{\alpha=1}^{q-1}J_{ij}\hat{\eta}_{i,\alpha}\hat{\eta}^{\dagger}_{j,\alpha}-\frac{2\Gamma}{q}\sum_{j}\sum_{\alpha=1}^{q-1}\hat{M}_j^{\alpha}, \quad (1)$$

where J_{ij} and Γ denote the ferromagnetic exchange coupling and a transverse field, respectively, and the sums on i and j are assumed to be over the N sites of a regular d-dimensional hypercubic lattice.

In Eq. (1), $\hat{\eta}_{j,\alpha}=\hat{\Omega}_j^{\alpha}$ $(\alpha=1,\ldots,q-1)$, the spin operators $\hat{\Omega}_j$ and $\hat{M}_j$ commute for different sites and, on a given site, they obey the $\mathbb{Z}_q$ algebra[3-7]

$$\hat{M}_j\hat{\Omega}_j=\omega^{-1}\hat{\Omega}_j\hat{M}_j,\ \ \hat{M}_j^{\dagger}\hat{\Omega}_j=\omega\hat{\Omega}_j\hat{M}_j^{\dagger},\ \ \hat{\Omega}_j^{q}=\hat{M}_j^{q}=\mathbb{1}, \quad (2)$$

where $\mathbb{1}$ is the identity operator, $\omega=e^{2\pi i/q}$ and the dagger denotes the Hermitian conjugation. In the representation $\{|\sigma\rangle_j;\sigma=1,\ldots,q\}$ in which $\hat{\Omega}_j$ is diagonal, one has

$$\hat{\Omega}_j|\sigma\rangle_j=\omega^{\sigma-1}|\sigma\rangle_j\,, \quad (3)$$

$$\hat{M}_j|\sigma\rangle_j=|\sigma+1\rangle_j\,, \quad (4)$$

where $|\sigma\rangle_j$ $(\sigma=1,\ldots,q)$ denotes an eigenstate of the spin operator $\hat{\Omega}_j$ at the jth site satisfying the relations

$$|\sigma+nq\rangle_j=|\sigma\rangle_j\ \ (n=0,\pm1,\pm2,\ldots)\,. \quad (5)$$

It is easy to show that

$$\hat{\eta}^{\dagger}_{j,\alpha}=\hat{\eta}_{j,q-\alpha},\ \ (\hat{M}_j^{\dagger})^{\alpha}=\hat{M}_j^{q-\alpha}. \quad (6)$$

We shall find it convenient to work in the representation $\{|\lambda\rangle_j';\lambda=1,\ldots,q\}$ in which $\hat{M}_j$ is diagonal[3,5] with

$$\hat{M}_j|\lambda\rangle_j'=\omega^{-(\lambda-1)}|\lambda\rangle_j'\,, \quad (7)$$

$$\hat{\Omega}_j|\lambda\rangle_j'=|\lambda+1\rangle_j'\,, \quad (8)$$

where the eigenstates $|\lambda\rangle_j'$ of $\hat{M}_j$ in the representation where the Potts coupling is diagonal are expressed by

$$|\lambda\rangle_j' = q^{-1/2}\sum_{\sigma=1}^{q}\omega^{(\lambda-1)(\sigma-1)}|\sigma\rangle_j . \quad (9)$$

Here it is also $|\lambda+nq\rangle_j' = |\lambda\rangle_j'$ and we have

$$\hat{\eta}_{j,\alpha}|\lambda\rangle_j' = |\lambda+\alpha\rangle_j', \; \hat{\eta}_{j,\alpha}^{\dagger}|\lambda\rangle_j' = |\lambda-\alpha\rangle_j' . \quad (10)$$

Notice that, owing to Eq. (6), one can put the Hamiltonian in the form

$$\hat{H} = -\frac{1}{2q}\sum_{i,j}\sum_{\alpha,\beta=1}^{q-1} J_{ij}\lambda_{\alpha,\beta}\hat{\eta}_{i,\alpha}\hat{\eta}_{j,\beta} - \frac{2\Gamma}{q}\sum_j\sum_{\alpha=1}^{q-1}\hat{M}_j^{\alpha} , \quad (11)$$

with $\lambda_{\alpha,\beta} = \delta_{\alpha+\beta,q}$.

The microscopic model (11) can be mapped in an imaginary-time path-integral representation. Indeed, the partition function can be written as[11]

$$Z = \left\{\exp\left[\frac{1}{2q}\left(\frac{\delta}{\delta h}, J\lambda\frac{\delta}{\delta h}\right)\right] Z_0[h]\right\}_{h=0} , \quad (12)$$

where $h \equiv \{h_{j\alpha}\}$ is an auxiliary field,

$$\left(\frac{\delta}{\delta h}, J\lambda\frac{\delta}{\delta h}\right) = \sum_{i,j}\sum_{\alpha,\beta=1}^{q-1} J_{ij}\lambda_{\alpha,\beta}\int_0^{1/T} d\tau \frac{\delta}{\delta h_{i\alpha}(\tau)}\frac{\delta}{\delta h_{j\beta}(\tau)} \quad (13)$$

and

$$Z_0[h] = T_r\left[e^{-\hat{H}_0/T} T_\tau \exp\left[\sum_j\sum_{\alpha=1}^{q-1}\int_0^{1/T} d\tau h_{j\alpha}(\tau)\hat{\eta}_{i,\alpha}(\tau)\right]\right] , \quad (14)$$

with

$$\hat{H}_0 = -\frac{2\Gamma}{q}\sum_j\sum_{\alpha=1}^{q-1}\hat{M}_j^{\alpha} . \quad (15)$$

In Eq. (14), T_τ denotes the τ-ordering operator and $\hat{\eta}_{j,\alpha}(\tau) = e^{\tau\hat{H}_0}\hat{\eta}_{j,\alpha}e^{-\tau\hat{H}_0}$.

Now, we use the functional identity[11]

$$\exp\left[\frac{1}{2q}\left(\frac{\delta}{\delta h}, J\lambda\frac{\delta}{\delta h}\right)\right] \equiv C^{-1}\int \mathcal{D}[\phi]\exp\left[-\frac{1}{2q}(\phi, J^{-1}\lambda\phi) + \frac{1}{q}\left(\phi, \frac{\delta}{\delta h}\right)\right] , \quad (16)$$

where C is a normalization constant,

$$\left(\phi, \frac{\delta}{\delta h}\right) = \sum_j\sum_{\alpha=1}^{q-1}\int_0^{1/T} d\tau \phi_{j\alpha}(\tau)\frac{\delta}{\delta h_{j\alpha}(\tau)} \quad (17)$$

and J^{-1} denotes the inverse of the exchange matrix $J = (J^{ij})$.

Then, apart from an inessential constant factor, we find

$$Z = \int \mathcal{D}[\phi] e^{-\mathcal{H}\{\phi\}} , \quad (18)$$

where the functional Hamiltonian $\mathcal{H}\{\phi\}$, expressed into functional power series of the fields $\{\phi_{j\alpha}(\tau)\}$, has the general form

$$\mathcal{H}\{\phi\} = \frac{1}{2q}\sum_{i,j}\sum_{\alpha,\beta=1}^{q-1}\int_0^{1/T} d\tau (J^{-1})_{ij}\lambda_{\alpha\beta}\phi_{i\alpha}(\tau)\phi_{j\beta}(\tau) + \sum_{n=1}^{\infty}\frac{1}{n!q^n}\sum_j\sum_{\alpha_1,\ldots,\alpha_n=1}^{q-1}\int_0^{1/T} d\tau_1 \cdots \int_0^{1/T} d\tau_n u_n^{(j)}(\tau_1,\ldots,\tau_n)\phi_{j\alpha_1}(\tau_1)\cdots\phi_{j\alpha_n}(\tau_n) . \quad (19)$$

Here $\phi_{j\alpha}(\tau)$, with $\phi_{j\alpha}(\tau+n/T) = \phi_{j\alpha}(\tau)$ $(n = 0, \pm 1, \pm 2, \ldots)$, represents the order-parameter field on the d-dimensional hypercubic lattice,

$$u_n^{(j)}(\tau_1,\ldots,\tau_n) = -\langle T_\tau\hat{\eta}_{j,\alpha_1}(\tau_1)\cdots\hat{\eta}_{j,\alpha_n}(\tau_n)\rangle_0^{(\mathrm{irr})} , \quad (20)$$

and $\langle\cdots\rangle_0^{(\mathrm{irr})}$ is the irreducible part of

$$\langle\cdots\rangle_0 = T_r\left[\exp\left(\frac{2\Gamma}{q}\sum_{\alpha=1}^{q-1}\hat{M}_j^{\alpha}\right)(\cdots)\right] \Big/ T_r\left[\exp\left(\frac{2\Gamma}{q}\sum_{\alpha=1}^{q-1}\hat{M}_j^{\alpha}\right)\right] . \quad (21)$$

By neglecting higher-order terms which are irrelevant in the RG sense, with some algebra taken into account [Eqs. (5)–(10)], the functional (19) reduces in the low temperature ($\Gamma/T \gg 1$) and continuum limits to

$$\mathcal{H}\{\psi\} = \int d^dx \int_0^{1/T} d\tau \left[\frac{1}{2}\sum_{\alpha_1,\alpha_2=1}^{q-1}\lambda_{\alpha_1,\alpha_2}\left(\nabla\psi_{\alpha_1}\cdot\nabla\psi_{\alpha_2} + c_0\frac{\partial\psi_{\alpha_1}}{\partial\tau}\frac{\partial\psi_{\alpha_2}}{\partial\tau} + r_0\psi_{\alpha_1}\psi_{\alpha_2}\right) - \frac{w_0}{3!}\sum_{\alpha_1,\ldots,\alpha_3=1}^{q-1}\lambda_{\alpha_1,\alpha_2,\alpha_3}\psi_{\alpha_1}\psi_{\alpha_2}\psi_{\alpha_3} + \frac{1}{4!}\sum_{\alpha_1,\ldots,\alpha_4=1}^{q-1}(u_0\Lambda_{\alpha_1,\ldots,\alpha_4} - v_0\lambda_{\alpha_1,\ldots,\alpha_4})\psi_{\alpha_1}\cdots\psi_{\alpha_4}\right] , \quad (22)$$

with

$$\lambda_{a_1,\ldots,a_k}=\sum_{n=1}^{k-1}\delta_{a_1+\cdots+a_k,\,nq}\,,$$
$$\Lambda_{a_1,\ldots,a_4}=\tfrac{1}{3}(\lambda_{a_1,a_2}\lambda_{a_3,a_4}+\lambda_{a_1,a_3}\lambda_{a_2,a_4}+\lambda_{a_1,a_4}\lambda_{a_2,a_3})\,. \tag{23}$$

In Eq. (22), $\psi_a=\psi_a(\mathbf{x},\tau)$ is the order parameter field in the $(\mathbf{x},\tau)$ space and the coefficients c_0, r_0, w_0, and v_0 are expressed in a simple way in terms of the lattice spacing a, the transverse field Γ, the number of states q, and the defined positive parameters $J_0=\sum_j J_{ij}$ and

$$J_1=J_0^{-2}\sum_j[(\mathbf{x}_i-\mathbf{x}_j)/a]^2J_{ij}\,.$$

It is immediately seen that, for the two-state quantum Potts model, one has $a_\nu=1$ $(\nu=1,\ldots,4)$, $\lambda_{a_1,a_2}=1$, $\lambda_{a_1,a_2,a_3}=\lambda_{a_1,a_2,a_3,a_4}=0$, and the functional (22) reduces to that well known[10] for the transverse Ising model.

As $T\to 0$, the quantum nature of the model becomes relevant and the Matsubara frequencies $\omega_l=2\pi lT$ $(l=0,\pm1,\pm2,\ldots)$ in the Fourier representation of the fields

$$\psi_a(\mathbf{x},\tau)=T^{1/2}\sum_{\omega_l}\int_{|\mathbf{p}|<\Lambda=\pi/a}\frac{d^dp}{(2\pi)^d}e^{-i(\mathbf{p}\cdot\mathbf{x}+\omega_l\tau)}\psi_a(\mathbf{p},\omega_l) \tag{24}$$

become a continuous variable. Then, with an appropriate rescaling of the fields, at $T=0$ Eq. (22) reduces to

$$\begin{aligned}\mathcal{H}\{\psi\}={}&\tfrac{1}{2}\sum_{a_1,a_2=1}^{q-1}\lambda_{a_1,a_2}\int_{\mathbf{p}}\int_{\omega}(r_0+p^2+\omega^2)\psi_{a_1}(\mathbf{p},\omega)\psi_{a_2}(-\mathbf{p},-\omega)\\&-\frac{w_0}{3!}\sum_{a_1,\ldots,a_3=1}^{q-1}\lambda_{a_1,a_2,a_3}\prod_{\nu=1}^{3}\int_{\mathbf{p}_\nu}\int_{\omega_\nu}\mathcal{E}\left[\sum_{\nu=1}^{3}\mathbf{p}_\nu\right]\mathcal{E}\left[\sum_{\nu=1}^{3}\omega_\nu\right]\psi_{a_1}(\mathbf{p}_1,\omega_1)\cdots\psi_{a_3}(\mathbf{p}_3,\omega_3)\\&+\frac{1}{4!}\sum_{a_1,\ldots,a_4=1}^{q-1}(u_0\Lambda_{a_1,\ldots,a_4}-v_0\lambda_{a_1,\ldots,a_4})\prod_{\nu=1}^{4}\int_{\mathbf{p}_\nu}\int_{\omega_\nu}\mathcal{E}\left[\sum_{\nu=1}^{4}\mathbf{p}_\nu\right]\mathcal{E}\left[\sum_{\nu=1}^{4}\right]\psi_{a_1}(\mathbf{p}_1,\omega_1)\cdots\psi_{a_4}(\mathbf{p}_4,\omega_4)\,,\end{aligned} \tag{25}$$

where

$$\int_{\mathbf{p}}\cdots=\int_{|\mathbf{p}|<\Lambda}d^dp/(2\pi)^d\cdots,\quad \int_{\omega}\cdots=\int_{-\infty}^{+\infty}d\omega/2\pi\cdots,\quad \mathcal{E}(\mathbf{p})=(2\pi)^d\delta(\mathbf{p}),\quad \mathcal{E}(\omega)=2\pi\delta(\omega)\,.$$

The functional Hamiltonian (25) has two borderline dimensions $d^*=3$ and $d^*=5$. For a RG study of the critical properties of the $(T=0)$ quantum Potts model in the vicinity of five dimensions it is sufficient to preserve in (25) the cubic terms only. The quartic couplings become relevant near three dimensions. Here we limit ourselves to analyze quantum criticality near $d^*=5$ within a field-theoretic RG approach via the minimal subtraction method.

The functional Hamiltonian (25), considered up to $O(\psi^3)$ terms, is renormalizable for $d\le 5$. In order to formulate the corresponding renormalized theory without ultraviolet divergences, we introduce the renormalized parameter w related to the bare w_0 as

$$\mu^{-\epsilon/2}w_0=Z_\psi^{-3/2}(w)Z_w(w)w\equiv g(w)$$

and the rescaled field $\psi_a^R(\mathbf{p},\omega)=Z_\psi^{-1/2}(w)\psi_a(\mathbf{p},\omega)$. Here μ denotes an arbitrary renormalized momentum unit, $\epsilon=5-d$ and Z_ψ and Z_w are renormalization constants.

Now, we define the RG parameter $-\infty<l<+\infty$ by the relation

$$r_0-r_{0c}=Z_2(\bar w)\bar\mu^2he^l\,, \tag{26}$$

with $r_0-r_{0c}=\text{const}$, where $h=(\Gamma-\Gamma_c)/\Gamma_c$ is the transverse field critical deviation, Z_2 denotes the renormalization constant and $r_{0c}(\Gamma_c)$ is determined by the equation $\Gamma_2(\mathbf{p}=0,\ \omega=0)=0$ for the two-point vertex function $\Gamma_2(\mathbf{p},\omega)$. Here $\bar w$ and $\bar\mu$ depend on the RG parameter l.

Then, using the minimal subtraction method[12] and Eq. (26), one obtains at the one loop level the following RG equations:

$$\bar\mu\frac{d\bar w}{d\bar\mu}=-\frac{\epsilon}{2}\bar w\left[1-\frac{3K_d}{32\epsilon}(10-3q)\bar w^2\right], \tag{27}$$

$$h\frac{d\bar\mu}{dh}=\nu(\bar w)\bar\mu\,, \tag{28}$$

with the initial conditions $\bar\mu=\mu$ and $\bar w=w$ for $h=1$ and $K_d=2^{1-d}\pi^{-d/2}/\Gamma(d/2)$. Here,

$$\nu(\bar w)=\frac{1}{2}\left[1-\frac{10(q-2)}{32}K_d\bar w^2\right], \tag{29}$$

and we have assumed h as the scaling variable by using the condition $he^l=1$.

Equation (27) has two fixed points: (i) the Gaussian fixed point, stable for $d>5$, with $w^*=0$ and scaling exponent $y_{w^*}^G=-\epsilon/2$; (ii) a nontrivial fixed point, stable for $d\le 5$, with $w^{*2}=[32K_d/3(10-3q)]\epsilon$ and $y_{w^*}=\epsilon$. It has physical sense for $q<\frac{10}{3}$. In this case, the nonlinear solution of Eq. (27) reads

$$\bar w^2=\begin{cases}w^2w^{*2}/[w^2+(w^{*2}-w^2)(\bar\mu/\mu)\epsilon], & d<5,\\ w^2/[1-\frac{3}{32}w^2(10-3q)\ln(\bar\mu/\mu)], & d=5,\end{cases} \tag{30}$$

and the correlation length exponent for $d\le 5$ is given by

$$\nu=\nu(w^*)=\tfrac{1}{2}\left[1-\frac{10(q-2)}{3(10-3q)}\epsilon+O(\epsilon^2)\right]. \tag{31}$$

Other critical exponents can be obtained in the usual way.[10]

Notice that for $q > \frac{10}{3}$ only the Gaussian fixed point is physical, but it is unstable for $d < 5$ and no continuous phase transition may occur. Thus, for the quantum Potts model at $T = 0$ one has $q_c\ (d < 5) = \frac{10}{3}$.

Comparing the quantum critical exponents with the corresponding classical ones,[13] it immediately follows that the dimensional crossover $d \rightarrow d+1$ occurs also in the present quantum model near five dimensions.[10] Preliminary RG calculations indicate that a different picture of the quantum fluctuation effects may appear in the vicinity of $d^* = 3$.

Two of us (K.L.-W. and K.W.) appreciate the warm hospitality of the Department of Theoretical Physics of Salerno University during the preparation of this paper. The financial support of Polish Academy of Science Contract No. C.P.B.P. 01.12 is also acknowledged.

*Permanent address: Institute of Physics, Pedagogical University of Zielcna Gòra, plac Slowianski 6, 65-069 Zielcna Gòra, Poland.

[1]F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
[2]E. Fradkin and L. Susskind, Phys. Rev. D **17**, 2637 (1978); J. B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
[3]J. Solyom and P. Pfeuty, Phys. Rev. B **24**, 218 (1981).
[4]J. B. Kogut and D. K. Sinclair, Phys. Lett. **81A**, 149 (1981); *ibid.* **86A**, 38 (1981).
[5]Y. Y. Goldschmidt and J. Shigemitsu, Nucl. Phys. **B200 [FS4]**, 149 (1982).
[6]A. Cant and P. A. Pearce, Commun. Math. Phys. **90**, 373 (1983).
[7]P. A. Pearce, Physica **125A**, 247 (1984).
[8]R. J. Baxter, J. Phys. C **6**, L445 (1973).
[9]C. J. Hamer, J. Phys. A **14**, 2981 (1981).
[10]J. Hertz, Phys. Rev. B **14**, 1164 (1976); G. Busiello and L. De Cesare, Nuovo Cimento **59B**, 327 (1980); G. Busiello, L. De Cesare, and I. Rabuffo, Physica **117A**, 445 (1983); K. Lukierska-Walasek and K. Walasek, J. Phys. C **16**, 3149 (1983); K. Lukierska-Walasek, Physica **127A**, 613 (1984).
[11]J. Rzewuski, *Field Theory* (Ilife Books LTD, London, 1969), Vol. II.
[12]G. t'Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972).
[13]J. Hertz, Phys. Scr. **T10**, 1 (1985).